IMS

MATHS

BOOK-04

4

MECHANICS

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Moments and Products of Inertia

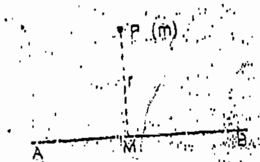
§ 1.1. Definitions :

(a) Rigid Body. *A rigid body is a collection of particles such that the distance between any two particles of the body remains always the same.*

(b) Moment of Inertia of a particle.

The moment of inertia of a particle of mass m at the point P, about the line AB is defined by

$$I = mr^2,$$



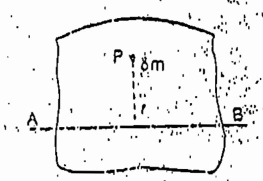
where r is the perpendicular distance of P from the line AB.

(c) Moment of Inertia of a system of particles. The moment of inertia of a system of particles of masses $m_1, m_2, \ldots, m_n$ at distances $r_1, r_2, \ldots, r_n$ respectively from the line AB, about the line AB is defined by

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots\ldots + m_n r_n^2$$

$$= \sum_{i=1}^{n} m_i r_i^2.$$

(d) Moment of Inertia of a body. Let δm be the mass of an elementary portion of the body and r its distance from the line AB, then the moment of inertia of the mass δm about the line AB is $r^2 \delta m$.



$\therefore$ The moment of inertia of the body about the line AB is given by

$$I = \int r^2 dm,$$

where the integration is taken over the whole body.

(e) Radius of Gyration. The moment of inertia of a body about the line AB is given by

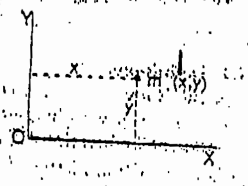
$$I = \int r^2 dm$$

If the total mass of the body is M and K a quantity such that

$$I = MK^2,$$

then K is called the *radius of gyration* of the body about the line AB.

(ii) **Product of Inertia.** Let (x, y) be the coordinates of a mass m with respect to two mutually perpendicular lines OX and OY as axes. Then the product of inertia of mass m with respect to the lines OX and OY is defined by mxy.



If (x, y) be the coordinates of the mass m of an elementary portion of the body with respect to the perpendicular axes OX and OY, then the product of inertia of the body about these axes OX and OY is defined by Σmxy.

§ 1.2. Moment and Product of Inertia with respect to three mutually perpendicular axes.

Let (x, y, z) be the coordinates of the mass m of a body with respect to three mutually perpendicular axes OX, OY, OZ in space. Then we shall denote by A, B, C the moments of inertia of the body about the coordinate axes OX, OY, OZ respectively and by D, E, F the products of inertia about the axes OY, OZ; OZ, OX and OX, OY respectively. These moments and products of inertia are given by

$A = \Sigma m (y^2 + z^2)$, $\quad B = \Sigma m (z^2 + x^2)$, $\quad C = \Sigma m (x^2 + y^2)$

$D = \Sigma myz$, $\quad E = \Sigma mzx$, $\quad F = \Sigma mxy$.

§ 1.3. Some Simple Propositions:

Prop. I. *If A, B, C denote the moments and D, E, F the products of inertia about three mutually perpendicular axes, the sum of any two of them is greater than the third.*

We have, $\quad A = \Sigma m (y^2 + z^2), B = \Sigma m (z^2 + x^2), C = \Sigma m (x^2 + y^2)$

then $A + B - C = \Sigma m (y^2 + z^2) + \Sigma m (z^2 + x^2) - \Sigma m (x^2 + y^2)$

$$= 2\Sigma mz^2 = +\text{ ve}.$$

$\therefore A + B > C.$

Prop. II. *The sum of the moments of inertia about any three rectangular axes meeting at a given point is always constant and is equal to twice the moment of inertia about that point.*

We have

$$A + B + C = \Sigma m (y^2 + z^2) + \Sigma m (z^2 + x^2) + \Sigma m (x^2 + y^2)$$

$$= 2\,\Sigma m (x^2 + y^2 + z^2) = 2\Sigma mr^2$$

$$= 2\,(M.I. \text{ of the body about the given point})$$

$\because r = \sqrt{(x^2 + y^2 + z^2)} =$ distance of the mass m at (x, y, z) from the given point O as origin.

Thus the sum $A + B + C$ is independent of the directions of axes and is equal to twice the moment of inertia about the given point.

Prop. III. *The sum of the moments of inertia of a body with reference to any plane through a given point and its normal at that point is constant and is equal to the moment of inertia of the body with respect to the point.*

Let the given point O be taken as the origin and the plane as XY plane. If C' is the moment of inertia of the body about the XY plane and C the moment of inertia of the body about its normal at O, which is Z-axis, then

$$C' = \Sigma mz^2 \text{ and } C = \Sigma m (x^2 + y^2).$$

$$\therefore C' + C = \Sigma m (x^2 + y^2 + z^2) = \Sigma mr^2$$

$$= M.I. \text{ of the body about } O.$$

Thus $C' + C$ is independent of the plane through O and is constant equal to the moment of inertia of the body about the point.

Note. By Prop. II, we have $A + B + C = 2\Sigma mr^2$

and by prop. III, we have $C + C' = \Sigma mr^2$

$\therefore C + C'_1 = \frac{1}{2}(A + B + C)$ or $C' = \frac{1}{2}(A + B - C)$.

Thus if A', B', C' denote the moments of inertia of the body with respect to the planes YZ, ZX and XY respectively, then

$A' = \frac{1}{2}(B + C - A)$, $B' = \frac{1}{2}(C + A - B)$ and $C' = \frac{1}{2}(A + B - C)$

Prop. IV. $A > 2D$, $B > 2E$ and $C > 2F$.

we know that $A.M. > G.M.$

$\therefore \dfrac{y^2 + z^2}{2} > \sqrt{(y^2 \cdot z^2)}$ or $y^2 + z^2 > 2yz$

or $\Sigma m(y^2 + z^2) > 2\Sigma\, myz$

i.e. $A > 2D$.

Similarly $B > 2E$ and $C > 2F$.

MOMENTS OF INERTIA IN SOME SIMPLE CASES.

§ 1.4. Moment of Inertia of a uniform rod of length $2a$

(i) *About a line through an end and perpendicular to the rod.*

Let M be the mass of a rod AB of length $2a$, then mass of the rod per unit length $= \rho = M/2a$.

Consider an element PQ of breadth δx at a distance x from the end A.

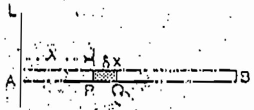
Mass of the element. $PQ = \dfrac{M}{2a}\,\delta x = \delta m.$

M.I. of this element PQ about the line LM passing through the end A and perpendicular to the rod AB

$= x^2 \delta m = \dfrac{M}{2a}\, x^2 \delta x.$

$\therefore$ M.I. of the rod AB about LM

$= \int_0^{2a} \frac{M}{2a} x^2 dx = \frac{M}{2a}\left[\frac{1}{3}x^3\right]_0^{2a} = \frac{4}{3}Ma^2$



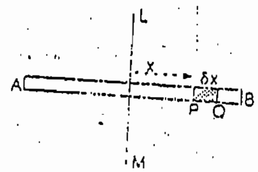
(ii) *About a line through the middle point and perpendicular to the rod.*

Let LM be the line passing through the middle point C and perpendicular to the rod AB.

Consider an element PQ of breadth δx at a distance x from the middle point C.

Mass of the element

$PQ = \frac{M}{2a}\delta x = \delta m \qquad (\because \rho = M/2a)$



M.I. of the element PQ about the line LM

$= x^2 \delta m = x^2 \frac{M}{2a}\delta x.$

$\therefore$ M.I. of the rod AB about LM

$= \int_{-a}^{a} \frac{M}{2a} x^2 dx = \frac{M}{2a}\left[\frac{1}{3}x^3\right]_{-a}^{a} = \frac{1}{3}Ma^2$

§ 1.5 Moment of Inertia of a rectangular lamina. [Meerut TDC 96 (BP)]

(i) *About a line through its centre and paralles to a side.*

Let M be the mass of a rectangular lamina $ABCD$ such that $AB = 2a$ and $BC = 2b$.

$\therefore$ Mass per unit area of the rectangle $= \rho = \frac{M}{4ab}$.

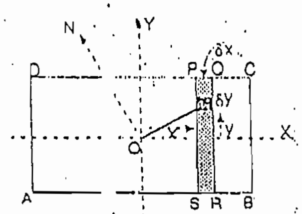
Let OX and OY be the lines parallel to the sides AB and BC of the rectangle through its centre C.

Consider an elementary strip $PQRS$ of breadth δx at a distance x from O and parallel to BC.

Mass of the strip

$PQRS = \rho . 2b\delta x$

$= \frac{M}{4ab} 2b\delta x = \frac{M}{2a}\delta x = \delta m.$



M.I. of the strip about $OX = \frac{1}{3}b^2 \delta m$.

[see § 1.4 (ii)]

$= \frac{1}{3}b^2 . \frac{M}{2a}\delta x = \frac{1}{6}\frac{M b^2}{a}\delta x.$

$\therefore$ M.I. of the rectangle $ABCD$ about OX

$= \int_{-a}^{a} \frac{Mb^2}{6a} dx = \frac{Mb^2}{6a}[x]_{-a}^{a} = \frac{1}{3}Mb^2.$

Similarly M.I. of the rectangle $ABCD$ about $OY = \frac{1}{3}Ma^2$

Alter. Consider an elementary area $\delta x \delta y$ at a point (x, y) of the lamina.

Mass of the elementary area $= \rho \delta x \delta y = \frac{M}{4ab}\delta x \delta y = \delta m.$

M.I. of this elementary mass about $OX = y^2 \delta m = \frac{My^2}{4ab}\delta x \delta y.$

$\therefore$ M.I. of the rectangular lamina $ABCD$ about OX

$= \int_{-a}^{a}\int_{-b}^{b} \frac{M}{4ab} y^2 \, dx dy = \frac{M}{4ab}\int_{-a}^{a}\left[\frac{1}{3}y^3\right]_{-b}^{b} dx$

$= \frac{M}{4ab}\cdot\frac{2}{3}b^3 \int_{-a}^{a} dx = \frac{M}{6a}b^2 [x]_{-a}^{a} = \frac{1}{3}Mb^2.$

(ii) *About a line through its centre and perpendicular to its plane.*

Let ON be the line through the centre O and perpendicular to the plane of the rectangular lamina $ABCD$.

Consider an elementary area $\delta x \delta y$ at a point (x, y) of the lamina.

Mass of the elementary area $= \rho \delta x \delta y = \frac{M}{4ab}\delta x \delta y = \delta m;$

Distance of this elementary area from $ON = \sqrt{(x^2 + y^2)}$.

$\therefore$ M.I. of this elementary mass about ON

$= ON^2 \, \delta m = (x^2 + y^2) \cdot \frac{M}{4ab}\delta x \delta y.$

Hence M.I. of the rectangular lamina about ON

$= \int_{-a}^{a}\int_{-b}^{b} \frac{M}{4ab}(x^2 + y^2) \, dx dy$

$= \frac{M}{4ab}\int_{-a}^{a}\left[x^2 y + \frac{1}{3}y^3\right]_{-b}^{b} dx = \frac{M}{4ab}\int_{-a}^{a} 2\left(bx^2 + \frac{1}{3}b^3\right) dx$

$= \frac{M}{4ab}\left[2\left(\frac{b}{3}x^3 + \frac{1}{3}b^3 x\right)\right]_{-a}^{a} = \frac{M}{4ab}\cdot\frac{4}{3}(ba^3 + b^3 a)$

$= \frac{M}{3}(a^2 + b^2).$

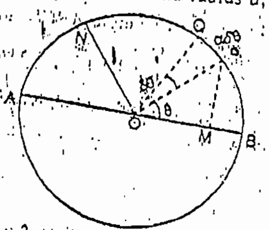
Note. M.I. about $ON = \frac{M}{3}(a^2 + b^2) = \frac{1}{3}Ma^2 + \frac{1}{3}Mb^2$

$=$ M.I. about OY + M.I. about OX.

§ 1.6. Moment of Inertia of a Circular wire. [Meerut TDC 95 (BP)]

(i) *About a diameter.*

Let M be the mass of the circular wire of centre O and radius a, then mass per unit length of the wire $= \rho = M/2\pi a$.



Consider an elementary arc $PQ = a\,\delta\theta$ of the wire, then its mass $= \rho a \delta\theta = \delta m$.

Distance of this element from the diameter $AB = PM = a \sin\theta$.

$\therefore$ M.I. of this element about the diameter AB

$= (a \sin\theta)^2 \cdot \delta m = a^2 \sin^2\theta \cdot \rho a \delta\theta = \rho a^3 \sin^2\theta \cdot \delta\theta$.

Hence M.I. of the circular wire about the diameter AB

$= \int_0^{2\pi} = \rho a^3 \sin^2\theta \, d\theta = \frac{1}{2}\rho a^3 \int^{2\pi} (1 - \cos 2\theta)\, d\theta$

$= \frac{1}{2}\rho a^3 \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \frac{1}{2} \cdot \frac{M}{2\pi a} \cdot a^3 \cdot 2\pi$ $\qquad (\because \rho = M/2\pi a)$

$= \frac{1}{2} Ma^2$.

(ii) *About a line through the centre and perpendicular to its plane.*

Let ON be the line through the centre O and perpendicular to the plane of the circular wire.

$\therefore$ M.I. of the elementary arc PQ about ON

$= OP^2 \cdot \delta m = a^2 \cdot \rho a\, \delta\theta = \rho a^3\, \delta\theta$.

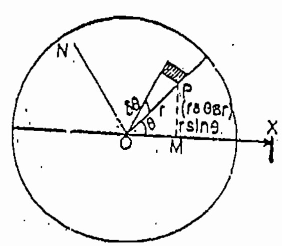
Hence M.I. of the wire about ON

$= \int_0^{2\pi} \rho a^3 d\theta = \frac{M}{2\pi a} \cdot a^3 [\theta]_0^{2\pi} = Ma^2$. $\qquad (\because \rho = M/2\pi a)$

§ 1.7. Moment of Inertia of a Circular plate.

(i) *About a diameter.*

Let M be the mass of a circular plate of centre O and radius a; then mass per unit area of the plate $= \rho = M/\pi a^2$.



Consider an elementary area $r\delta\theta\delta r$ at the point $P(r, \theta)$ of the plate referred to the centre O as the pole and OX as the initial line.

Mass of the element $= \rho \cdot r\delta\theta\delta r = \delta m$.

Distance of this element from $OX = OM = r \sin\theta$.

$\therefore$ M.I. of the element about OX

$= (r \sin\theta)^2 \cdot \delta m = r^2 \sin^2\theta \cdot \rho r\, \delta\theta\delta r = \rho r^3 \sin^2\theta\, \delta\theta\delta r$.

Hence M.I. of the circular plate about OX

$= \int_0^{2\pi}\int_0^{a} \rho r^3 \sin^2\theta\, d\theta dr = \rho \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_0^a \sin^2\theta\, d\theta$

$= \frac{1}{4}\rho a^4 \cdot \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \frac{1}{8}\rho a^4 \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{8} \cdot \rho a^4 \cdot 2\pi = \frac{1}{4} \cdot \frac{M}{\pi a^2} \cdot \pi a^4 = \frac{1}{4} Ma^2$ $\qquad (\because \rho = M/\pi a^2)$

(ii) *About a line through the centre and perpendicular to its plane.*

Let ON be the line through the centre O and perpendicular to the plane of the plate.

$\therefore$ M.I. of the elementary area about ON

$= OP^2 \cdot \delta m = r^2 \cdot \rho r\delta\theta\delta r = \rho r^3\, \delta\theta\delta r$.

Hence M.I. of the circular plate about O

$= \int_0^{2\pi}\int_0^{a} \rho r^3\, d\theta dr = \rho \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_0^a d\theta$

$= \frac{1}{4}\rho a^4 [\theta]_0^{2\pi} = \frac{1}{4} \cdot \frac{M}{\pi a^2} \cdot a^4 \cdot 2\pi = \frac{1}{2} Ma^2$ $\qquad (\because \rho = M/\pi a^2)$

§ 1.8. Moment of Inertia of an elliptic disc. [Meerut TDC 95 (P)]

Let M be the mass of an elliptic disc of axes $2a$ and $2b$, then mass per unit area of the disc

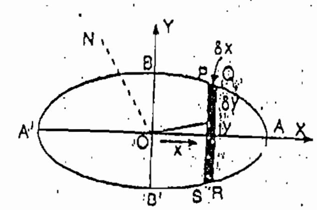
$$= \rho = \frac{M}{\pi ab}.$$

Consider an elementary area $\delta x\, \delta y$ at the point (x, y), then its mass $= \rho\delta x\, \delta y = \delta m$.

$\therefore$ M.I. of the elementary mass about OX

$= y^2\, \delta m = y^2\, \rho \delta x \delta y$.

Hence moment of inertia of the elliptic disc about OX



$= \int_{-a}^{a}\int_{-y'}^{y'} y^2\, \rho\, dx dy$

$= \rho \int_{-a}^{a} \left[\frac{1}{3} y^3\right]_{-y'}^{y'} dx$

$= \frac{2}{3}\rho \int_{-a}^{a} y'^3\, dx$

$= \frac{4}{3}\rho\int_{-a}^{a} b^3\left(1-\frac{x^2}{a^2}\right)^{3/2} dx$ $\left(\because \text{ Equation of the ellipse is } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$

$= \frac{4}{3}\rho b^3\int_{-\pi/2}^{\pi/2}(1-\sin^2\theta)^{3/2}\, a\cos\theta\, d\theta$. Putting $x = a\sin\theta$

$= \frac{4}{3}\rho b^3 a\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta = \frac{4}{3}\rho b^3 a \cdot 2\int_0^{\pi/2}\cos^4\theta\, d\theta$

$= \frac{4}{3}\rho b^3 a\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{2\Gamma(3)} = \frac{1}{3}\rho b^3 a\cdot\frac{1}{16}\cdot\frac{3}{4}\pi = \frac{1}{4}\rho\pi b^3 a$

$= \frac{1}{4}\cdot\frac{M}{\pi ab}\cdot\pi b^3 a = \frac{1}{4}Mb^2$ $(\because \rho = M/\pi ab)$

Similarly M. I. of the an elliptic disc about the minor axis $BB' = \frac{1}{4}Ma^2$.

And M.I. of the disc about the line ON through the centre O and perpendicular to its plane

= M.I. about OA + M.I. about OB

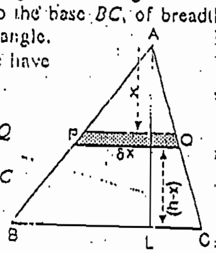
$= \frac{1}{4}Mb^2 + \frac{1}{4}Ma^2 = \frac{1}{4}M(a^2+b^2)$.

§ 1.9. Moment of Inertia of a uniform triangular lamina about one side. [Meerut TDC 95 (P)]

Let M be the mass and $h = AL$, the height of a triangular lamina ABC. Let PQ be an elementary strip parallel to the base BC, of breadth δx and at a distance x from the vertex A of the triangle.

From similar triangles APQ and ABC, we have

$x/AL = PQ/BC$.



$\therefore$ $PQ = ax/h$, where $BC = a$.

$\therefore$ δm = mass of the elementary strip PQ

$= \rho PQ\,\delta x = \rho\,(ax/h)\,\delta x$

$\therefore$ M. I. of the elementary strip about BC

$= (x-h)^2\,\delta m = \frac{\rho a}{h}(h-x)^2 x\delta x$.

$\therefore$ M. I. of the triangle ABC about BC

$= \int_0^h \frac{\rho a}{h}(h-x)^2 x dx = (\rho a/h)\int_0^h (h^2 x - 2hx^2 + x^3)\, dx$

$= (\rho a/h)\left[\frac{1}{2}h^2x^2 - \frac{2}{3}hx^3 + \frac{1}{4}x^4\right]_0^h = \frac{1}{12}\rho a h^3 = \frac{1}{6}Mh^2$.

$[\because M = \text{mass of } \Delta ABC = \rho.(\frac{1}{2}ah)]$

§ 1.10 Moment of Inertia of a rectangular parallelopiped about an axis through its centre and parallel to one of its edges.

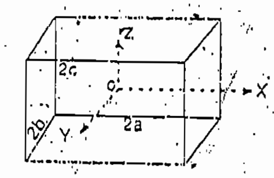
Let O be the centre and $2a, 2b, 2c$ the lengths of the edges of a rectangular parallelopiped. If M is the mass of the parallelopiped the mass per unit volume

$= \rho = \frac{M}{2a.2b.2c} = \frac{M}{8abc}$.

Let OX, OY, OZ be the axes through the centre and parallel to the edges of the rectangular parallelopiped.

Consider an elementary volume $\delta x\,\delta y\,\delta z$ of the parallelopiped at the point $P(x, y, z)$, then its mass

$= \rho\,\delta x\,\delta y\,\delta z = \delta m$.



Distance of the point $P(x, y, z)$ from OX is $\sqrt{(y^2+z^2)}$.

$\therefore$ M.I. of the elementary volume of mass δm at P about OX

$= \rho\,(y^2+z^2)\,\delta x\,\delta y\,\delta z$.

Hence M.I. of the rectangular parallelopiped about OX (which is parallel to $2a$).

$= \int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}\rho.(y^2+z^2)\,dx\,dy\,dz$

$= \int_{-a}^{a}\int_{-b}^{b}\left[y^2z+\frac{1}{3}z^3\right]_{-c}^{c} dx\,dy = \rho\int_{-a}^{a}\int_{-b}^{b} 2\,(y^2c+\frac{1}{3}c^3)\,dx\,dy$

$= 2\rho\int_{-a}^{a}\left[\frac{1}{3}y^3c+\frac{1}{3}c^3y\right]_{-b}^{b} dx = \frac{4}{3}\rho\int_{-a}^{a} 2\,(b^3c+c^3b)\,dx$.

$= \frac{4\rho}{3}bc\,(b^2+c^2)\,[x]_{-a}^{a} = \frac{4\rho}{3}bc\,(b^2+c^2).\,2a$

$= \frac{4}{3}\cdot\frac{M}{8abc}\cdot bc\,(b^2+c^2).\,2a$ $\because \rho = \frac{M}{8abc}$

$= \frac{1}{3}M(b^2+c^2)$.

Similarly M.I of the rectangular parallelopiped about the lines OY, OZ, through centre O and parallel to $2b$ and $2c$ are $\frac{1}{3}M(c^2+a^2)$ and $\frac{1}{3}M(a^2+b^2)$ respectively.

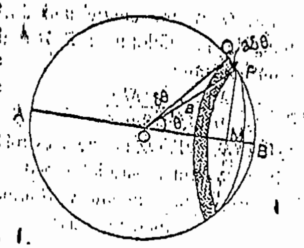
Note : For cube of side $2a, 2b = 2c = 2a$.

$\therefore$ M.I. of a cube about a line through its centre and parallel to one edge $= \frac{2}{3}Ma^2$.

§ 1.11. M.I of a spherical shell (i.e. hollow sphere) about diameter. [Meerut TDC 92]

A spherical shell (i.e. hollow sphere) of radius a is formed by the revolution of a semi-circular arc of radius a about its diameter.

Consider an elementary arc $PQ = a\delta\theta$, at the point P, of the semi-circular arc. A circular ring of radius $PM = a \sin\theta$ will be fromed by the revolution of this arc PQ about the diameter AB.



$\therefore$ Mass of this elementary ring

$= \delta m = \rho \,.\, 2\pi\, PM \,.\, a\delta\theta$.

$= \rho \,.\, 2\pi a \sin\theta \,.\, a\,\delta\theta = \rho\, 2\pi a^2 \sin\theta\, \delta\theta$,

where $\rho = \dfrac{M}{4\pi a^2}$, M is the mass of the shell.

M.I. of this elementary ring about AB

(a line through the centre of the ring and perpendicular to its plane)

$= PM^2 .\, \delta m = a^2 \sin^2\theta \,.\, \rho\, 2\pi a^2 \sin\theta\, \delta\theta$. (see § 1.6)

$= 2\pi\rho a^4 \sin^3\theta\delta\theta$

$\therefore$ M.I. of the shell about the diameter AB

$= \int_0^{\pi} 2\pi\rho a^4 \sin^3\theta\, d\theta = 2\pi\rho a^4 \int_0^{\pi} (1-\cos^2\theta) \sin\theta\, d\theta$

$= -2\pi\rho a^4 \int_1^{-1} (1-t^2)\, dt$ Putting $\cos\theta = t$, so that $-\sin\theta\, d\theta = dt$

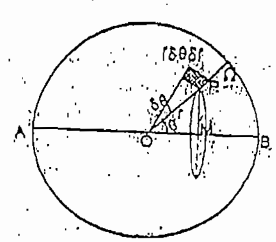
$= -2\pi \,.\, \dfrac{M}{4\pi a^2} \,.\, a^4 \left[t - \frac{1}{3} t^3 \right]_1^{-1} = \frac{2}{3} Ma^2$.

§ 1.12. M.I. of a solid sphere about a diameter.

[Meerut TDC 94, 94(P), 97]

A solid sphere of radius a is formed by the revolution of a semi-circular area of radius a about its diameter.

Consider an elementary area $r\delta\theta\delta r$ at the point $P(r, \theta)$ of the semi-circular area. When this element is revolved about the diameter AB, a circular ring of radius $PM = r\sin\theta$ and cross-section $r\delta\theta\delta r$ is formed.



Mass of this elementary ring

$= \delta m = \rho \,.\, 2\pi \,.\, r\sin\theta \,.\, r\delta\theta\delta r$

$= \rho 2\pi r^2 \sin\theta\, \delta\theta\delta r$

where $\rho = \dfrac{M}{\left(\frac{4}{3}\pi a^3\right)}$, M is the mass of the sphere. M.I. of this elementary ring about AB (a line through the

centre of the ring and perpendicular to its plane)

$= PM^2 .\, \delta m = r^2 \sin^2\theta \,.\, \rho 2\pi r^2 \sin\theta\delta\theta\delta r$ (see § 1.6)

$= 2\pi\rho r^4 \sin^3\theta\delta\theta\delta r$.

$\therefore$ M.I. of the sphere about the diameter AB

$= \int_{\theta=0}^{\pi} \int_{r=0}^{a} 2\pi\rho r^4 \sin^3\theta\, d\theta dr = 2\pi\rho\, \frac{1}{5} a^5 \int_0^{\pi} \sin^3\theta\, d\theta$

$= \dfrac{2\pi}{5} \,.\, \dfrac{M}{\frac{4}{3}\pi a^3} \,.\, a^5 \,.\, \left(\frac{4}{3}\pi\right) = \frac{2}{5} Ma^2$.

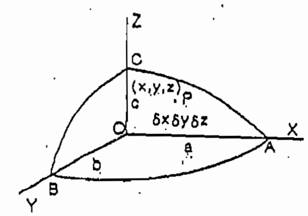
§ 1.13. M.I. of an ellipsoid.

[Meerut TDC, 93]

Let the equation of the ellipsoid be

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Consider an elementary volume $\delta x\, \delta y\, \delta z$ at the point $P(x, y, z)$ of the ellipsoid in the positive octant.



$\therefore$ Mass of this element

$= \rho \delta x\, \delta y\, \delta z$

where ρ = Mass per unit volume

$= \dfrac{M}{\frac{4}{3}\pi abc} = \dfrac{3M}{4\pi abc}$, M is the mass of the ellipsoid.

Distance of the point $P(x, y, z)$ from $OX = \sqrt{(y^2 + z^2)}$.

$\therefore$ M.I. of this elementary volume about OX

$= (y^2 + z^2)\, \rho\, \delta x\, \delta y\, \delta z$.

$\therefore$ M.I. of the ellipsoid about OX

$= 8 \iiint (y^2 + z^2)\, \rho\, dx\, dy\, dz$, where, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$

the integration being extended over positive octant of the ellipsoid.

Putting $\dfrac{x^2}{a^2} = u$, $\dfrac{y^2}{b^2} = v$, $\dfrac{z^2}{c^2} = w$

i.e. $x = au^{1/2}$, $y = bv^{1/2}$, $z = cw^{1/2}$

so that $dx = \frac{1}{2} au^{-1/2}\, du$, $dy = \frac{1}{2} bv^{-1/2}\, dv$, $dz = \frac{1}{2} cw^{-1/2}\, dw$, we have

M.I. of the ellipsoid about OX (i.e. the axis $2a$)

$= 8 \,.\, \dfrac{abc}{8} \,.\, \rho \iiint (b^2 v + c^2 w)\, u^{-1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw$

$= abc \cdot \rho \cdot$
$$\left[b^2 \iiint u^{1/2-1} v^{3/2-1} w^{1/2-1}\, du\, dv\, dw + c^2 \iiint u^{1/2-1} v^{1/2-1} w^{3/2-1}\, du\, dv\, dw\right]$$

where $u + v + w \le 1$

$$= abc\,\rho \left[b^2 \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(1+\frac{1}{2}+\frac{3}{2}+\frac{1}{2})} + c^2 \cdot \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(1+\frac{1}{2}+\frac{1}{2}+\frac{3}{2})}\right] \text{ By Dirichlet's theorem.}$$

$$= abc \cdot \frac{3M}{4\pi abc}(b^2+c^2) \cdot \frac{\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi} \cdot \sqrt{\pi}}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}} = \frac{1}{5} M(b^2+c^2).$$

§ 1.14. Reference Table.

The moments of inertia of some standard rigid bodies considered in § 1.4 to § 1.13 are given in the following table. The students are advised to remember all these as they will be used frequently.

Rigid body	M.I.
1. Uniform thin rod of length $2a$ and mass M.	
(i) About a line through the *middle point* and perpendicular to its length	$\frac{1}{3} Ma^2$
(ii) About a line through one *end* and perpendicular to its length	$\frac{4}{3} Ma^2$
2. Rectangular plate of sides $2a$, $2b$ and mass M.	
(i) About a line through the centre and parallel to the side $2a$	$\frac{1}{3} Mb^2$
(ii) About a line through the centre and parallel to the side $2b$	$\frac{1}{3} Ma^2$
(iii) About a line through the centre and perpendicular to the plate	$\frac{1}{3} M(a^2+b^2)$
3. Rectangular parallelopiped of edges $2a$, $2b$, $2c$ and mass M. About a line through its centre and parallel to the edge $2a$	$\frac{1}{3} M(b^2+c^2)$
4. Circular ring of radius a and mass M.	
(i) About its diameter	$\frac{1}{2} Ma^2$
(ii) About a line through the centre and perpendicular to the plane of the ring	Ma^2
5. Circular plate of radius a and mass M.	
(i) About its diameter	$\frac{1}{4} Ma^2$
(ii) About a line through the centre and perpendicular to its plane	$\frac{1}{2} Ma^2$
6. Elliptic disc of axes $2a$ and $2b$ and mass M.	
(i) About the axis $2a$	$\frac{1}{4} Mb^2$
(ii) About the axis $2b$	$\frac{1}{4} Ma^2$
(iii) About a line through the centre and perpendicular to its plane	$\frac{1}{4} M(a^2+b^2)$
7. Spherical shell of radius a and mass M. About a diameter	$\frac{2}{3} Ma^2$
8. Solid sphere of radius a and mass M. About a diameter	$\frac{2}{5} Ma^2$
9. Ellipsoid of axis $2a$, $2b$, $2c$ and mass M. About the axis $2a$.	$\frac{1}{5} M(b^2+c^2)$.

Routh's Rule. All the above M.I. may be remembered with the help of the following Routh's Rule.

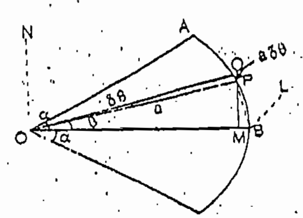
M.I. about an axis of symmetry

$$= \text{Mass} \times \frac{\text{Sum of squares of perpendicular axis}}{3, 4 \text{ or } 5}$$

The denominator is 3, 4 or 5 according as the body is rectangular (including rod), elliptical (including circular) or ellipsoid (including sphere).

EXAMPLES

Ex. 1. *Find the M.I. of the arc of a circle about*
(i) *the diameter bisecting the arc*
(ii) *an axis through the centre, perpendicular to its plane*
(iii) *an axis through its middle point perpendicular to its plane.*



Sol. Let OB be the diameter bisecting the circular arc ABC subtending an angle 2α at the centre O. Let a be the radius of the arc.

Consider an elementary arc $PQ = a\,\delta\theta$ at the point P of the arc.

$\therefore$ Its Mass $\delta m = \rho a\,\delta\theta$

where ρ = mass per unit length of the arc

$= \frac{M}{2\alpha a}$, M is the mass of the arc ABC.

(i) Distance of P from diameter $OB = PM = a \sin \theta$.

$\therefore$ M.I. of the elementary arc about OB

$= PM^2 . \delta m = (a \sin \theta)^2 \rho a \, \delta\theta$

$= \rho a^3 \sin^2 \theta \, \delta\theta$.

$\therefore$ M.I. of the arc ABC about the diameter OB

$= \int_{-\alpha}^{\alpha} \rho a^3 \sin^2 \theta \, d\theta = \frac{1}{2} \rho a^3 \int_{-\alpha}^{\alpha} (1 - \cos 2\theta) \, d\theta$

$= \frac{1}{2} \rho a^3 \left[\theta - \frac{1}{2} \sin 2\theta \right]_{-\alpha}^{\alpha} = \frac{1}{2} \frac{M}{2\alpha a} a^3 [2\alpha - \sin 2\alpha]$

$= \frac{Ma^2}{2\alpha} (\alpha - \sin \alpha \cos \alpha)$.

(ii) Distance of the point P from ON, an axis through the centre and perpendicular to the plane of the arc $= OP = a$.

$\therefore$ M.I. of the elementary mass δm at P about ON

$= a^2 . \delta m = \rho a^3 \, \delta\theta$

$\therefore$ M.I. of the arc ABC about $ON = \int_{-\alpha}^{\alpha} \rho a^3 \, d\theta = \rho a^3 [\theta]_{-\alpha}^{\alpha}$

$= \frac{M}{2\alpha a} a^3 . 2\alpha = Ma^2$.

(iii) Distance of the point P from BL, an axis through the middle point B of the arc ABC and perpendicular to its plane

$= PB = \sqrt{(OP^2 + OB^2 - 2OP . OB \cos \theta)} = \sqrt{(a^2 + a^2 - 2a^2 \cos \theta)}$.

$= a\sqrt{[2(1 - \cos \theta)]} = a\sqrt{[2.2 \sin^2 \frac{1}{2}\theta]} = 2a \sin \frac{1}{2}\theta$

$\therefore$ M.I. of the elementary mass δm at P about $BL = PB^2 . \delta m$

$= (2a \sin \frac{1}{2}\theta)^2 \rho a \, \delta\theta = 4a^3 \rho \sin^2 \frac{1}{2}\theta \, \delta\theta$.

$\therefore$ M.I. of the arc ABC about $BL = \int_{-\alpha}^{\alpha} 4a^3 \rho \sin^2 \frac{1}{2}\theta \, d\theta$

$= 2a^3 \rho \int_{-\alpha}^{\alpha} (1 - \cos \theta) \, d\theta = 2a^3 . \frac{M}{2\alpha a} [\theta - \sin \theta]_{-\alpha}^{\alpha}$

$= \frac{2Ma^2}{\alpha} (\alpha - \sin \alpha)$.

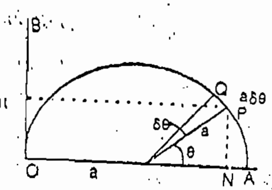
Ex. 2. *Find the product of inertia of a semicircular wire about diameter and tangent at its extremity.*

Sol. Let M be the mass, a the radius and OA the diameter of a semi-circular arc. Let OB be the tangent at the extremity O.

Consider an elementary arc $PQ = a \, \delta\theta$ at the point P of the wire.

$\therefore$ Its mass $= \delta m = \rho a \, \delta\theta$

where ρ = mass per unit length $= \frac{M}{\pi a}$.



P.I. of this elementary mass about OA and $OB = PN . PL . \delta m$

$= a \sin \theta (a + a \cos \theta) \rho a \, \delta\theta$

$= \rho a^3 (\sin \theta + \sin \theta \cos \theta) \, \delta\theta$

$\therefore$ P.I. of the wire about OA and OB

$= \int_0^{\pi} \rho a^3 (\sin \theta + \sin \theta \cos \theta) \, d\theta = \rho a^3 \left[-\cos \theta + \frac{1}{2} \sin^2 \theta \right]_0^{\pi}$

$= \frac{M}{\pi a} a^3 (2) = \frac{2Ma^2}{\pi}$.

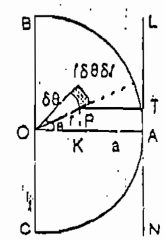
Ex. 3. *Show that the M.I. of a semi-circular lamina about a tangent parallel to the bounding diameter is $Ma^2 \left(\frac{5}{4} - \frac{8}{3\pi} \right)$ where a is the radius and M is the mass of lamina.* (Meerut TDC 90 (P) 96 (P))

Sol. Let LN be the tangent parallel to the bounding diameter BC of a semi-circular lamina of radius a and mass M.

Consider an elementary area $r \, \delta\theta \, \delta r$ at the point P of the lamina, then its mass $\delta m = \rho r \, \delta\theta \, \delta r$.

Where ρ = Mass per unit area

$= \frac{M}{\frac{1}{2}\pi a^2} = \frac{2M}{\pi a^2}$



Distance of the point P from $LN = PT$

$= KA = OA - OK = a - r \cos \theta$

$\therefore$ M.I. of the elementary mass δm at about LN

$= PT^2 . \delta m = (a - r \cos \theta)^2 . \rho r \, \delta\theta \, \delta r$

$\therefore$ M.I. of the lamina about LN

$= \int_{\theta = -\pi/2}^{\pi/2} \int_{r=0}^{a} (a - r \cos \theta)^2 \rho r \, d\theta \, dr$

$= \rho \int_{-\pi/2}^{\pi/2} \int_0^{a} (a^2 r - 2ar^2 \cos \theta + r^3 \cos^2 \theta) \, d\theta \, dr$

$= \rho \int_{-\pi/2}^{\pi/2} \left[\frac{a^2}{2} r^2 - \frac{2}{3} a r^3 \cos \theta + \frac{1}{4} r^4 \cos^2 \theta \right]_{r=0}^{a} d\theta$

$$= \rho \int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{2}a^4 - \tfrac{2}{3}a^4 \cos\theta + \tfrac{1}{4}a^4 \cos^2\theta\right] d\theta$$

$$= 2\rho a^4 \int_0^{\pi/2} \left(\tfrac{1}{2} - \tfrac{2}{3}\cos\theta + \tfrac{1}{4}\cos^2\theta\right) d\theta$$

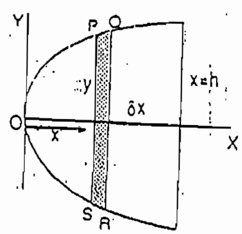
$$= 2\rho a^4 \left[\left[\tfrac{1}{2}\theta - \tfrac{2}{3}\sin\theta\right]_0^{\pi/2} + \tfrac{1}{4} \cdot \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{2\Gamma(2)}\right]$$

$$= 2 \cdot \frac{2M}{\pi a^2} a^4 \left[\tfrac{1}{4}\pi - \tfrac{2}{3} + \tfrac{1}{4} \cdot \frac{\pi}{4}\right] = Ma^2\left(\frac{5}{4} - \frac{8}{3\pi}\right)$$

Ex. 4. *Show that the M.I. of parabolic area (of latus rectum 4a) cut off by an ordinate at distance h from the vertex is $\frac{3}{7}Mh^2$ about the tangent at the vertex and $\frac{4}{5}M\,ah$ about the axis.* [Meerut TDC 90, 92 (P)]

Sol. Let the equation of the parabola of latus rectum $4a$ be $y^2 = 4ax$.

Let OAB be the portion of the parabola cut off by an ordinate at a distance h from the vertex.



Consider an elementary strip $PQRS$ of width δx, parallel to Oy.

$\therefore$ Mass of the strip $\delta m = \rho \cdot 2y\,\delta x$, where ρ is the mass per unit area.

$\therefore$ M = Mass of the portion $OABO$ of the parabola

$$= \int_0^h \rho\, 2y\, dx$$

$$= 2\rho \int_0^h 2\sqrt{(ax)}\, dx = 4\rho\sqrt{a} \cdot \tfrac{2}{3}h^{3/2} = \tfrac{8}{3}\rho a^{1/2} h^{3/2}$$

Now, the distance of every point of the strip from Oy, the tangent at the vertex, is x.

$\therefore$ M.I. of the strip about $Oy = x^2\,\delta m = \rho\, 2x^2 y\,\delta x$;

M.I. of the whole area $OABO$ about $Oy = \int_0^h 2\rho\, x^2 y\, dx$

$$= 2\rho \int_0^h x^2\, 2\sqrt{(ax)}\, dx = 4\rho a^{1/2} \int_0^h x^{5/2}\, dx = \tfrac{8}{7}\rho a^{1/2} h^{7/2}$$

$$= \tfrac{3}{7}\left(\tfrac{8}{3}\rho a^{1/2} h^{3/2}\right) h^2 = \tfrac{3}{7}Mh^2.$$

Again M.I. of the strip PQRS about $OX = \frac{1}{3}y^2\,\delta m$

$$= \tfrac{1}{3}y^2 \cdot \rho \cdot 2y\,\delta x = \tfrac{2}{3}\rho y^3 \delta x$$

$\therefore$ M.I. of the whole area OABO about $OX = \int_0^h \frac{2}{3}\rho y^3\, dx$

$$= \tfrac{2}{3}\rho \int_0^h (4ax)^{3/2}\, dx = \tfrac{16}{3}a^{3/2}\rho\, \tfrac{2}{5}h^{5/2} = \tfrac{4}{5}\left(\tfrac{8}{3}\rho a^{1/2} h^{3/2}\right) ah = \tfrac{4}{5}Mah.$$

Ex. 5. *Find the M.I. of the area of the lemniscate $r^2 = a^2\cos 2\theta$*
(i) about its axis [Meerut TDC 96, Rohilkhand 83]
(ii) about a line through the origin in its plane and perpendicular to its axis. [Meerut 90 (S)]
(iii) about a line through the origin and perpendicular to its plane.

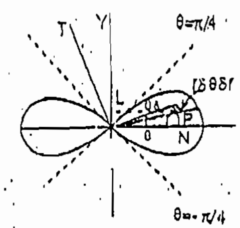
Sol. The loop of the lemniscate is formed between $\theta = -\pi/4$ and $\theta = \pi/4$. The curve is as shown in the fig.

Consider an elementary area $r\,\delta\theta\,\delta r$ at the point $P(r, \theta)$ of the curve, then its mass $\delta m = \rho r\,\delta\theta\,\delta r$

$\therefore$ The mass of the whole area is given by

$$M = 2\int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr = \rho \int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta\, d\theta$$

$$= \rho a^2 \left[\tfrac{1}{2}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = \rho a^2. \qquad ...(1)$$



(i) M.I. of elementary mass δm at P about the axis OX

$= PN^2 \cdot \delta m = (r\sin\theta)^2 \rho r\,\delta\theta\,\delta r$

$= \rho r^3 \sin^2\theta\,\delta\theta\,\delta r.$

$\therefore$ M.I. of the lemniscate about OX

$$= 2\int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2\theta\, d\theta\, dr$$

$$= 2\rho \int_{-\pi/4}^{\pi/4} \tfrac{1}{4}a^4 \cos^2 2\theta \sin^2\theta\, d\theta$$

$$= 2 \cdot \frac{2\rho a^4}{4} \int_0^{\pi/4} \tfrac{1}{2}\cos^2 2\theta\,(1 - \cos 2\theta)\, d\theta$$

$$= \tfrac{1}{4}\rho a^4 \int_0^{\pi/2} \cos^2 t\,(1 - \cos t)\, dt, \qquad \text{Putting } 2\theta = t, \text{ so that } d\theta = \tfrac{1}{2}dt$$

$$= \tfrac{1}{4}\rho a^4 \left[\int_0^{\pi/2} \cos^2 t\, dt - \int_0^{\pi/2} \cos^3 t\, dt\right]$$

$$= \tfrac{1}{4}\rho a^4 \left[\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{2\Gamma(2)} - \frac{\Gamma(2)\,\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{2})}\right] = \frac{1}{4}Ma^2\left(\frac{\pi}{4} - \frac{2}{3}\right) \qquad \text{from (1)}$$

$$= \frac{Ma^2}{16}\left(\pi - \tfrac{8}{3}\right).$$

(ii) Distance of the point $P(r, \theta)$ from OY a line through the origin in the plane of the lemniscate and perpendicular to its axis $= PL = r\cos\theta$.

$\therefore$ M.I. of δm at P about OY

$= PL^2 . \delta m = r^2 \cos^2\theta \, \rho^2 \delta\theta \, \delta r = \rho r^3 \cos^2\theta \, \delta\theta \, \delta r$.

$\therefore$ M.I of the lemniscate about OY

$$= 2\int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r^3 \cos^2\theta \, d\theta \, dr$$

$$= \frac{2\rho}{4}\int_{-\pi/4}^{\pi/4} a^4 \cos^2 2\theta \cos^2\theta \, d\theta$$

$$= \frac{\rho}{2} 2a^4 \int_0^{\pi/4} \frac{1}{2}\cos^2 2\theta \, (1+\cos 2\theta) \, d\theta$$

$$= \frac{1}{2} Ma^2 . \frac{1}{2}\int_0^{\pi/2} \cos^2 t \, (1+\cos t) \, dt, \qquad \text{Putting } 2\theta = t.$$

$$= \frac{1}{4} Ma^2 \left(\frac{\pi}{4} + \frac{2}{3}\right) \qquad \text{as in case (i)}$$

$$= \frac{1}{48} Ma^2 (3\pi + 8).$$

(iii) Let OT be the line through the origin and prependicular to the plane of the lemniscate.

Distance of δm at P from $OT = OP = r$

$\therefore$ M.I. of δm at P about $OT = OP^2 . \delta m = r^2 . \rho r \, \delta\theta \, \delta r = \rho r^3 \, \delta\theta \, \delta r$

$\therefore$ M.I. of the lemniscate about OT

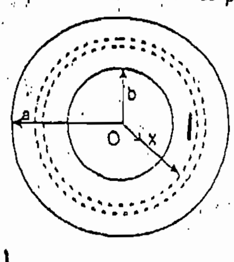
$$= 2\int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r^3 \, d\theta \, dr = \frac{2\rho}{4}\int_{-\pi/4}^{\pi/4} a^4 \cos^2 2\theta \, d\theta$$

$$= \frac{1}{2}\rho a^4 . 2\int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta) \, d\theta = \frac{1}{2} Ma^2 \left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{1}{8}\pi Ma^2.$$

Ex. 6. *Find the M.I. of a hollow sphere about a diameter, its external and internal radii being a and b respectively.* [Meerut TDC. 90(P)]

Sol. If M is the mass of the given hollow sphere, then mass per unit volume

$$\rho = \frac{M}{(\frac{4}{3}\pi a^3 - \frac{4}{3}\pi b^3)} = \frac{3M}{4\pi(a^3 - b^3)}$$



Consider a concentric spherical shell of radius x (s.t. $b < x < a$) and thickness δx

Mass of this elementary shell

$= \delta m = \rho . 4\pi x^2 \delta x$

M.I. of this shell about a diameter

$= \frac{2}{3} x^2 . \delta m$

$= \frac{2}{3} x^2 . \rho 4\pi x^2 \delta x = \frac{8}{3}\rho\pi x^4 \delta x.$

$\therefore$ M.I. of the given hollow sphere about a diameter

$$= \int_b^a \frac{8}{3}\rho\pi x^4 dx = \frac{8}{3}\rho\pi \frac{1}{5}(a^5 - b^5)$$

$$= \frac{8}{15}\pi . \frac{3M}{4\pi(a^3 - b^3)} . (a^5 - b^5)$$

$$= \frac{2M}{5} . \frac{a^5 - b^5}{a^3 - b^3}.$$

Ex. 7. *Show that the M.I. of a paraboloid of revolution about its axis is M/3 × the square of the radius of its base.*

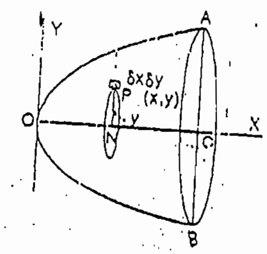
Sol. Let the praboloid of revolution be generated by the revolution of the area bounded by the parabola $y^2 = 4ax$, and x-axis about the axis OX. Let b be the radius of its base.

$\therefore$ for the point A,

$y = AC = b$

$\therefore$ from $y^2 = 4ax$,

$x = \dfrac{b^2}{4a} = OC$.



Consider an elementary area $\delta x \, \delta y$ at the point $P(x, y)$ of the area $OACO$.

By the revolution of this area $\delta x \, \delta y$ about OX, a circular ring of radius y and area of cross-section $\delta x \, \delta y$ is formed.

Mass of this elementary ring,

$\delta m = \rho 2\pi y \, \delta x \, \delta y$.

where ρ is the mass per unit volume.

$\therefore$ Mass of the paraboloid of revolution

$$M = \int_{x=0}^{b^2/4a}\int_{y=0}^{\sqrt{(4ax)}} \rho \, 2\pi y \, dx \, dy = 2\pi\rho . \frac{1}{2}\int_0^{b^2/4a} \left[y^2\right]_0^{\sqrt{(4ax)}} dx$$

$$= \pi\rho\int_0^{b^2/4a} 4ax \, dx = 4\pi\rho a . \left[\frac{1}{2}x^2\right]_0^{b^2/4a} = \frac{\pi\rho b^4}{8a} \qquad ...(1)$$

Now M.I. of the elementary ring of mass δm about OX (a line through its centre and perpendicular to its plane)

$= y^2 \delta m = y^2 . \rho \, 2\pi y \, \delta x \, \delta y = 2\pi\rho \, y^3 \, \delta x \, \delta y$

$\therefore$ M.I. of the paraboloid of revolution about OX

$$= \int_{x=0}^{b^2/4a}\int_{y=0}^{2\sqrt{ax}} 2\pi\rho\, y^3\, dx\, dy = \frac{2\pi\rho}{4}\int_0^{b^2/4a} 16a^2x^2\, dx$$

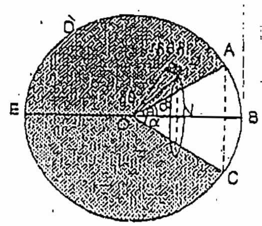
$$= 8\pi\rho a^2 \cdot \frac{1}{3}\left(\frac{b^2}{4a}\right)^3 = \frac{\pi\rho b^6}{24a} = \frac{1}{3}\left(\frac{\pi\rho b^4}{8a}\right)b^2$$

$= \frac{1}{3}M$. (square of the radius of the base).

Ex. 8. *From a uniform sphere of radius a, spherical sector of vertical angle 2α is removed. Show that the M.I. of the remainder of mass M about the axis of symmetry is*

$$\tfrac{1}{5}Ma^2(1+\cos\alpha)(2-\cos\alpha).$$

Sol. Let the spherical sector $OABCO$ of vertical angle 2α be removed from the sphere of radius a and centre O. This may be generated by the revolution of the area $OADEO$ of the circle of radius a and centre at O about the diameter EB.



Consider an elementary area $r\delta\theta\delta r$ at the point P of this area. By the revolution of this elementary area about EB a circular ring of radius $PN = r\sin\theta$ and area of cross-section $r\delta\theta\,\delta r$ is formed. Mass of this elementary ring, $\delta m = \rho \cdot 2\pi r\sin\theta \cdot r\delta\theta\delta r$

$= 2\pi\rho r^2 \sin\theta\, \delta\theta\delta r.$

∴ Mass of the remainder

$$M = \int_{\theta=\alpha}^{\pi}\int_{r=0}^{a} 2\pi\rho r^2 \sin\theta\, d\theta dr = \frac{2\pi\rho a^3}{3}\int_\alpha^\pi \sin\theta\, d\theta$$

$$= \frac{2\pi\rho}{3}a^3(1+\cos\alpha) \quad \therefore \quad \rho = \frac{3M}{2\pi a^3(1+\cos\alpha)} \qquad \ldots(1)$$

Now M.I. of the elementary ring about EB, the line through the centre and perpendicular to its plane

$= PN^2 \cdot \delta m = r^2\sin^2\theta \cdot 2\pi\rho r^2 \sin\theta\delta\theta\delta r = 2\pi\rho r^4 \sin^3\theta\, \delta\theta\, \delta r$

∴ M.I of the remainder about EB (the exis of symmetry)

$$= \int_{\theta=\alpha}^{\pi}\int_{r=0}^{a} 2\pi\rho r^4 \sin^3\theta\, d\theta dr = \tfrac{2}{5}\pi\rho a^5 \int_\alpha^\pi \sin^3\theta\, d\theta$$

$$= \tfrac{2}{5}\pi\rho a^5 \int_\alpha^\pi \tfrac{1}{4}(3\sin\theta - \sin 3\theta)\, d\theta$$

$$= \frac{\pi\rho a^5}{10}\left[-3\cos\theta + \frac{1}{3}\cos 3\theta\right]_\alpha^\pi$$

$$= \frac{\pi\rho a^5}{10}\left[\frac{8}{3} + 3\cos\alpha - \tfrac{1}{3}\cos 3\alpha\right]$$

$$= \frac{\pi a^5}{30}\cdot\rho\,[8 + 9\cos\alpha - (4\cos^3\alpha - 3\cos\alpha)]$$

$$= \frac{4\pi a^5}{30}\rho\,[2 + 3\cos\alpha - \cos^3\alpha]$$

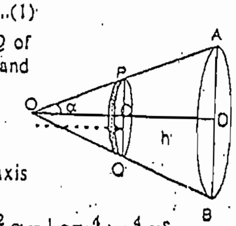
$$= \frac{2\pi a^5}{15}\cdot\frac{3M}{2\pi a^3(1+\cos\alpha)}\cdot(1+\cos\alpha)(2+\cos\alpha-\cos^2\alpha) \qquad \text{[from (1)]}$$

$= \frac{1}{5}Ma^2(1+\cos\alpha)(2-\cos\alpha)$

Ex. 9. *Find the M.I. of a right solid cone of mass M, height h and radius of whose base is a, about its axis.*

Sol. Let O be the vertex of the right solid cone of mass M, height h and radius of whose base is a. If α is the semi-vertical angle and ρ the density of the cone, then

$M = \frac{1}{3}\pi\rho h^3\tan^2\alpha.$ ...(1)



Consider an elementery disc PQ of thickness δx, parallel to the base AB and at a distance x from the vertex O.

∴ Mass of the disc,

$\delta m = \rho\pi x^2\tan^2\alpha\delta x.$

M.I. of this elementary disc about axis OD:

$= \frac{1}{2}\delta m CP^2 = \frac{1}{2}(\rho\pi x^2\tan^2\alpha\delta x)\, x^2\tan^2\alpha = \frac{1}{2}\rho\pi x^4\tan^4\alpha\delta x.$

∴ M.I. of the cone about axis OD.

$$= \int_0^h \tfrac{1}{2}\rho\pi x^4 \tan^4\alpha\, dx = \rho\frac{\pi}{10}h^5\tan^4\alpha = \tfrac{3}{10}M h^2\tan^2\alpha, \qquad \text{from (1)}$$

$= \frac{3}{10}Ma^2.$ $(\because \tan\alpha = a/h)$

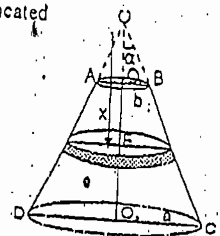
Ex. 10. *Find the M.I. of a truncated cone about its axis, the radii of its ends being a and b.* [Meerut TDC 93 (BP)]

Sol. Let $ABCD$ be the truncated cone with the vertex at O and of semi-vertical angle α. Also let $O_1B = b$ and $O_2C = a$.

Consider an elementary strip perpendicular to the axis at a distance x from O and of thickness δx.

∴ *Its Mass* $= \delta m = \rho\pi(x\tan\alpha)^2\delta x.$

If M is the total mass of the truncated cone then



$$M = \int_{x=b\cot\alpha}^{a\cot\alpha} \rho\pi x^2 \tan^2\alpha\, dx$$

$\because OO_1 = b\cot\alpha,\ OO_2 = a\cot\alpha$

$= \frac{1}{3}\rho\pi \tan^2\alpha\,(a^3 - b^3)\cot^3\alpha$

$= \frac{1}{3}\rho\pi \cot\alpha\,(a^3 - b^3)$

$$\therefore \rho = \frac{3M\tan\alpha}{\pi(a^3 - b^3)} \qquad ...(1)$$

Now M.I. of the elementary disc about O_1O_2, a line through the centre and perpendicular to its plane

$= \frac{1}{2}(x\tan\alpha)^2,\ \delta m = \frac{1}{2}x^2\tan^2\alpha \cdot \rho\pi x^2\tan^2\alpha\,\delta x$

$= \frac{1}{2}\rho\pi x^4 \tan^4\alpha\,\delta x.$

$\therefore$ M.I. of the truncated cone about its axis O_1O_2.

$$M = \int_{x=b\cot\alpha}^{a\cot\alpha} \tfrac{1}{2}\rho\pi x^4 \tan^4\alpha\, dx = \frac{1}{10}\rho\pi(a^5 - b^5)\cot^5\alpha \cdot \tan^4\alpha$$

$$= \frac{1}{10} \cdot \frac{3M\tan\alpha}{\pi(a^3 - b^3)}\pi(a^5 - b^5)\cot\alpha = \frac{3M}{10} \cdot \frac{a^5 - b^5}{a^3 - b^3} \qquad \text{from (1)}$$

Ex. 11. (a) *Find the M.I. about the x-axis of the portion of ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, which lies in the positive octant, supposing the law of volume density to be $\rho = \mu xyz$.*

Sol. (Refer fig of § 1.13 on page 11).

Consider an elementary volume $\delta x\,\delta y\,\delta z$ at the point $P(x, y, z)$ where $\rho = \mu xyz$.

$\therefore$ Mass of this element $= \rho\,\delta x\,\delta y\,\delta z = \mu xyz\,\delta x\,\delta y\,\delta z$.

$\therefore M$ = Mass of the octant $= \iiint \mu xyz\, dx\, dy\, dz$,

$$\text{where } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

the integration being extended over the positive octant.

Putting $\frac{x^2}{a^2} = u,\ \frac{y^2}{b^2} = v,\ \frac{z^2}{c^2} = w$

i.e. $x = au^{1/2},\ y = bv^{1/2},\ z = cw^{1/2}$, so that $dx = \frac{1}{2}au^{-1/2}\,du$ etc.

$$\therefore M = \frac{\mu}{8}a^2b^2c^2 \iiint u^{1/2}v^{1/2}w^{1/2} \cdot u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw,$$

where $u + v + w \le 1$.

$$= \frac{1}{8}\mu a^2b^2c^2 \iiint u^{1-1} \cdot v^{1-1} \cdot w^{1-1}\,du\,dv\,dw,\ u + v + w \le 1.$$

$$= \frac{1}{8}\mu a^2b^2c^2 \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)}, \text{ By Dirichlet's theorem}$$

$$= \frac{1}{48}\mu a^2b^2c^2. \qquad ...(1)$$

Now M.I. of the elementary mass δm at P, about OX

$= (y^2 + z^2)\,\delta m$

$\because$ Distance of $P(x, y, z)$ from OX is $\sqrt{(y^2 + z^2)}$

$= \mu xyz\,(y^2 + z^2)\,\delta x\,\delta y\,\delta z$

$\therefore$ M.I. of the octant of the ellipsoid about OX

$$= \iiint \mu xyz\,(y^2 + z^2)\,dx\,dy\,dz, \quad \text{where } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

The integration being extended over the positive octant.

Putting $x = au^{1/2},\ y = bv^{1/2},\ z = cw^{1/2}$, so that $dx = \frac{1}{2}au^{-1/2}\,du$ etc.

$$\therefore \text{M.I.} = \frac{1}{8}\mu a^2b^2c^2 \iiint u^{1/2}v^{1/2}w^{1/2}(b^2v + c^2w)\,u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw$$

where $u + v + w \le 1$

$$= \frac{1}{8}\mu a^2b^2c^2 \left[b^2 \iiint u^{1-1}v^{2-1}w^{1-1}\,du\,dv\,dw + c^2 \iiint u^{1-1}v^{1-1}w^{2-1}\,du\,dv\,dw\right]$$

$$= \frac{1}{8}\mu a^2b^2c^2\left[b^2\frac{\Gamma(1)\,\Gamma(2)\,\Gamma(1)}{\Gamma(1+1+2+1)} + c^2\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(2)}{\Gamma(1+1+1+2)}\right]$$

$= 6M(b^2 + c^2) \cdot \frac{1}{24} = \frac{1}{4}M(b^2 + c^2)$. By Dirichlet's theorem

(b) *Show that the M.I. of an ellipsoid of mass M and semi-axes a, b, c, with regard to a diametral plane whose direction-cosines referred to principal planes are (l, m, n) is $\frac{1}{5}M(a^2l^2 + b^2m^2 + c^2n^2)$.*

Sol. From § 1.12, on page (11), the moments of inertia of the ellipsoid with regard to the principal axes are

$\frac{1}{5}M(b^2 + c^2),\ \frac{1}{5}M(c^2 + a^2),\ \frac{1}{5}M(a^2 + b^2)$.

$\therefore$ By prop. II of § 1.3 on page (2), the moments of inertia with regard to principal planes are

$\frac{1}{5}Ma^2,\ \frac{1}{5}Mb^2,\ \frac{1}{5}Mc^2$.

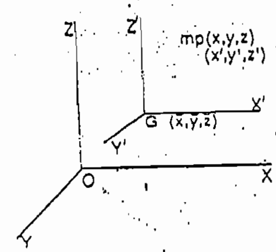
$\therefore$ M.I. of the ellipsoid about the diametral plane whose d.c's referred to principal planes are l, m, n is

$\frac{1}{3}Ma^2 \cdot l^2 + \frac{1}{3}Mb^2 \cdot m^2 + \frac{1}{3}Mc^2 \cdot n^2 = \frac{1}{3}M(a^2l^2 + b^2m^2 + c^2n^2)$.

§ 1.15. Theorem of Parallel Axis :

The moments and products of inertia about axes through the centre of gravity are given, to find the moments and products of inertia about parallel axes.

Let $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of the centre of gravity G of the body referred to the rectangular axes OX, OY, OZ through a fixed point O. Let GX', GY', GZ' be the axes through G parallel to the axes OX, OY, OZ respectively.



If (x, y, z) and (x', y', z') are the coordinates of a particle of mass m at P referred to the coordinate axes OX, OY, OZ and parallel axes GX', GY', GZ' respectively, then

$x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$.

M.I. of the body about OX

$= \Sigma m(y^2 + z^2) = \Sigma m\{(\bar{y} + y')^2 + (\bar{z} + z')^2\}$

$= \Sigma m(y'^2 + z'^2) + (\bar{y}^2 + \bar{z}^2)\Sigma m + 2\bar{y}\,\Sigma my' + 2\bar{z}\Sigma mz'$ ...(1)

Now referred to GX', GY', GZ' as axes the coordinates of G are $(0, 0, 0)$

$\therefore \frac{\Sigma mx'}{\Sigma m}$ or $\Sigma mx' = 0$. Similarly $\Sigma my' = 0$, $\Sigma mz' = 0$.

$\therefore$ From (1), M.I. of the body about OX

$= \Sigma m(y'^2 + z'^2) + M(\bar{y}^2 + \bar{z}^2)$.

$=$ *M.I. of the body about GX' + M.I. of the total mass M at G about OX.*

Also, Product of Inertia (P.I) of the body about OX and OY

$= \Sigma mxy = \Sigma m[(\bar{x} + x')(\bar{y} + y')]$

$= \Sigma mx'y' + \bar{x}\,\bar{y}\Sigma m + \bar{x}\Sigma my' + \bar{y}\Sigma mx'$

$= \Sigma mx'y' + M\bar{x}\,\bar{y}$

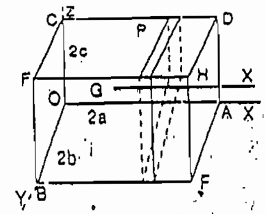
$=$ P.I. about GX' and GY' + P.I. of the total mass M at G about OX and OY

EXAMPLES

Ex. 12. *Find the M.I. of a rectangular parallelopiped about an edge.*

Sol. Let $2a, 2b, 2c$ be the lengths of the edges of a rectangular parallelopiped of mass M.

$\therefore$ M.I. of the rectangular parallelopiped about the edge OA = M.I. of the rectangular parallelopiped about a parallel axis GX' through its C. G. 'G' + M.I. of total mass M at C. G. 'G' about OA.



$= \frac{M}{3}(b^2 + c^2)$

+ M (perpendicular distance of G from $OA)^2$

$= \frac{M}{3}(b^2 + c^2) + M(b^2 + c^2) = \frac{4}{3}M(b^2 + c^2)$.

Alter. Consider an element $\delta x\,\delta y\,\delta z$ at the point P whose coordinates referred to the rectangular axes along edges OA, OB, OC are (x, y, z).

$\therefore$ M.I. of this element about OA

$= (\rho\delta x\,\delta y\,\delta z) \cdot (y^2 + z^2)$.

$\therefore$ M.I. of the rectangular parallelopiped about OA

$= \int_{x=0}^{2a}\int_{y=0}^{2b}\int_{z=0}^{2c} \rho(y^2 + z^2)\,dx\,dy\,dz = \frac{4}{3}M(b^2 + c^2)$

$\because \rho = \frac{M}{8abc}$.

Ex. 13. *Find the M.I. of a right circular cylinder about*

(i) *its axis,* [Meerut TDC 96]

(ii) *a staight line through its C. G. and perpendicular to its axis.*

Sol. Let a be the radius, h the height and M the mass of a right circular cylinder. If ρ is the density of the cylinder then $M = \rho\pi a^2 h$.

Consider an elementary disc, of breadth δx, perpendicular to the axis O_1O_2 and at a distance x from the centre of gravity O of cylinder.

$\therefore$ Mass of the disc $\delta m = \rho \cdot \pi a^2\delta x$.

M.I. of the disc about $O_1O_2 = \frac{1}{2}a^2\delta m = \frac{1}{2}a^2 \cdot \rho\pi a^2\delta x = \frac{1}{2}\rho\pi a^4\delta x$.

$\therefore$ M.I. of the cylinder about O_1O_2

$= \int_{-h/2}^{h/2} \frac{1}{2}\rho\pi a^4 dx = \frac{1}{2}\rho\pi a^4 h = \frac{1}{2}Ma^2$

$(\because M = \rho\pi a^2 h)$

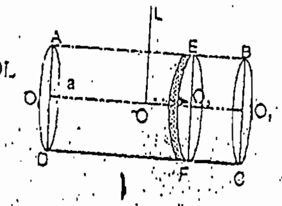
(ii) Let OL be the line through the C. G. 'O' and perpendicular to the axis of the cylinder.

M.I. of the elementary disc about OL.

= M.I. of the disc about the parallel line EF through its C. G. O_3 + M.I. of the total M at O_3 about OL

$= \frac{1}{4}a^2\delta m + x^2\delta m = (\frac{1}{4}a^2 + x^2)\,\delta m$

$= (\frac{1}{4}a^2 + x^2)\,\rho\pi a^2 \delta x.$



$\therefore$ M.I. of the cylinder about OL

$= \int_{-h/2}^{h/2} (\frac{1}{4}a^2 + x^2)\,\rho\pi a^2 dx$

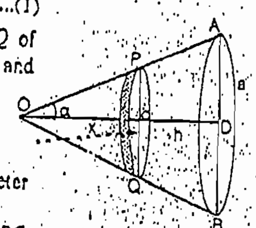
$= \rho\pi a^2 \left[\frac{1}{4}a^2 x + \frac{1}{3}x^3\right]_{-h/2}^{h/2}$

$= \frac{1}{4}\rho\pi a^2 h\,(a^2 + \frac{1}{3}h^2) = \frac{1}{4}M\,(a^2 + \frac{1}{3}h^2).$

Ex. 14. *Prove that the M.I. of a uniform right circular solid cone of mass M, height h and base-radius a, about a diameter of its base is* $\frac{M}{20}(3a^2 + 2h^2)$. **[Meerut TDC 92(P), 93(P), 95(BP)]**

Sol. Let O be the vertex of a right circular cone of mass M, height h and base-radius a. If α is the semi-vertical angle and ρ the density of the cone, then

$M = \frac{1}{3}\pi h^3 \tan^2 \alpha\rho.$ ...(1)



Cosider an elementary disc PQ of thickness δx, parallel to the base AB and at a distance x from the vertex O.

$\therefore$ Mass of the disc

$= \delta m = \rho\pi x^2 \tan^2 \alpha\,\delta x.$

$\therefore$ M.I. of the disc about the diameter AB of the base of the cone

= Its M.I. about parallel diameter PQ of the disc + M.I. of the total mass δm at centre C about AB

$= \frac{1}{4}\delta m\,.\,CP^2 + \delta m\,.\,CD^2 = \rho\pi x^2 \tan^2 \alpha\,[\frac{1}{4}x^2 \tan^2 \alpha + (h - x)^2]\,\delta x.$

$\therefore$ M.I. of the cone about the diameter of the base

$= \int_0^h \rho\pi x^2 \tan^2 \alpha\,[\frac{1}{4}x^2 \tan^2 \alpha + (h - x)^2]\,\delta x.$

$= \frac{1}{4}\rho\pi \tan^2 \alpha \int_0^h (x^4 \tan^2 \alpha + 4h^2x^2 - 8hx^3 + 4x^4)\,dx$

$= \frac{1}{4}\rho\pi \tan^2 \alpha \left[\frac{1}{5}h^5 \tan^2 \alpha + \frac{4}{3}h^5 - 2h^5 + \frac{4}{5}h^5\right]$

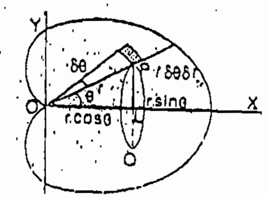
$= \frac{1}{60}\rho\pi h^5 \tan^2 \alpha\,(3\tan^2 \alpha + 2) = \frac{1}{20}Mh^2\,(3\tan^2 \alpha + 2).$ [from (1)]

$= \frac{1}{20}Mh^2\left(3\,.\,\frac{a^2}{h^2} + 2\right) = \frac{M}{20}\left(3a^2 + 2h^2\right)$ $(\because \tan \alpha = a/h).$

Ex. 15. *A solid body of density ρ is in the shape of the solid formed by the revolution of the cardiod $r = a\,(1 + \cos \theta)$ about the initial line; show that its M.I. about a straight line through the pole and perpendicular to the initial line is* $\frac{352}{105}\pi\rho a^5$.

Sol. Let OX be the initial line (axis of the cardiod) and OY the line perpendicular to it through the pole O.

Consider an elementary area $r\delta\theta\delta r$ at the point $P(r, \theta)$. Then the mass of the elementary ring of radius PL obtained by the revolution of element $r\delta\theta\delta r$ about OX,



$\delta m = \rho\,.\,2\pi\,PL\,.\,r\delta\theta\delta r$

$= 2\pi\rho r \sin \theta\,.\,r\delta\theta\delta r$

$= 2\pi\rho r^2 \sin \theta\,\delta\theta\delta r,$

where ρ is the mass per unit volume of the solid formed by the revolution of the cardiod about the initial line OX.

M.I. of this elementary ring about OY

= Its M.I. about the diameter PQ + M.I. of mass δm at centre L about OY

$= \frac{1}{2}\delta m\,.\,PL^2 + \delta m\,.\,OL^2 = (\frac{1}{2}PL^2 + OL^2)\,\delta m$

$= (\frac{1}{2}r^2 \sin^2 \theta + r^2 \cos^2 \theta)\,2\pi\rho r^2 \sin \theta\,\delta\theta\delta r$

$= \pi\rho\,(\sin^2 \theta + 2\cos^2 \theta)\,r^4 \sin \theta\,\delta\theta\delta r$

$= \pi\rho\,(1 + \cos^2 \theta)\,r^4 \sin \theta\,\delta\theta\delta r.$

$\therefore$ M.I. of the soild of revolution about OY

$= \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} \pi\rho\,(1 + \cos^2 \theta)\,r^4 \sin \theta\,d\theta\,dr$

$= \frac{1}{5}\pi\rho a^5 \int_0^{\pi} (1 + \cos^2 \theta)\,(1 + \cos \theta)^5 \sin \theta\,d\theta\,.$

$= -\frac{1}{5}\pi\rho a^5 \int_2^0 \{1 + (t - 1)^2\}\,.\,t^5\,dt$ Putting $1 + \cos \theta = t$

$= \frac{1}{3} \pi \rho a^5 \int_0^1 (2t^5 - 2t^6 + t^7)\, dt$

$= \frac{1}{3} \pi \rho a^5 \left[\frac{1}{3} t^6 - \frac{2}{7} t^7 + \frac{1}{8} t^8\right]_0^2 = \frac{1}{3} \pi \rho a^5 \cdot \left(\frac{352}{21}\right) = \frac{352}{105} \pi \rho a^5.$

Ex. 16. *Find the M.I. of a triangle ABC about a perpendicular to the plane through A.* [Meerut TDC 96(P), 97]

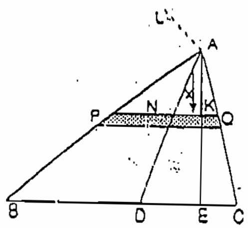
Sol. Let AL be a line through A and perpendicular to the plane of the triangle ABC of mass M and density ρ.

Let the height of the triangle, $AE = h$.

$\therefore\ M = \frac{1}{2} \rho BC \cdot AE$

$= \frac{1}{2} \rho a h$ ...(1)

Consider an elementary strip PQ of thickness δx at a distance x from A and parallel to BC. Let the median AD and the perpendicular AE meet PQ at N and K respectively. Clearly N will be the middle point of PQ.



From similar triangles APQ and ABC, we have

$\frac{PQ}{BC} = \frac{AK}{AE}$ or $\frac{PQ}{a} = \frac{x}{h}$ or $PQ = \frac{ax}{h}$.

Also from similar triangle ANK and ADE, we have

$\frac{AN}{AD} = \frac{AK}{AE}$ or $\frac{AN}{AD} = \frac{x}{h}$ or $AN = \frac{x}{h} AD$.

In ΔADE, we have

$AD^2 = AE^2 + DE^2 = AE^2 + (BE - BD)^2$

$= (AE^2 + BE^2) + BD^2 - 2\, BE \cdot BD$

$= AB^2 + (\frac{1}{2} BC)^2 - 2 \cdot AB \cos B \cdot \frac{1}{2} BC$

$= c^2 + \frac{1}{4} a^2 - c \cdot \frac{a^2 + c^2 - b^2}{2ac} \cdot a$

or $AD^2 = \frac{1}{4} (2b^2 + 2c^2 - a^2)$ ...(2)

Now, mass of the elementary strip PQ,

$\delta m = \rho PQ \delta x = \rho \frac{ax}{h} \delta m.$

M.I. of strip PQ about the line AL.

$=$ M.I. of strip PQ about the line parallel to AL through its C. G. N + M.I. of mass δm at N about AL

$= \frac{1}{3} \delta m \cdot \left(\frac{1}{2} PQ\right)^2 + \delta m\, AN^2 = \left(\frac{1}{12} \frac{a^2 x^2}{h^2} + \frac{x^2}{h^2} AD^2\right) \delta m$

$= \frac{x^2}{12h^2} \left(a^2 + 12AD^2\right) \frac{\rho a x}{h} \delta x$

$= \frac{\rho a}{12h^3} [a^2 + 12 \cdot \frac{1}{4} (2b^2 + 2c^2 - a^2)] \cdot x^3 \delta x$

$= \frac{\rho a}{6h^3} (3b^2 + 3c^2 - a^2)\, x^3 \delta x.$

$\therefore$ M.I. of the ΔABC about AL

$= \int_0^h \frac{\rho a}{6h^3} (3b^2 + 3c^2 - a^2)\, x^3\, dx = \frac{\rho a h}{24} (3b^2 + 3c^2 - a^2)$

$= \frac{M}{12} (3b^2 + 3c^2 - a^2).$ [from (1)]

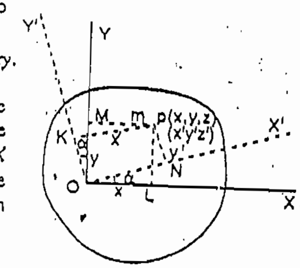
§ 1.16. Moment and Product of Inertia of a Plane Lamina about a Line.

If the moments and products of inertia of a plane lamina about two perpendicular axes in its plane are given, to find the moment and product of inertia about any perpendicular lines through their point of intersection.

Let A and B be the moments of inertia and F the product of inertia of a plane lamina about the perpendicular axes OX and OY in its plane.

Consider an element of mass m of the lamina at the point P whose co-ordinates are (x, y) with reference to the axes OX and OY.

$\therefore\ A = \Sigma my^2,\ B = \Sigma mx^2$ and $F = \Sigma mxy.$



Let OX' and OY' be the perpendicular axes in the plane of the lamina and inclined at an angle α to OX and OY respectively. If (x', y') are the co-ordinates of the point P with reference to these axes, then

$x' = PK = x \cos\alpha + y \sin\alpha$

and $y' = PN = y \cos\alpha - x \sin\alpha.$

$\therefore$ M.I. of the lamina about OX'

$= \Sigma mPN^2 = \Sigma my'^2 = \Sigma m (y \cos\alpha - x \sin\alpha)^2$

$= (\Sigma my^2) \cos^2\alpha + (\Sigma mx^2) \sin^2\alpha - 2 (\Sigma mxy) \sin\alpha \cos\alpha$

$= A \cos^2\alpha + B \sin^2\alpha - F \sin 2\alpha.$ ...(1)

Also P.I. of the lamina about OX' and OY'

$= \Sigma\, mPN \,.\, PK = \Sigma\, my'\,x'$

$= \Sigma\, m\,(y\cos\alpha - x\sin\alpha)\,(x\cos\alpha + y\sin\alpha)$

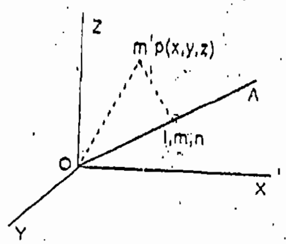
$= (\Sigma\, my^2 - \Sigma\, mx^2)\sin\alpha\cos\alpha + (\Sigma\, mxy)(\cos^2\alpha - \sin^2\alpha)$

$= \frac{1}{2}(A - B)\sin 2\alpha + F\cos 2\alpha.$ ...(2)

§ 1.17. M.I. of a Body about a Line.

Given the moments and products of inertia of a body about three mutually perpendicular axes, to find the M.I. about any line through their meeting point.

Let OX, OY, OZ be three mutually perpendicular axes. Consider an element of mass m' of the body at the point $P(x, y, z)$ then



A = M.I. about OX

$= \Sigma\, m'(y^2 + z^2)$

B = M.I. about OY

$= \Sigma\, m'(z^2 + x^2)$

C = M.I. about OZ

$= \Sigma\, m'(x^2 + y^2)$

D = P.I. about OY and OZ

$= \Sigma\, m'\,yz$

E = P.I. about OZ and OX

$= \Sigma\, m'\,zx$

F = P.I. about OX and OY

$= \Sigma\, m'\,xy.$

Let OA be a line through the point O, (meeting point of the axes), and l, m, n its direction cosines.

If PL is the perpendicular from P on OA, then

$PL^2 = OP^2 - OL^2 = (x^2 + y^2 + z^2) - (lx + my + nz)^2$

$= x^2(1 - l^2) + y^2(1 - m^2) + z^2(1 - n^2) - 2mnyz - 2nlxz - 2lmxy$

$= x^2(m^2 + n^2) + y^2(n^2 + l^2) + z^2(l^2 + m^2) - 2mnyz - 2nlxz - 2lmxy$

$(\because\ l^2 + m^2 + n^2 = 1)$

$= (y^2 + z^2)\,l^2 + (z^2 + x^2)\,m^2 + (x^2 + y^2)\,n^2 - 2mnyz - 2nlxz - 2lmxy$

$\therefore$ M.I. of the body about OA

$= \Sigma\, m'\,PL^2 = l^2\,\Sigma\, m'(y^2 + z^2) + m^2\,\Sigma\, m'(z^2 + x^2)$
$+ n^2\,\Sigma\, m'(x^2 + y^2) - 2mn\,\Sigma\, m'\,yz - 2nl\,\Sigma\, m'\,xz - 2lm\,\Sigma\, m'\,xy$

$= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm.$ ...(1)

Note. § 1.16. is a special case of § 1.17.

For a plane lamina $n = 0$, $l = \cos\alpha$ and $m = \cos(90° - \alpha) = \sin\alpha.$

Putting $n = 0$ in (1), we get the M.I. of the lamina about OA.

$= A\cos^2\alpha + \beta\sin^2\alpha - F\sin 2\alpha.$

EXAMPLES

Ex. 17. *Show that M.I. of a rectangle of mass M and sides 2a, 2b about a diagonal is* $\frac{2M}{3}\,\frac{a^2b^2}{a^2 + b^2}$.

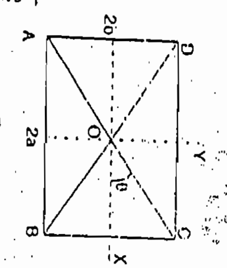
Deduce that in case of a square.

Sol. Let $ABCD$ be a rectangle of mass M and $AB = 2a$, $BC = 2b$

Then M.I. of rectangle about $OX = A = \frac{1}{3}Mb^2$,

and M.I. of rectangle about $OY = B = \frac{1}{3}Ma^2$.

P.I. of the rectangle about OX and $OY = F = 0$. (By symmetry)



If diagonal AC make an angle θ with AB, then

$\cos\theta = \frac{AB}{AC} = \frac{2a}{\sqrt{(4a^2 + 4b^2)}} = \frac{a}{\sqrt{(a^2 + b^2)}}$

and $\sin\theta = \frac{BC}{AB} = \frac{b}{\sqrt{(a^2 + b^2)}}$

$\therefore$ M.I. of the rectangle about AC

$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$ (see equation (1), § 1.16)

$= \frac{1}{3}Mb^2 \cdot \frac{a^2}{a^2 + b^2} + \frac{1}{3}Ma^2 \cdot \frac{b^2}{a^2 + b^2} - 0 = \frac{2M}{3} \cdot \frac{a^2b^2}{a^2 + b^2}.$

Deduction. For a square, $2b = 2a$.

$\therefore$ M.I. of square about AC

$= \frac{2M}{3} \cdot \frac{a^4}{a^2 + a^2} = \frac{1}{3}Ma^2.$

Ex. 18. *Show that the M.I. of an elliptic area of mass M and semi-axes a and b about a diameter of length 2r is* $\frac{1}{4}M\frac{a^2b^2}{r^2}$.

Sol. Let PP' be the diameter of length $2r$ of an elliptic area of mass M and semi-axes a and b. Equation of the ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ ...(1)

If PP' make an angle θ with OX then co-ordinates of P are $(r\cos\theta, r\sin\theta)$.

Since P lie on equation (1).

$\therefore\ (r^2/a^2)\cos^2\theta + (r^2/b^2)\sin^2\theta = 1$

or $b^2\cos^2\theta + a^2\sin^2\theta = \frac{a^2b^2}{r^2}$ ...(2)

Now, M.I. of the ellipse about $OX = A = \frac{1}{4} Mb^2$,

and M.I. of the ellipse about $OY = B = \frac{1}{4} Ma^2$.

Also P.I. of the ellipse about OX and OY:

$\therefore F = 0$. (By symmetry)

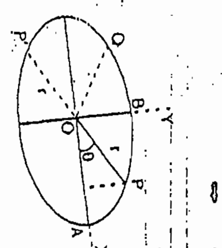
$\therefore$ M.I. of the ellipse about the diameter PP'

$= A \cos^2 \theta + B \sin^2 \theta - F \sin 2\theta$.

$= \frac{1}{4} Mb^2 \cos^2 \theta + \frac{1}{4} Ma^2 \sin^2 \theta - 0$

$= \frac{1}{4} M (b^2 \cos^2 \theta + a^2 \sin^2 \theta)$

$= \frac{M}{4} \cdot \frac{a^2 b^2}{r^2}$ [from (2)]



Ex. 19. *If k_1 and k_2 be the radii of gyration of an elliptic lamina about two conjugate diameter, then*

$$\frac{1}{k_1^2} + \frac{1}{k_2^2} = 4\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

Sol. Let $OP = r_1$ and $OQ = r_2$ be two conjugate semi-diameters of an elliptic lamina of mass M and semi-axes a, b.

M.I. of the ellipse about OP

$= Mk_1^2 = \frac{M}{4} \frac{a^2 b^2}{r^2}$. (See Ex. 18)

$\therefore \frac{1}{k_1^2} = \frac{4r_1^2}{a^2 b^2}$. Similarly, $\frac{1}{k_2^2} = \frac{4r_2^2}{a^2 b^2}$.

$\therefore \frac{1}{k_1^2} + \frac{1}{k_2^2} = \frac{4}{a^2 b^2}(r_1^2 + r_2^2) = \frac{4}{a^2 b^2}(a^2 + b^2)$.

($\because r_1^2 + r_2^2 = a^2 + b^2$. By property)

$= 4\,(1/a^2 + 1/b^2)$.

Ex. 20. *Show that the M.I. of an elliptic area of mass M and equation, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, about a diameter parallel to the axis of x is $\frac{-aM\Delta}{4(ab - h^2)^2}$,*

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

Sol. Equation of the ellipse is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. ...(1)

Shifting the origin to the centre of the ellipse, the equation of the ellipse becomes

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0. \quad ...(2)$$

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$. (By geometry)

Putting $y = 0$ in (2), we have $x^2 = -\frac{\Delta}{a(ab - h^2)}$.

$\therefore$ If r is the length of the semi-diameter of the ellipse parallel to the axis of x, then

$$r^2 = -\frac{\Delta}{a(ab - h^2)}. \quad ...(3)$$

Now, the equation (2) of the ellipse can be written as

$$-\frac{a}{c'} x^2 - \frac{2h}{c'} xy - \frac{b}{c'} y^2 = 1, \quad ...(4)$$

where $c' = \Delta/(ab - h^2)$.

Which is of the standard form $Ax^2 + 2Hxy + By^2 = 1$.

The squares of the lengths of the semi-axes of the ellipse, are given by the values R^2 in the equation

$$\left(A - \frac{1}{R^2}\right)\left(B - \frac{1}{R^2}\right) = H^2$$

or $\left(-\frac{a}{c'} - \frac{1}{R^2}\right)\left(-\frac{b}{c'} - \frac{1}{R^2}\right) = \left(-\frac{h}{c'}\right)^2$.

or $\frac{1}{R^4} + \left(\frac{a + b}{c'}\right) \cdot \frac{1}{R^2} + \frac{ab - h^2}{c'^2} = 0$. ...(5)

If α and β are the lengths of semi-axes of ellipse then $1/\alpha^2$, $1/\beta^2$ are the roots of (5).

$\therefore \frac{1}{\alpha^2} \cdot \frac{1}{\beta^2} = \frac{ab - h^2}{c'^2}$ or $\alpha^2 \beta^2 = \frac{c'^2}{ab - h^2} = \frac{\Delta^2}{(ab - h^2)^3}$

$\therefore$ From Ex. 18, M.I. of the ellipse about the diameter

$$= \frac{M}{4} \frac{\alpha^2 \beta^2}{r^2} = \frac{M}{4} \frac{\Delta^2}{(ab - h^2)^3} \cdot \left[-\frac{a(ab - h^2)}{\Delta}\right] = -\frac{aM\Delta}{4(ab - h^2)^2}$$

Ex. 21. *Show that the M.I. of an ellipse of mass M and semi-axes a and b about a tangent is $\frac{5}{4} Mp^2$, where p is the perpendicular from the centre on the tangent.*

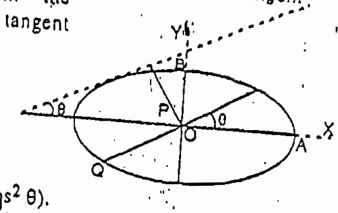
Sol. Let the equation of an ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore$ Equation of the tangent to the ellipse is

$y = mx + \sqrt{(a^2 m^2 + b^2)}$. ...(1)

where $m = \tan \theta$, if tangent is inclined at an angle θ to the axis of x.

If p is the perpendicular from the centre (0, 0) on the tangent (1), then



$$p = \frac{\sqrt{(a^2 m^2 + b^2)}}{\sqrt{(1+m^2)}}$$
$$= \frac{\sqrt{(a^2 \tan^2 \theta + b^2)}}{\sqrt{(1+\tan^2 \theta)}}$$
$$= \sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}. \quad \ldots(2)$$

M.I. of the ellipse about the diameter PQ which is parallel to the tangent

$= A \cos^2 \theta + B \sin^2 \theta - F \sin 2\theta$

$= \frac{1}{4} Mb^2 \cos^2 \theta + \frac{1}{4} Mp^2 \sin^2 \theta - 0$

$= \frac{1}{4} M(b^2 \cos^2 \theta + a^2 \sin^2 \theta) = \frac{1}{4} Mp^2$, from (2).

$\therefore$ M.I. of the ellipse about the tangent

= Its M.I. about the parallel line through C.G. O

+ M.I. of mass M at O about the tangent

$= \frac{1}{4} Mp^2 + Mp^2 = \frac{5}{4} Mp^2$.

Ex. 22. *Show that the sum of the moments of inertia of an elliptic area about any two perpendicular tangents is always the same.*

Sol. M.I. of an elliptic area about a tangent inclined at an angle θ to the major axis

$= \frac{5}{4} Mp^2$ (See last Ex. 21)

$= \frac{5}{4} M(a^2 \sin^2 \theta + b^2 \cos^2 \theta)$.

Replacing θ by $\theta + \pi/2$, the M.I. of the elliptic area about a perpendicular tangent

$= \frac{5}{4} M(a^2 \cos^2 \theta + b^2 \sin^2 \theta)$

$\therefore$ Sum of the moments of inertia about any two perpendicular tangent

$= \frac{5}{4} M(a^2 \sin^2 \theta + b^2 \cos^2 \theta) + \frac{5}{4} M(a^2 \cos^2 \theta + b^2 \sin^2 \theta)$

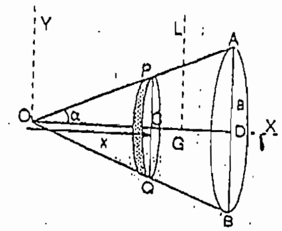
$= \frac{5}{4} M(a^2 + b^2)$,

which is always the same as it is independent of θ.

Ex. 23. *Show that the M.I. of a right solid cone whose height is h and radius of whose base is a, is $\frac{3Ma^2}{20} \cdot \frac{6h^2 + a^2}{h^2 + a^2}$ about a slant side, and $\left(\frac{3M}{80}\right)(h^2 + 4a^2)$ about a line through the centre of gravity of the cone perpendicular to its axis.* [Meerut TDC 93 (P), 96 (BP)]

Sol. Let M be the mass of a right circular cone of height h and radius of whose base is a. If α is the semi-vertical angle and ρ the density of the cone, then

$M = \frac{1}{3} \rho \pi h^3 \tan^2 \alpha$. ...(1)



Take the vertex of the cone as the origin, x-axis along the axis OD of the cone and y-axis perpendicular to OD.

The slant side OA make an angle α with OX.

$\therefore$ M.I. of the cone about $OA = A \cos^2 \alpha + B \sin^2 \alpha - F \sin 2\alpha$. ...(2)

where, A = M.I. of the cone about OX, B = M.I. of the cone about OY and F = P.I. of the cone about OX and OY.

Now consider an elementary disc PQ parallel to the base AB of the cone, of thickness δx and at a distance x from O.

Mass of this elementary disc $= \delta m = \rho \pi x^2 \tan^2 \alpha \delta x$.

(i) M.I. of the elementary disc about $OX = \frac{1}{2} \delta m \cdot CP^2 = \frac{1}{2} \rho \pi x^4 \tan^4 \alpha \delta x$

$\therefore A$ = M.I. of the cone about $OX = \int_0^h \frac{1}{2} \rho \pi x^4 \tan^4 \alpha \, dx$

$= \frac{1}{10} \rho \pi h^5 \tan^4 \alpha = \frac{3M}{10} h^2 \tan^2 \alpha$, from (1)

$= \frac{3}{10} Ma^2$. $\because \tan \alpha = \frac{a}{h}$.

(ii) M.I. of the elementary disc about OY

= Its M.I. about parallel diameter PQ

+ M.I. of mass δm at C about OY

$= \frac{1}{4} \delta m \cdot CP^2 + \delta m \cdot OC^2 = (\frac{1}{4} x^2 \tan^2 \alpha + x^2) \cdot \rho \pi x^2 \tan^2 \alpha \, \delta x$

$= \frac{1}{4} (\tan^2 \alpha + 4) \rho \pi x^4 \tan^2 \alpha \, \delta x$

$\therefore B$ = M.I. of the cone about OY

$= \int_0^h \frac{1}{4} (\tan^2 \alpha + 4) \rho \pi x^4 \tan^2 \alpha \, dx$

$= \frac{1}{20} \rho \pi h^5 (\tan^2 \alpha + 4) \tan^2 \alpha = \frac{3}{20} Mh^2 (\tan^2 \alpha + 4)$, from (1)

$= \frac{3}{20} M(a^2 + 4h^2)$. $\because \tan \alpha = \frac{a}{h}$.

(iii) F = P.I. of the cone about OX and $OY = 0$. By symmetry about OX.

Also $\cos\alpha = \frac{OD}{OA} = \frac{OD}{\sqrt{(OD^2 + AD^2)}} = \frac{h}{\sqrt{(h^2 + a^2)}}$

and $\sin\alpha = \frac{AD}{OA} = \frac{a}{\sqrt{(h^2 + a^2)}}$.

$\therefore$ from (2) M.I. of the cone about slant side

$$= \frac{3}{10} Ma^2 \cdot \frac{h^2}{h^2 + a^2} + \frac{3}{20} M(a^2 + 4h^2) \frac{a^2}{h^2 + a^2} = \frac{3Ma^2}{20} \cdot \frac{6h^2 + a^2}{h^2 + a^2}$$

Second Part. Let GL be a line through the C.G. G of the cone and perpendicular to its axis OD. Then

M.I. of the cone about OY = M.I. of the cone about parallel line GL through C.G. G + M.I. of total mass M at G about OY.

$\therefore$ M.I. of the cone about the line GL

= M.I. of the cone about OY − M.I. of total mass M at G about OY.

$$= \frac{3}{20} M(a^2 + 4h^2) - M \cdot OG^2 = \frac{3}{20} M(a^2 + 4h^2) - M \cdot (\tfrac{3}{4}h)^2, \quad (\because OG = \tfrac{3}{4}h)$$

$$= \frac{3M}{80}(h^2 + 4a^2).$$

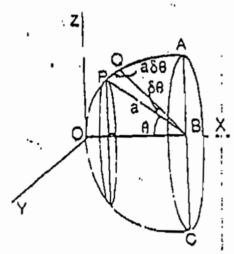
Ex. 24. *Show that for a thin hemispherical shell of mass M and radius a, the M.I. about any line through the vertex is $\frac{2}{3}Ma^2$.*

Sol. A hemispherical shell with vertex at the origin O is generated by the revolution of the arc OA of quadrant OAB of the circle of radius a.

If ρ is the density of the shell, then

$M = 2\pi\rho a^2$ ...(1)

Take the x-axis along the symmetrical radius OB of the shell and axes OY and OZ perpendicular to OX.



Consider an elementary arc $a\,\delta\theta$ at the point P of the arc OA.

The mass of the elementary ring obtained by the revolution of this elementary arc $a\,\delta\theta$ at P about OX.

$= \delta m = \rho \cdot 2\pi PL \cdot a\delta\theta = 2\pi\rho a^2 \sin\theta\,\delta\theta$, $\quad (\because PL = a\sin\theta)$

(i) M.I. of the elmentary ring about OX

$= \delta m \cdot PL^2 = 2\rho\pi a^2 \sin\theta\,\delta\theta\, a^2 \sin^2\theta = 2\rho\pi a^4 \sin^3\theta\,\delta\theta$

$\therefore$ A = M.I. of the shell about OX

$$= \int_0^{\pi/2} 2\rho\pi a^4 \sin^3\theta\, d\theta = 2\rho\pi a^4 \cdot \frac{\Gamma(2)\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{5}{2})} = \frac{4}{3}\rho\pi a^4 = \frac{2}{3}Ma^2, \quad \text{from (1)}$$

(ii) M.I. of the elementary ring about OY

= Its M.I. about parallel diameter through L

+ M.I. of δm at L about OY

$= \frac{1}{2}\delta m\, PL^2 + \delta m\, OL^2 = [\frac{1}{2}a^2 \sin^2\theta + (a - a\cos\theta)^2] \cdot 2\rho\pi a^2 \sin\theta\,\delta\theta$

$= \rho\pi a^4 [\sin^2\theta + 2(1 - \cos\theta)^2] \sin\theta\,\delta\theta$

$= \rho\pi a^4 [\sin^2\theta + 2 + 2\cos^2\theta - 4\cos\theta] \sin\theta\,\delta\theta$

$= \rho\pi a^4 (3 + \cos^2\theta - 4\cos\theta) \sin\theta\,\delta\theta$

$\therefore$ B = M.I. of the shell about OY

$$= \int_0^{\pi/2} \rho\pi a^4 (3 + \cos^2\theta - 4\cos\theta)\sin\theta\, d\theta$$

$$= -\rho\pi a^4 \int_1^0 (3 + t^2 - 4t)\, dt, \text{ Putting } \cos\theta = t.$$

$$= \tfrac{4}{3}\pi\rho a^4 = \tfrac{2}{3}Ma^2, \qquad \text{from (1).}$$

And C = M.I. of the shell about $OZ = B = \frac{2}{3}Ma^2$, (By Summetry)

(iii) Since the coordinates of

$\therefore$ C.G. are G $(a/2, 0, 0)$

$\therefore$ D = P.I. of the shell about OY and OZ

= P.I. of the shell about lines through C. G., 'G' parallel to OY and OZ + P.I. of the total mass M at G about OY and OZ.

$= O + M \cdot O \cdot O = O$

(Since shell is symmetrical about lines through G, parallel to OY and OZ)

Similarly $E = O = F$.

If l, m, n are the direction cosines of any line through the vertex O, then M.I. of the shell about this line

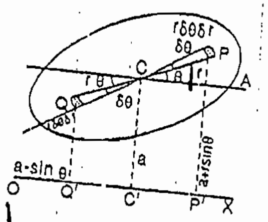
$= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm$

$= \frac{2}{3}Ma^2 l^2 + \frac{2}{3}Ma^2 \cdot m^2 + \frac{2}{3}Ma^2 n^2 = \frac{2}{3}Ma^2 (l^2 + m^2 + n^2) = \frac{2}{3}Ma^2.$

§ 1.18. Theorem I. *A closed curve revolves round any line OX in its own plane which does not intersect it. Show that the M.I. of the solid of revolution so formed about OX is equal to $M(a^2 + 3k^2)$, where M is the mass of the solid generated, a is the distance from OX of the centre C of the cruve, and k is the radius of gyration of the curve about a line through C parallel to OX.*

Prof. Let C be the centre of the closed curve which revolve round any line OX in its own plane which does not intersect it. Given that the distance of C from OX, $CC' = a$.

If M is the mass of the solid of revolution fromed about OX, then by Pappus Theorem, we have $M = 2\pi a \rho S$, where S is the area of the closed surface.



Consider an element $r\delta\theta\delta r$ at $P(r, \theta)$ taking C as the pole and the line CA parallel to OX as the initial line. For this element $r\delta\theta\delta r$ at P there will be an equal element for the same value of θ at Q in the opposite direction.

The distances of P and Q from OX are given by $PP' = a + r\sin\theta$ and $QQ' = a - r\sin\theta$.

Now, the area of the closed curve

$S = 2\iint r\,d\theta\,dr$ ...(1)

the intrgeration being taken to cover the upper half of the area.

$\therefore$ M.I. of the area S about CA is $S\rho k^2$

and $S\rho k^2 = 2\iint (r\sin\theta)^2 \cdot \rho r\,d\theta\,dr$

the intrgration being taken to cover the upper half of the area.

$= 2\rho \iint r^3 \sin^2\theta\,d\theta\,dr$ ...(2)

$\therefore$ M.I. of the solid of revolution about OX.

$= \iint [2\pi(a + r\sin\theta)\cdot(a + r\sin\theta)^2 + 2\pi(a - r\sin\theta)\cdot(a - r\sin\theta)^2]\cdot \rho r\,d\theta\,dr$

$= \iint 4\pi\rho\,(a^3 + 3ar^2 \sin^2\theta)\,r\,d\theta\,dr$

$= 4\pi\rho a^2 \iint r\,d\theta\,dr + 12\pi\rho a \iint r^3 \sin^2\theta\,d\theta\,dr$

$= 4\pi\rho a^3 \cdot S + 6\pi\rho a \cdot S\rho k^2$ [By (1) and (2)]

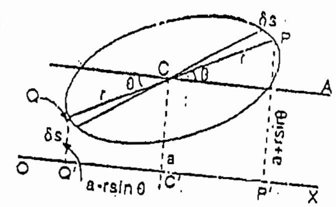
$= 2\pi\rho a S(a^2 + 3k^2) = M(a^2 + 3k^2)$. $\because M = 2\pi\rho a S$.

Theorem II. A closed curve revolves round any line OX in its own plane which does not intersect it. Show that the M.I. of the surface of revolution so formed about OX is equal to $M(a^2 + 3k^2)$, where M is the mass of the surface generated, a is the distance from OX of the centre C of the curve and k is the radius of gyration of the arc of the curve about a line through C parallel to OX.

Prof. Let l be the length of the arc of the closed curve, then

$l = 2\int ds$ ...(1)

the integration being taken to cover the upper half of the arc



By Pappus theroem, the mass M of the solid of revolution is given by

$M = 2\pi a\rho l$.

If k is the radius of gyration of the arc of the curve about OX then its M.I. about CA (a line parallel to OX) is ρk^2.

Consider an element δs at $P(r, \theta)$ of the arc taking C as centre and CA as initial line. For this element δs at $P(r, \theta)$ on the arc ther will be an equal arc δs for the same value of θ in opposite direction at Q on the arc.

we have $PP' = a + r\sin\theta$ and $QQ' = a - r\sin\theta$.

$\therefore$ M.I. of the arc of the curve about CA,

$l\rho k^2 = 2\iint (r\sin\theta)^2 \cdot \rho ds$

$= 2\rho \int r^2 \sin^2\theta\,ds$ ...(2)

the integration being taken to cover the upper half of the arc.

Now, M.I. of the surface of revolution about OX

$= \int [2\pi(a + r\sin\theta)\cdot(a + r\sin\theta)^2 + 2\pi(a - r\sin\theta)\cdot(a - r\sin)^2]\cdot\rho ds$

$= \int 4\pi\rho\,(a^3 + 3ar^2 \sin^2\theta)\,ds$

$= 4\pi\rho a^3 \int ds + 12\pi\rho a \int r^2 \sin^2\theta\,ds$

$= 2\pi\rho a^3 l + 6\pi a\rho l k^2$ [By (1) and (2)]

$= 2\pi\rho a l\,(a^2 + 3k^2) = M(a^2 + 3k^2)$. $(\because M = 2\pi a\rho l)$

EXAMPLES

Ex. 25. *The M.I. a out its axis, of a solid rubber tyre, of mass M and circular cross-section of radius a is $(M/4)(4b^2 + 3a^2)$, where b is the radius of the curve.*

Sol. Let OX be the axis of the solid tyre of mass M and circular cross-section of radius a. Solid tyre is obtained by the revolution of the circle of radius a and centre C about OX, where $CC' = b$.

Let CA be the line through C, parallel to OX.

Then M.I. of the circular area of mass M' (say) about CA

$M'k^2 = \frac{1}{4}M'a^2$. $\therefore k^2 = \frac{1}{4}a^2$.

From Theorem I of § 1.17, M.I. of the solid tyre about OX

$= M(b^2 + 3k^2)$ here a is equal to b

$= M(b^2 + \frac{3}{4}a^2) = (M/4)(4b^2 + 3a^2)$.

Ex. 26. *The M.I. about its axis of a hollow tyre, of mass M and circular corss-section of radius a is (M/2) $(2b^2 + 3a^2)$, where b is the radius of the curve.*

Sol. Refer figure of last Ex. 25.

Here the hollow tyre is obtained by the revolution of the arc of the circle of radius a and centre C about OX, where $CC' = b$.

M.I. of the arc of mass M' (say) of the circle about CA.

$M'k^2 = \frac{1}{2} M' a^2$. $\therefore k^2 = \frac{1}{2} a^2$.

$\therefore$ From Theorem II of § 1.18, M.I. of the hollow tyre about OX

$= M(b^2 + 3k^2)$, here a is equal to b

$= M(b^2 + \frac{3}{2} a^2) = (M/2)(2b^2 + 3a^2)$.

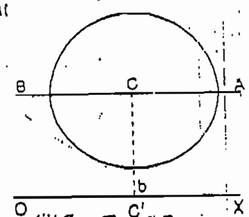


FIG. Ex 25

§ 1.19. M.I. by the Method of Differentiation.

If y is a function of x and δx, δy are small increments in the values of x and y respectively, then we know that

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$ i.e. $\frac{\delta y}{\delta x} = \frac{dy}{dx}$ approximately,

or $\delta y = \frac{dy}{dx} \delta x$.

For example :

(i) Area of a circle, $A = \pi r^2$, then

$\delta A = \left(\frac{d}{dr} A \right) . \delta r = \frac{d}{dr} (\pi r^2) . \delta r = (2\pi r) \delta r$

= (Circumference of a circle of radius r) × thickness δr.

(ii) Volume of sphere, $V = \frac{4}{3} \pi r^3$, then

$\delta V = \left(\frac{d}{dr} V \right) . \delta r = \frac{d}{dr} (\frac{4}{3} \pi r^3) . \delta r = (4\pi r^2) \delta r$

= (Surface of the spherical shell of radius r) × thickness δr.

This method of differentiation can be used in finding the moments of inertia in some cases. For this see the following examples.

EXAMPLES

Ex. 27. *Show that the M.I. of a thin homogeneous ellipsoidal shell (bounded by similar and similarly situated concentric ellipsoids) about an axes is (M/3) $(b + c^2)$, where M is the mass of the shell.*

Sol. We know that the M.I. of an ellipsoid of density ρ and semi-axis a, b, c about x-axis is equal to

$\left(\frac{4}{3} \pi abc\rho \right) . \frac{b^2 + c^2}{5}$.

Let the ellipsoid decrease indefinitely small in size.

$\therefore$ M.I. of the enclosed ellipsoidal shell

$$= d \left\{ \frac{4}{3} \pi abc\rho . \frac{b^2 + c^2}{5} \right\} \quad ...(1)$$

Since the shell is bounded by similar and similarly situated concentric ellipsoids, therefore if a', b', c' are the semi-axes of the similar ellipsoid, then we have

$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

$\therefore b = \frac{b'}{a'} a = pa$ and $c = \frac{c'}{a'} a = qa$.

$\therefore$ From (1), M.I. of the ellipsoidal shell

$$= d \left[\frac{4}{3} \pi \rho pq . \frac{p^2 + q^2}{5} a^5 \right]$$

$$= \frac{4}{3} \pi \rho pq . (p^2 + q^2) a^4 da. \quad ...(2)$$

But the mass of the ellipsoid $= \frac{4}{3} \pi abc\rho = \frac{4}{3} \pi \rho pqa^3$

$\therefore$ Mass of the ellipsoidal shell

$M = d(\frac{4}{3} \pi \rho pqa^3) = 4\pi \rho pqa^2 da$.

Hence from (2), we have

M.I. of the ellipsoidal shell

$= \frac{M}{3} (p^2 + q^2) a^2 = \frac{M}{3} (b^2 + c^2)$

§ 1.20. M.I. of Hetrogeneous Bodies.

The method of differentiation can be used in finding the M.I. of a hetrogenous body whose boundary is a surface of uniform density. For this proceed as follows :

(i) Find the M.I. of homogenous solid body of density ρ.

(ii) Express this M.I. in terms of a single parameter α (say) i.e. M.I. $= \rho\phi(\alpha)$.

(iii) Then by differentiation, the M.I. of a shell which is considered to be made of a layor of uniform density ρ

$= \rho\phi'(\alpha) d\alpha$.

(iv) Replace ρ by the variable deisity σ.

(v) Thus the M.I. of the given hetrogeneous body is given by

M.I. $= \int \rho\phi'(\alpha) d\alpha$.

For illustration see the following examples.

EXAMPLES

Ex. 28. *Show that the M.I. of a hetrogenous ellipsoid about the major axis is $\frac{2}{9}M(b^2+c^2)$, the strata of uniform density being similar concentric ellipsoids and the density along the major axis varying as the distance from the centre.*

Sol. (i) We know that the M.I. of an ellipsoid of density ρ and semi-axes a, b, c about x-axis is equal to

$$\left(\frac{4}{3}\pi abc\rho\right)\cdot\frac{b^2+c^2}{5}.$$

Also the mass of the ellipsoid $=\frac{4}{3}\pi abc\rho$.

(ii) Since the boundary surfaces are similar concentric ellipsoid, therefore, if a', b', c' are the semi-axes of the similar ellipsoid then we have

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$$

i.e. $b=\frac{b'}{a'}a=pa$ and $c=\frac{c'}{a'}a=qa$.

$\therefore$ M.I. of the ellipsoid about x-axis

$=\frac{4}{3}\pi abc\rho\cdot\frac{b^2+c^2}{5}=\frac{4}{3}\pi\rho pq\cdot\frac{p^2+q^2}{5}a^5$.

(iii) Differentiating the above M.I., the M.I. of a shell of uniform density ρ

$=d\left(\frac{4}{3}\pi\rho pq\cdot\frac{p^2+q^2}{5}a^5\right)$

$=\frac{4}{3}\pi\rho pq(p^2+q^2)a^4da$.

(iv) Since the density varies as the distance from the centre,

$\therefore\ \sigma=\lambda a$.

Replacing ρ by $\sigma=\lambda a$ and then integrating the M.I. of the hetrogenous ellipsoid about the major axis

$=\int_0^a\frac{4}{3}\pi\lambda apq(p^2+q^2)a^4da$

$=\frac{4}{3}\pi\lambda pq(p^2+q^2)\int_0^a a^5da=\frac{2}{9}\pi\lambda pq(p^2+q^2)\cdot a^6$ ...(1)

Also the mass of the ellipsoid $=\frac{4}{3}\pi abc\rho=\frac{4}{3}\pi\rho pqa^3$.

$\therefore$ Mass of the ellipsoidal shell $=d(\frac{4}{3}\pi\rho pqa^3)$

$=4\pi\rho pqa^2da$.

Replacing ρ by $\sigma=\lambda a$ and then integrating, the mass of the hetrogeneous ellipsoid is given by

$M=\int_0^a 4\pi\lambda apqa^2da=\pi\lambda pqa^4$.

Hence from (1), M.I. of the hetrogeneous ellipsoid

$=\frac{2}{9}M(p^2+q^2)a^2=\frac{2}{9}M(b^2+c^2)$.

Ex. 29. *The M.I. of a hetrogeneous ellipse about minor axis, the strata of uniform density being confocal ellipses and density along minor axis, varying as the distance from the centre is*

$$\frac{3M}{20}\cdot\frac{4a^5+c^5-5a^3c^2}{2a^3+c^3-3ac^2}.$$

Sol. For confocal ellipses, we have

$a^2e^2=a^2-b^2=$ Constant.

$\therefore$ Taking $a^2-b^2=c^2$, the euation of the confocal ellipse is

$\frac{x^2}{b^2+c^2}+\frac{y^2}{b^2}=1$, where $a^2=b^2+c^2$. ...(1)

The M.I. of homogenous ellipse of uniform density ρ about minor axis is

$(\rho\pi ba)\cdot\frac{a^2}{4}=\rho\pi b\sqrt{(b^2+c^2)}\cdot\frac{b^2+c^2}{4}=\frac{1}{4}\rho\pi b(b^2+c^2)^{3/2}$.

Differentiating, the M.I. of an elliptic strata of uniform density ρ

$=d\{\frac{1}{4}\rho\pi b(b^2+c^2)^{3/2}\}$

$=\frac{1}{4}\rho\pi\{1.(b^2+c^2)^{3/2}+b.\frac{3}{2}(b^2+c^2)^{1/2}.2b\}db$

$=\frac{1}{4}\pi\rho\sqrt{(b^2+c^2)}.(4b^2+c^2)db$.

Since the density varies as the distance from the centre, therefore replacing ρ by λb and integrating the M.I. of the hetrogeneous ellipse about minor axis

$=\int_0^b\frac{1}{4}\pi\lambda b\sqrt{(b^2+c^2)}.(4b^2+c^2)db$

$=\frac{1}{4}\pi\lambda\left[\int_0^b 4(b^2+c^2)^{3/2}bdb-3\int_0^b c^2(b^2+c^2)^{1/2}bdb\right]$

$=\frac{1}{4}\pi\lambda\left[\frac{4}{5}(b^2+c^2)^{5/2}-c^2(b^2+c^2)^{3/2}\right]_0^b$

$=\frac{1}{4}\pi\lambda[\frac{4}{5}\{(b^2+c^2)^{5/2}-c^5\}-c^2\{(b^2+c^2)^{3/2}-c^3\}]$

$=\frac{1}{4}\pi\lambda[\frac{4}{5}(a^5-c^5)-c^2(a^3-c^3)]$. ...(2)

Also the mass of the ellipse $=\pi\rho ba=\rho\pi b\sqrt{(b^2+c^2)}$

$\therefore$ Mass of the elliptic strata of uniform density ρ

$= d\,[\rho\pi b\sqrt{(b^2+c^2)}]$

$= \rho\pi\,[1.\sqrt{(b^2+c^2)} + b.\frac{1}{2}(b^2+c^2)^{-1/2}.2b]\,db$

$= \rho\pi \cdot \dfrac{2b^2+c^2}{\sqrt{(b^2+c^2)}}\,db.$

Replacing ρ by λb and integrating the mass of the heterogeneous ellipse

$$M = \int_0^b \pi\lambda b \cdot \frac{2b^2+c^2}{\sqrt{(b^2+c^2)}}\,db$$

$$= \pi\lambda\left[\int_0^b 2\sqrt{(b^2+c^2)}\;b\,db - c^2\int_0^b \frac{b\,db}{\sqrt{(b^2+c^2)}}\right]$$

$$= \pi\lambda\left[\tfrac{2}{3}(b^2+c^2)^{3/2} - c^2\sqrt{(b^2+c^2)}\right]_0^b$$

$$= \pi\lambda\,[\tfrac{2}{3}\{(b^2+c^2)^{3/2} - c^3\} - c^2\{(b^2+c^2)^{1/2} - c\}]$$

$$= \pi\lambda\,[\tfrac{2}{3}(a^3-c^3) - c^2(a-c)]. \qquad \because b^2+c^2=a^2.$$

Hence from (2) the M.I. of the heterogeneous ellipse about the minor axis

$$= \frac{M}{4}\cdot\frac{\frac{4}{5}(a^5-c^5) - c^2(a^3-c^3)}{\frac{2}{3}(a^3-c^3) - c^2(a-c)}$$

$$= \frac{3M}{20}\cdot\frac{4a^5+c^5-5a^3c^2}{2a^3+c^3-3ac^2}.$$

§ 1.21. Momental Ellipsoid.

The M.I. of a body about a line OQ whose direction cosines are l, m, n is given by

$Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm,$

where A, B, C, D, E, F are the moments and products of inertia of the body about the axes.

Let P be a point on OQ such that the M.I. of the body about OQ may be inversely proportional to OP^2

i.e. $Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm \propto \dfrac{1}{OP^2}$

or $Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm = \dfrac{Mk^4}{r^2}$, where $OP = r$.

or $Al^2r^2 + Bm^2r^2 + Cn^2r^2 - 2Dmr.nr - 2Enr.lr - 2Flr.mr = Mk^4$

or $Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = Mk^4$. ...(1)

Since A, B, C are essentially positive, therefore equation (1) represent an ellipsoid. *This is called the momental ellipsoid of the body at O.*

By solid geometry, we can find three mutually perpendicular diameters, such that with these diameters as coordinate axes, the equation of the ellipsoid is transformed into the form

$A_1x^2 + B_1y^2 + C_1z^2 = Mk^4.$ ...(2)

The product of inertia with respect to these new axes will vanish. *These three new axes are called the principal axes of the body at the point O. And a plane through any two of these axes is called a principal plane of the body.*

Hence for every body there exists at every point O, a set of three mutually perpendicular axes, which are the three principal diameters of the momental ellipsoid at O, such that the products of inertia of the body about them taken two at a time vanish.

Note. When the three principal moments of inertia at any point O are the same, the ellipsoid becomes a sphere. In this case every diameter is a principal diameter and all radii vectors are the same.

§ 1.22. Momental Ellipse.

Let OX and OY be two mutually perpendicular axes and OQ a line through O, all in the plane of a lamina. Then M.I. of the plane lamina, about OQ is given by

$$A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta,$$

where A, B denote the moments of inertia about OX, OY and F the product of inertia about OX and OY.

Let P be a point on OQ such that the M.I. of the lamina about OQ may be inversely proportional to OP^2.

i.e. $A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta \propto \dfrac{1}{OP^2}$

or $A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta = \dfrac{Mk^4}{r^2}$ where $OP = r$,

or $Ar^2\cos^2\theta - 2Fr\cos\theta\, r\sin\theta + Br^2\sin^2\theta = Mk^4$

or $Ax^2 - 2Fxy + By^2 = Mk^4$ ...(1)

Since A and B are essentially positive, therefore equation (1) represent an ellipse. *This is called a* momental ellipse *of the lamina at O.*

Note. The section of the momental ellipsoid at O by the plane of the lamina is the momental ellipse.

EXAMPLES

Ex. 30. *Find the momental ellipsoid at any point O of a material straight rod AB of mass M and length 2a.*

Sol. Let G be the centre of gravity of a material straight rod AB of mass M and length $2a$. Let O be a point on the rod s.t. $OG = c$.

Consider the axis OX along the rod and axis OY perpendicular to the rod.

$\therefore$ A = M.I. of the rod about $OX = 0$.

B = M.I. of the rod about OY = M.I. of the rod about parallel axis GY' + M.I. of mass M at G about OY

$= \frac{1}{3} Ma^2 + Mc^2 = M(\frac{1}{3} a^2 + c^2)$

Similarly C = M.I. of the rod about $OZ = M(\frac{1}{3} a^2 + c^2)$.

The coordinates of the C.G. 'G' of the rod are $(c, 0, 0)$:

$\therefore$ $D = O = E = F$.

Hence equation of the momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy$ = Const.

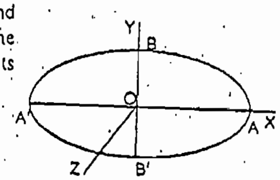
or $O + M(\frac{1}{3} a^2 + c^2) y^2 + M(\frac{1}{3}a^2 + c^2) z^2$ = Const.

or $M(\frac{1}{3}a^2 + c^2)(y^2 + z^2)$ = Const.

or $y^2 + z^2$ = const

Ex. 31. *Show that the momental ellipsoid at the centre of an elliptic plate is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = const.$

Sol. Let M be the mass of an elliptic plate of semi-axes a and b. Let the axes OX and OY be taken along the major and minor axes of the elliptic plate in its plane and the axes OZ perpendicular to its plane. Then



A = M.I. of the plate about OX

$= \frac{1}{4} Mb^2$

B = M.I. of the plate about $OY = \frac{1}{4} Ma^2$

C = M.I. of the plate about OZ.

$= \frac{1}{4} M(a^2 + b^2)$

and since plate is symmetrical about OX and OY

$\therefore$ $D = O = E = F$.

$\therefore$ Equation of the momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dmn - 2Enl - 2Flm$ = Const.

or $\frac{1}{4} Mb^2x^2 + \frac{1}{4} Ma^2y^2 + \frac{1}{4} M(a^2 + b^2) z^2$ = Const.

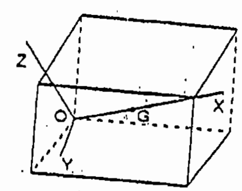
or $\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$ = Const.

Ex. 32. *Show that the equation of the momental ellipsoid at the corner of a cube of side 2a referred to its principal axes is*

$$2x^2 + 11(y^2 + z^2) = C,$$

where C is constant.

Sol. Let G be the centre of gravity of a cube of side $2a$. Let O be a corner of the cube at which we have to determine the equation of the momental ellipsoid.



Take the line OX through G as the axis of x and two mutually perpendicular lines OY and OZ through O as the axis of y and z.

The coordinates of G referred to OX, OY, OZ as axis are $(a\sqrt{3}, 0, 0)$ and the products of inertia of the cube about any two mutually perpendicular lines through G is zero.

$\therefore$ the product of inertia about the axes OX, OY, OZ taken in pairs is zero. Thus OX, OY, OZ are the principal axes of the momental ellipsoid at O.

Since the M.I. of the cube about any axis (parallel to an edge) through $G = \frac{2}{3} Ma^2$.

$\therefore$ A = M.I. about $OX = A'l^2 + B'm^2 + C'n^2 = \frac{2}{3} Ma^2$

$\because$ $A' = B' = C' = \frac{2}{3} Ma^2$.

B = M.I. about OY = M.I. about parallel axis through G + M.I. of total mass M at G about OY

$= \frac{2}{3} Ma^2 + M.OG^2 = \frac{2}{3} Ma^2 + M(a\sqrt{3})^2 = \frac{11}{3} Ma^2$.

Similarly, C = M.I. about $OZ = \frac{11}{3} Ma^2$.

and $D = O = E = F$.

Hence equation of the momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dmn - 2Enl - 2Flm$ = Const.

or $\frac{2}{3} Ma^2x^2 + \frac{11}{3} Ma^2y^2 + \frac{11}{3} Ma^2z^2$ = Const.

or $2x^2 + 11(y^2 + z^2) = C$, where C is a constant.

Ex. 33. *Show that the momental ellipsoid at the centre of an ellipsoid is* $(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2 = const.$

Sol. The equation of an ellipsoid, referred to the principal axes is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

$\therefore A =$ M.I. about $OX = \frac{1}{5}M(b^2+c^2)$

$B =$ M.I. about $OY = \frac{1}{5}M(c^2+a^2)$

$C =$ M.I. about $OZ = \frac{1}{5}M(a^2+b^2)$

and $D = O = E = F$.

Hence equation of the momental ellipsoid at the centre of the ellipsoid is

$Ax^2+By^2+Cz^2-2Dmn-2Eml-2Flm=$ const.

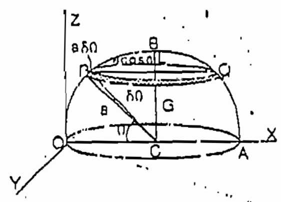
or $\frac{1}{5}M(b^2+c^2)x^2+\frac{1}{5}M(c^2+a^2)y^2+\frac{1}{5}M(a^2+b^2)z^2=$ const.

or $(b^2+c^2)x^2+(c^2+a^2)y^2+(a^2+b^2)z^2=$ const.

Ex. 34. *Show that the momental ellipsoid at a point on the edge of the circular base of a thin hemispherical shell is*

$$2x^2+5(y^2+z^2)-3zx=\text{const.}$$

Sol. Let O be a point on the the edge of the circular base of a thin hemispherical shell of radius a and mass M. Take the axis OX along the diameter OA of base of the shell, axis OY perpendicular to OX through O in the plane of the base and axis OZ perpendicular to the base. The thin hemispherical shell radius a is obtained by the revolution of arc OB of the quadrant of a circle of radius a about the line CB which is parallel to OZ and at a distance a from it.



Consider an element of arc $a\ \delta\theta$ at P. By the revolution of this arc about CB a circular ring of radius $PL = a\cos\theta$ and cross-section $a\ \delta\theta$ is obtained.

Mass of this elementary ring

$=\delta m=\rho\,.\,2\pi a\cos\theta\,.\,a\delta\theta=2\pi a^2\rho\cos\theta\ \delta\theta.$

M.I. of this elementary ring about OA

$=$ Its M.I. about $PQ+$ M.I. of mass δm at centre L about OA.

$=PL^2\,\delta m+CL^2\,\delta m=(\frac{1}{2}a^2\cos^2\theta+a^2\sin^2\theta).2\pi a^2\rho\cos\theta\delta\theta$

$=\pi\rho a^4(\cos^2\theta+2\sin^2\theta)\cos\theta\delta\theta=\pi\rho a^4(1+\sin^2\theta)\cos\theta\delta\theta.$

$\therefore A =$ M.I. of the hemispherical shell about OX

$=\int_0^{\pi/2}\pi\rho a^4(1+\sin^2\theta)\cos\theta d\theta$

$=\pi\rho a^4\int_0^1(1+t^2)\,dt,$ Putting $\sin\theta=t$

$=\pi\rho a^4\left[t+\frac{1}{3}t^3\right]_0^1=\frac{4}{3}\pi\rho a^4=\frac{2}{3}Ma^2,$ $\because M=2\pi a^2\rho$

$B =$ M.I. of the hemi-spherical shell about OY

$=$ Its M.I. about parallel diameter through C

+ M.I. of total mass M at C about OY

$=\frac{2}{3}Ma^2+Ma^2=\frac{5}{3}Ma^2$

Also M.I. of the elementary ring about OZ

$=$ Its M.I. about $BC+$ M.I. of its mass δm at L about OZ

$=PL^2\,\delta m+OC^2\,\delta m=(a^2\cos^2\theta+a^2)\,2\pi a^2\rho\cos\theta\delta\theta$

$=2\pi a^4\rho\,(\cos^3\theta+\cos\theta)\,\delta\theta$

$\therefore C =$ M.I. of the hemispherical shell about OZ

$=\int_0^{\pi/2}2\pi a^4\rho(\cos^3\theta+\cos\theta)\,d\theta=2\pi a^4\rho\,.\left[\frac{\Gamma(2)\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{5}{2})}+(\sin\theta)_0^{\pi/2}\right]$

$=Ma^2\,(\frac{2}{3}+1)=\frac{5}{3}Ma^2.$

Coordinates of C.G. 'G' of the shell are $(a, 0, a/2)$

$D =$ P.I. of the shell about OY, OZ

$=$ P.I. of the shell about lines parallel to OY, OZ through $C+$ P.I. of mass M at G about OY, OZ

$=0+M.O.a/2=0.$

Similarly $E =$ P.I. of the shell about OZ, OX

$=0+M.a/2.a=\frac{1}{2}Ma^2$

and $F =$ P.I. of the shell about OX and $OY=0+M.a.O=0.$

Hence the equation of momental ellipsoid at O is

$Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy=$ const.

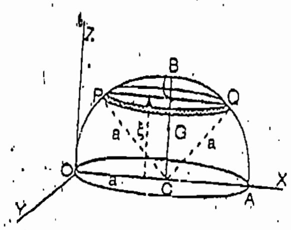
or $\frac{2}{3}Ma^2x^2+\frac{5}{3}Ma^2y^2+\frac{5}{3}Ma^2z^2-O-2.\frac{1}{2}Ma^2zx-O=$ const.

or $2x^2+5(y^2+z^2)-3zx=$ const.

Ex. 35. *Show that the momental ellipsoid at a point on the rim of a hemisphere is* $2x^2+7(y^2+z^2)-\frac{15}{4}xz=$ *Const.*

Sol. Let O be a point on the rim of a hemisphere of radius a and mass M. If ρ is the density then

$$M = \tfrac{2}{3}\pi a^3 \rho.$$



Take the axis OX along the diameter OA of the circular base axis OY perpendicular to OX through O in the plane of the base and axis OZ perpendicular to the base.

Consider an elementary strip PQ of thickness $\delta\xi$, parallel to the base and at a distance ξ from C, then

Mass of this elementary disc, $\delta m = \rho\pi PL^2 \delta\xi = \rho\pi(a^2 - \xi^2)\delta\xi$.

M.I. of the elementary disc about OX = Its M.I. about PQ + M.I. of mass δm at L about OX.

$= \frac{1}{4}PL^2\,\delta m + CL^2\delta m = [\frac{1}{4}(a^2 - \xi^2) + \xi^2]\,.\,\rho\pi(a^2 - \xi^2)\,\delta\xi$

$= \frac{1}{4}\pi\rho\,(a^4 + 2a^2\xi^2 - 3\xi^4)\,\delta\xi.$

$\therefore\ A$ = M.I. of the hemisphere about OX

$= \int_0^a \frac{1}{4}\pi\rho\,(a^4 + 2a^2\xi^2 - 3\xi^4)\,d\xi = \frac{4}{15}\pi\rho a^5 = \frac{2}{5}Ma^2.$

B = M.I. of the hemisphere about OY

= Its M.I. about the line through C (diameter of base) and parallel to OY + M.I. of total mass M at C about OY

$= \frac{2}{5}Ma^2 + Ma^2 = \frac{7}{5}Ma^2.$

Also M.I. of the elementary disc about OZ

= Its M.I. about CB + M.I. of its mass δm at L about OZ

$= \frac{1}{2}PL^2\,\delta m + OC^2\,\delta m = [\frac{1}{2}(a^2 - \xi^2) + a^2]\,\rho\pi\,(a^2 - \xi^2)\,d\xi$

$= \frac{1}{2}\rho\pi\,(3a^4 - 4a^2\xi^2 + \xi^4)\,d\xi$

$\therefore\ C$ = M.I. of the hemisphere about OZ

$= \int_0^a \frac{1}{2}\rho\pi\,(3a^4 - 4a^2\xi^2 + \xi^4)\,d\xi = \frac{14}{15}\rho\pi a^4 = \frac{7}{5}Ma^2.$

Coordinates of the C.G. 'G' of the hemisphere are $(a, 0, \frac{3}{8}a)$.

$\therefore\ D$ = P.I. of the hemisphere about OY and OZ

= Its P.I. about lines through G, parallel to OY and OZ + P.I. of mass M at G about OY and OZ

$= 0 + M.0.\frac{3}{8}a = 0$

Similarly E = P.I. of hemisphere about OZ and OX

$= 0 + M.\frac{3}{8}a.a = \frac{3}{8}Ma^2$

F = P.I. of hemisphere about OX and $OY = 0 + M.a.0 = 0.$

Hence the equation of momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{const.}$

or $\frac{2}{5}Ma^2x^2 + \frac{7}{5}Ma^2y^2 + \frac{7}{5}Ma^2z^2 - 0 - 2.\frac{3}{8}Ma^2zx - 0 = \text{const.}$

or $2x^2 + 7(y^2 + z^2) - \frac{15}{4}xz = \text{const.}$

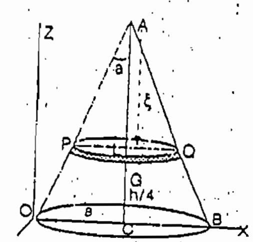
Ex. 36. *Prove that the equation of the momental ellipsoid at a point on the circular edge of a solid cone is*

$$(3a^2 + 2h^2)\,x^2 + (23a^2 + 2a^2)\,y^2 - 26a^2z^2 - 10ahxz = \text{const},$$

where h is the height and a the radius of the base.

Sol. Let O be a point on the circular edge of a solid cone of mass M, semi-vertical angle α, height h and radius of base a. If ρ is its density, then

$M = \frac{1}{3}\pi\rho h^3 \tan^2\alpha.$



Take the axis OX along the diameter OB of the base, axis OY perpendicular to OB in the plane of the base and OZ perpendicular to the base.

Consider an elementary disc PQ parallel to the base, at a distance ξ from the vertex A and of thickness $\delta\xi$.

$\therefore$ Mass of this elementary disc, $\delta m = \rho\pi PL^2\delta\xi$

$= \rho\pi\xi^2\tan^2\alpha\,d\xi.$

M.I. of this elementary disc about OX

= Its M.I. about PQ + M.I. of its mass δm at L about OX.

$= \frac{1}{4}PL^2\delta m + CL^2\delta m = [\frac{1}{4}\xi\tan s\,\alpha + (h - \xi)^2]\,\rho\pi\xi^2\tan^2\alpha d\xi.$

$\therefore\ A$ = M.I. of the cone about OX

$= \int_0^h [\frac{1}{4}\xi^2\tan^2\alpha + (h - \xi)^2]\,\rho\pi\xi^2\tan^2\alpha d\xi$

$$= \rho\pi \tan^2\alpha \int_0^h [\tfrac{1}{4}\xi^4 \tan^2\alpha + h^2\xi^2 - 2h\xi^3 + \xi^4]\, d\xi$$

$$= \rho\pi h^5 \tan^2\alpha \left[\frac{1}{20}\tan^2\alpha + \frac{1}{30}\right] = \frac{1}{60}\rho\pi h^5 \tan^2\alpha\,(3\tan^2\alpha + 2)$$

$$= \frac{1}{20} Mh^2 (3\tan^2\alpha + 2) = \frac{1}{20} M(3a^2 + 2h^2), \quad (\because \tan\alpha = a/h)$$

B = M.I. of the cone about OY

= Its M.I. about line parallel to OY through C (*i.e.* diameter of base) + M.I. of total mass M at C about OY

$$= \frac{1}{20} M(3a^2 + 2h^2) + Mh^2 = \frac{1}{20} M(23a^2 + 2h^2).$$

Now M.I. of the elementary disc about OZ

= Its M.I. about AC + M.I. of its mass δm at L about OZ

$$= \tfrac{1}{2} PL^2\delta m + OC^2 \delta m = (\tfrac{1}{2}\xi^2 \tan^2\alpha + a^2)\,\rho\pi\xi^2 \tan^2\alpha\, d\xi$$

$$= \rho\pi\,(\tfrac{1}{2}\xi^4 \tan^2\alpha + a^2\xi^2)\, d\xi.$$

$\therefore$ C = M.I. of the cone about OZ

$$= \int_0^h \rho\pi\,(\tfrac{1}{2}\xi^4 \tan^2\alpha + a^2\xi^2) \tan^2\alpha\, d\xi$$

$$= \rho\pi h^3 \left(\frac{1}{10} h^2 \tan^2\alpha + \frac{1}{3} a^2\right) \tan^2\alpha$$

$$= \frac{1}{10} M(3h^2 \tan^2\alpha + 10a^2) = \frac{13}{10} Ma^2, \quad (\because \tan\alpha = a/h)$$

The coordinates of C.G. G of the cone are $(a, 0, h/4)$.

$\therefore$ D = P.I. of the cone about OY and OZ

= P.I. of the cone about lines through G parallel to OY and OZ + P.I. of the mass M at G about OY and OZ

$= 0 + M \cdot 0 \cdot h/4 = 0.$

Similarly, E = P.I. of the cone about OZ and OX

$$= 0 + M \cdot \frac{h}{4} \cdot a = \frac{1}{4} Mah$$

and F = P.I. of the cone about OX and $OY = 0 + M \cdot a \cdot 0 = 0$

Hence the equation of the momental ellipsoid at O is

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}.$$

or $$\frac{1}{20} M(3a^2 + 2h^2)x^2 + \frac{1}{20} M(23a^2 + 2h^2)y^2 + \frac{13}{10} Ma^2z^2 - 0 - 2 \cdot \frac{1}{4} Mahzx - 0 = \text{constant}$$

or $$(3a^2 + 2h^2)x^2 + (23a^2 + 2h^2)y^2 + 26a^2z^2 - 10ahxz = \text{const.}$$

Ex. 37. *If $S \equiv Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy$ = constant, be the equation of the momental ellipsoid at the centre of gravity O of a body referred to any rectangular axes through O, then prove that momental ellipsoid at the point (p, q, r) is*

$$S + M[(qz - ry)^2 + (rx - pz)^2 + (py - qx)^2] = \text{const.}$$

where M is the mass of the body.

Sol. Since $S \equiv Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy$ = constant is the equation of the momental ellipsoid at the centre of gravity O of the body referred to the rectangular axes at O, therefore A, B, C are the moments and D, E, F are the products of inertia of the body about the rectangular axes through O.

Let A', B', C' be the moments and D', E', F' the products of inertia of the body about the parallel rectangular axes through (p, q, r). If M is the mass of the body, then

A' = M.I. about x-axis through C.G. O + M.I. of mass M at O about the axis parallel to x-axis through (p, q, r)

$= A + M(q^2 + r^2).$

Similarly, $B' = B + M(r^2 + p^2)$, $C' = C + M(p^2 + q^2)$

$D' = D + Mqr$, $E' = E + Mrp$, $F' = F + Mpq$.

Hence the equation of the momental ellipsoid at (p, q, r) is

$$A'x^2 + B'y^2 + C'z^2 - 2D'yz - 2E'zx - 2F'xy = \text{const.}$$

or $$[A + M(q^2 + r^2)]x^2 + [B + M(r^2 + p^2)]y^2 + [C + M(p^2 + q^2)]z^2 - 2(D + Mqr)yz - 2(E + Mrp)zx - 2(F + Mpq)xy = \text{const.}$$

or $$(Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy) + M[(q^2z^2 + r^2y^2 - 2qryz) + (r^2x^2 + p^2z^2 - 2prxz) + (p^2y^2 + q^2x^2 - 2pqxy)] = \text{const.}$$

or $$S + M[(qz - ry)^2 + (rx - pz)^2 + (py - qx)^2] = \text{const.}$$

§ 1.23. Equimomental Bodies.

Two systems or bodies are said to be equimomental or kinetically (or dynamically) equivalent when moments and products of inertia of one system or body about all axes are each equal to the moments and products of inertia of the other system or body about the same axes.

The necessary and sufficient conditions for two systems to be equimomental are that :

(i) the centre of gravity of the two systems is the same point;

(ii) both the systems have the same mass; and

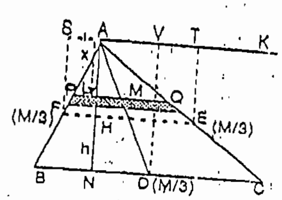
(iii) the two systems have the same principal axes and same principal moments about the centre of gravity.

§ 1.24. *The moments and products of inertia of a uniform triangle about any lines are the same as the moments and products of inertia of three particles placed at the middle points of the sides, each equal to one-third of the mass of the triangle.*

Let AD be the median of a triangle ABC of mass M. Let AN be the perpendicular on BC from A, AK perpendicular to AN in the plane of the triangle ABC and $AN = h$.

$\therefore M = \frac{1}{2} BC \cdot AN\, \rho = \frac{1}{2} ah\rho$

Consider an elementary strip PQ parallel to BC of thickness δx and at a distance x from A.



From similar triangles APQ and ABC, we have

$\frac{PQ}{BC} = \frac{AL}{AN}$,

$\therefore PQ = \frac{AL}{AN} \cdot BC = \frac{xa}{h}$

Now mass of the strip $PQ = \rho PQ \delta x = \frac{\rho a}{h} x \delta x$.

$\therefore$ M.I. of the strip about AK

= Its M.I. about PQ + M.I. of its mass δm at its C.G. (*i.e.* middle point of PQ) about AK

$= 0 + x^2 \delta m = \frac{\rho a}{h} x^3 \delta x.$

$\therefore$ M.I. of the ΔABC about $AK = \int_0^h \frac{\rho a}{h} x^3 dx$

$= \frac{1}{4} \rho a h^3 = \frac{1}{2} Mh^2.$ ...(1)

Also M.I. of the strip PQ about AN

= M.I. of the strip about parallel line through its C.G. M (middle point of BC) + M.I. of its mass δm at M about AN

$= \frac{1}{3}\left(\frac{1}{2} PQ\right)^2 \delta m + LM^2\, \delta m = \left[\frac{1}{3}\left(\frac{ax}{2h}\right)^2 + LM^2\right] \cdot \frac{\rho a}{h} x \delta x$

But from similar triangles ALM and AND, we have

$\frac{LM}{ND} = \frac{AL}{AN} = \frac{x}{h}$ $\quad \therefore \quad LM = \frac{x}{h} ND.$

$\therefore$ M.I. of the strip PQ about AN

$= \left[\frac{1}{3} \cdot \frac{a^2 x^2}{4h^2} + \frac{ND^2}{h^2} x^2\right] \frac{\rho a}{h} x \delta x$

$= \frac{\rho a}{12h^3} (a^2 + 12ND^2)\, x^3 \delta x.$

$\therefore$ M.I. of the triangle ABC about AN

$= \int_0^h \frac{\rho a}{12h^3} (a^2 + 12ND^2)\, x^3 dx$

$= \frac{\rho a h}{48} (a^2 + 12ND^2) = \frac{M}{24} [a^2 + 12(BD - BN)^2]$

$= \frac{M}{24}\left[a^2 + 12\left(\frac{a}{2} - c\cos B\right)^2\right]$

$= \frac{M}{24}\left[a^2 + 12\left(\frac{a}{2} - c\,\frac{a^2 + c^2 - b^2}{2ac}\right)^2\right]$

$= \frac{M}{24}\left[a^2 + \frac{3}{a^2}(b^2 - c^2)^2\right] = \frac{M}{24a^2}[a^4 + 3(b^2 - c^2)^2].$ ...(2)

and P.I. of the triangle ABC about AK and AN

$= \int_0^h (AL \cdot LM) \frac{\rho a}{h} x dx = \int_0^h \left(x \cdot \frac{x}{h} ND\right) \frac{\rho a}{h} x dx = \frac{1}{4} \rho a h^2 \cdot ND$

$= \frac{1}{2} Mh \cdot ND = \frac{1}{2} Mh\,(BD - BN) = \frac{1}{2} Mh\,(a/2 - c\cos B)$

$= \frac{1}{2} Mh\left(\frac{a}{2} - c \cdot \frac{a^2 + c^2 - b^2}{2ac}\right) = \frac{1}{4} \frac{Mh}{a} (b^2 - c^2)$ ...(3)

Now we shall consider a system of three particles each of mass $M/3$ placed at the middle points D, E, F of the sides of the ΔABC and find their moments and products of inertia about AK and AN.

M.I. of the three particles each of mass $M/3$ at D, E, F about AK

$\frac{M}{3} DV^2 + \frac{M}{3} ET^2 + \frac{1}{3} FS^2 = \frac{M}{3}\left[h^2 + \left(\frac{h}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right] = \frac{1}{2} Mh^2$ ...(4)

M.I. of the three particles each of mass $M/3$ at D, E, F about AN

$= \frac{M}{3} DN^2 + \frac{M}{3} EH^2 - \frac{M}{3} FH^2$

$= \frac{M}{3} [(BD - BN)^2 + (\frac{1}{2} CN)^2 + (\frac{1}{2} BN)^2]$

$= \frac{M}{3}\left[\left(\frac{a}{2} - c\cos B\right)^2 + \frac{1}{4}(b\cos C)^2 + \frac{1}{4}(c\cos B)^2\right]$

$= \frac{M}{12} [(a - 2c\cos B)^2 + b^2\cos^2 C + c^2\cos^2 B]$

$= \frac{M}{12} [(b\cos C + c\cos B - 2c\cos B)^2 + (b^2\cos^2 C + c^2\cos^2 B)]$

$= \frac{M}{12} [(b \cos C - c \cos B)^2 + (b \cos C - c \cos B)^2 + 2bc \cos B \cos C]$

$= \frac{M}{6} [(b \cos C - c \cos B)^2 + bc \cos B \cos C]$

$= \frac{M}{6} \left[\left(b \cdot \frac{a^2 + b^2 - c^2}{2ab} - c \cdot \frac{a^2 + c^2 - b^2}{2ac} \right)^2 + bc \cdot \frac{a^2 + c^2 - b^2}{2ac} \cdot \frac{a^2 + b^2 - c^2}{2ab} \right]$

$= \frac{M}{24a^2} [4(b^2 - c^2)^2 + a^4 - (b^2 - c^2)^2]$

$= \frac{M}{24a^2} [a^4 + 3(b^2 - c^2)^2]$, ...(5)

and P.I. of the three particles each of mass $M/3$ at D, E, F about AK and AN

$= \frac{M}{3} DN \cdot AN + \frac{M}{3} EH \cdot AH - \frac{M}{3} FH \cdot AH$

$= \frac{M}{3} \left[DN \cdot h + \frac{1}{2} CN \cdot \frac{h}{2} - \frac{1}{2} BN \cdot \frac{h}{2} \right] = \frac{1}{12} Mh (4DN + CN - BN)$

$= \frac{1}{12} Mh [4(BD - BN) + CN - BN] = \frac{1}{12} Mh \left[4 \cdot \frac{a}{2} + CN - 5BN \right]$

$= \frac{1}{12} Mh \left[4 \cdot \frac{a}{2} + b \cos C - 5 \cdot c \cos B \right]$

$= \frac{1}{12} Mh \left[4 \cdot \frac{a}{2} + b \cdot \frac{a^2 + b^2 - c^2}{2ab} - 5c \cdot \frac{a^2 + c^2 - b^2}{2ac} \right]$

$= \frac{Mh}{4a} (b^2 - c^2)$ ...(6)

From (1), (2), (3) and (4), (5), (6), it is clear that the moments and products of inertia of the ΔABC of mass M about AK and AN are the same as those of three particles each of mass $M/3$ placed at the middle points of the sides.

Note. Also the two systems have the same mass M and the same centre of gravity.

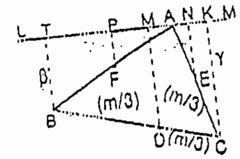
Hence the triangle of mass M is equimomental to three particles each of mass $M/3$ placed at the middle points of the sides.

EXAMPLES

Ex. 38. *Obtain the moment of inertia for a triangular lamina ABC about a straight line through A (or any vertex) in the plane of the triangle.*

Sol. Let m be the mass of the triangle ABC, then the triangle is equimomental to the three particles each of mass $m/3$ placed at the middle points D, E, F of its sides.

Let LM be any line through the vertex A and in the plane of the triangle ABC. Let β and γ be the distances of the vertices B and C from the line LM, *i.e.* $BT = \beta$ and $CK = \gamma$.



$\therefore$ Perpendicular distances of D, E, F from LM are as follows

$DM = \frac{1}{2}(\beta + \gamma)$, $EN = \frac{1}{2} CK = \frac{1}{2}\gamma$ and $FP = \frac{1}{2} BT = \frac{1}{2}\beta$.

$\therefore$ M.I. of the triangle ABC about LM

= Sum of M.I. of masses $m/3$ each at D, E, F about LM,

$= \frac{m}{3} \cdot DM^2 + \frac{m}{3} \cdot EN^2 + \frac{m}{3} \cdot FP^2$

$= \frac{m}{3} \left[\frac{1}{4} (\beta + \gamma)^2 + \frac{1}{4} \gamma^2 + \frac{1}{4} \beta^2 \right] = \frac{m}{6} (\beta^2 + \gamma^2 + \beta\gamma)$.

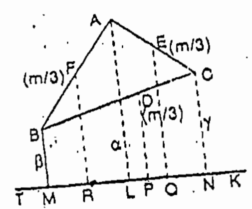
Ex. 39. *If α, β, γ be the distances of the vertices of a uniform triangular lamina of mass m from any line in its plane, prove that the M.I. about this line is $\frac{1}{6} m (\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta)$.*

Hence deduce that if h be the distance of the centre of inertia of the triangle from the line, then M.I. about this line is

$\frac{1}{12} m(\alpha^2 + \beta^2 + \gamma^2 + 9h^2)$.

Sol. Let ABC be the triangular lamina of mass m and AL, BM, CN the perpendiculars from A, B, C on a line TK in its plane, then

$AL = \alpha$, $BM = \beta$, $CN = \gamma$.



If DP, EQ, FR are the perpendiculars from the middle points D, E, F of sides on TK, then

$DP = \frac{1}{2}(BM + CN) = \frac{1}{2}(\beta + \gamma)$

$EQ = \frac{1}{2}(AL + CN) = \frac{1}{2}(\alpha + \gamma)$,

$FR = \frac{1}{2}(AL + BM) = \frac{1}{2}(\alpha + \beta)$.

Since the triangle is equimomental to the three particles each of mass $m/3$ placed at the middle points D, E, F of the triangle,

$\therefore$ M.I. of the ΔABC about TK

= Sum of M.I. of masses $\frac{1}{3}m$ each at D, E, F about TK

$= \left(\frac{m}{3}\right)\cdot(DP)^2 + \left(\frac{m}{3}\right)(EQ)^2 + \left(\frac{m}{3}\right)\cdot(FR)^2$

$= \frac{m}{3}\cdot\frac{1}{4}(\beta+\gamma)^2 + \frac{m}{3}\cdot\frac{1}{4}(\alpha+\gamma)^2 + \frac{m}{3}\cdot\frac{1}{4}(\alpha+\beta)^2$

$\frac{1}{6}m(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta)$ ...(1)

Deduction. If h is the distance of the centre of inertia of the ΔABC from TK, then $h = \frac{1}{3}(\alpha-\beta+\gamma)$.

$\therefore$ From (1), M.I. of the ΔABC about TK

$= \frac{1}{12}m(2x^2+2\beta^2+2\gamma^2+2\beta\gamma+2\gamma\alpha+2\alpha\beta)$

$= \frac{1}{12}m[\alpha^2+\beta^2+\gamma^2+(\alpha+\beta+\gamma)^2] = \frac{1}{12}m(\alpha^2+\beta^2+\gamma^2+9h^2)$.

Ex. 40. *Show that a uniform triangular lamina of mass m is equimomental with three particles, each of mass $m/12$ placed at the angular points and a particle of mass $\frac{3}{4}m$ placed at the centre of inertia of the triangle.*

Sol. (Refer fig of Ex. 39).

If α, β, γ are the distances of the vertices A, B, C of triangle ABC from a line TK in its plane, then

M.I. of the triangle ABC about TK

$= \frac{1}{6}m(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta)$ (see Ex. 39)

The C.G. of the masses $m/12$ each at the points A, B, C and a particle of mass $\frac{3}{4}m$ placed at the centre of inertia of the triangle is the same point as the C.G. of the triangular lamina.

Also, sum of the masses of the four particles.

$= \frac{1}{12}m + \frac{1}{12}m + \frac{1}{12}m + \frac{3}{4}m =$ mass of the ΔABC.

and M.I. of the four particles about the line TK

$= \frac{1}{12}m \cdot AL^2 + \frac{1}{12}m \cdot BM^2 + \frac{1}{12}m \cdot CN^2 + \frac{3}{4}mh^2$

$= \frac{1}{12}m(\alpha^2+\beta^2+\gamma^2+9h^2)$

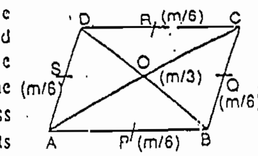
$= \frac{1}{6}m(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta)$, $\because h = \frac{1}{3}(\alpha+\beta+\gamma)$

$=$ M.I. of the ΔABC about the line TK.

Hence the triangular lamina and the four particles are equimomental.

Ex. 41. *$ABCD$ is a uniform parallelogram of mass M. At the middle points of the four sides are placed particles each of mass $M/6$ and at the intersection of the diagonals a particle of mass $M/3$, show that these five particles and the parallelogram are equimomental systems.*

Sol. Let $ABCD$ be a uniform parallelogram, of mass M, and P, Q, R, S the middle points of its sides.



Then mass of

$\Delta ABD =$ mass of $\Delta BCD = M/2$.

Now the ΔABD is equimomental to three particles each of mass equal to one third the mass of the triangle ABD, i.e. ΔABD is equimomental to three particle each of mass $\frac{1}{3}(\frac{1}{2}M) = \frac{1}{6}M$ at its the middle points P, O and S of its sides.

Similarly the ΔBCD is equimomental to three particles each of mass $\frac{1}{3}(\frac{1}{2}M) = \frac{1}{6}M$ at the middle points Q, R and O of its sides.

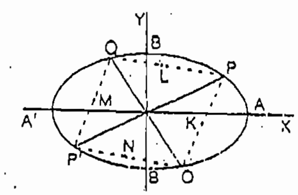
Hence the parallelogram $ABCD$ of mass M is equimomental to the particles each of mass $M/6$ at the middle points P, Q, R, S of the sides and particle of mass $\frac{1}{6}M + \frac{1}{6}M = \frac{1}{3}M$ at O (the point of intersection of the diagonals).

Ex. 42. *Particles each equal to one-quarter of the mass of an elliptic area are placed at the middle points of the chords joining the extremities of any pair of conjugate diameters. Prove that these four particles are equimomental to the elliptic area*

Sol. Let POP' and QOQ' be the conjugate diameters of an elliptic area of mass m. If ϕ is the eccentric angle of P then eccentric angle of Q is $(\pi/2+\phi)$.

$\therefore$ Coordinates of P are $(a\cos\phi, b\sin\phi)$ and coordinates of Q are $(a\cos(\phi+\pi/2), b\sin(\phi+\pi/2))$ or $(-a\sin\phi, b\cos\phi)$.

$\therefore$ Coordinates of P' are $(-a\cos\phi, -b\sin\phi)$ and that of Q' are $(a\sin\phi, -b\cos\phi)$.



If $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ are the coordinates of the middle points L, M, N, K of Chords $PQ, OP', P'Q', Q'P$ respectively, then

$x_1 = \frac{1}{2}a(\cos\phi - \sin\phi), y_1 = \frac{1}{2}b(\sin\phi+\cos\phi)$

$x_2 = -\frac{1}{2}c(\cos\phi+\sin\phi), y^2 = \frac{1}{2}b(\cos\phi - \sin\phi)$

$x_3 = \frac{1}{2}a(\sin\phi - \cos\phi), y_3 = -\frac{1}{2}b(\sin\phi+\cos\phi)$

and $x_4 = \frac{1}{2}a(\sin\phi+\cos\phi), y_4 = \frac{1}{2}b(\sin\phi - \cos\phi)$.

If $(\bar{x}, \bar{y})$ are the coordinates of four particles each of mass $m/4$ at L, M, N, K then

$\bar{x} = \frac{1}{4}(x_1 + x_2 + x_3 + x_4) = 0$ and $\bar{y} = \frac{1}{4}(y_1 + y_2 + y_3 + y_4) = 0$.

i.e. C.G. of the four particles is at O which is also the C.G. of the elliptic lamina.

Also M.I. of the four particles at L, M, N, K, about the major axis

$$= \frac{m}{4}(y_1^2 + y_2^2 + y_3^2 + y_4^2)$$

$$= \frac{m}{4} \cdot \frac{1}{4} b^2 [(\sin\phi + \cos\phi)^2 + (\cos\phi - \sin\phi)^2 + (\sin\phi + \cos\phi)^2 + (\sin\phi - \cos\phi)^2]$$

$= \frac{1}{4} mb^2$ = M.I. of the elliptic area about major axis.

Similarly M.I. of the four particles at L, M, N, K about the minor axis

$= \frac{1}{4} ma^2$ = M.I. of the elliptic area about minor axis.

and P.I. of the four particles at L, M, N, K about OX, OY

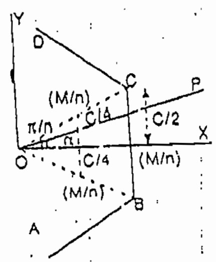
$$= \frac{1}{4} m (x_1 y_1 - x_2 y_2 + x_3 y_3 + x_4 y_4) = 0$$

= P.I. of the elliptic area about OX and OY.

Thus the four particles each of mass $m/4$ at L, M, N, K have the same mass, same C.G. and the same principal moments as that of the elliptic area. Hence the particles are equimomental to the elliptic area.

Ex. 43. *Show that the M.I. of a regular polygon of n sides about any straight line through its centre is* $\frac{Mc^2}{24} \cdot \frac{2 + \cos(2\pi/n)}{1 - \cos(2\pi/n)}$, *where n is the number of sides and c is the length of each side.*

Sol. Let $ABCD......A$ be a regular polygon of n sides each of length c. Let O be the centre of the polygon and lines OX (bisecting BC) and OY (perpendicular to OX) be taken in its plane as the axes of X and Y respectively.



If M is the mass of the polygon then it can be divided into n isosceles triangles each of mass M/n.

$\therefore$ mass of isosceles triangle $OBC = M/n$.

Also $\angle BOX = \angle COX = \frac{1}{2} \angle BOC$

$$= \frac{1}{2}(2\pi/n) = \pi/n.$$

Now the triangle OBC is equimomental to three particles each of mass $\frac{1}{3}(M/n)$ at the middle points of its sides.

$\therefore$ M.I. of the triangle OBC, about OX

$$= \frac{M}{3n} \cdot 0 + \frac{M}{3n} \cdot \left(\frac{c}{4}\right)^2 + \frac{M}{3n}\left(\frac{c}{4}\right)^2 = \frac{Mc^2}{24n} = A_1$$

$\therefore$ M.I. of the triangle OBC, about OY

$$= \frac{M}{3n}\left(\tfrac{1}{2}c\cot\frac{\pi}{n}\right)^2 + \frac{M}{3n}\left(\tfrac{1}{2}c\cot\frac{\pi}{n}\right)^2 + \frac{M}{3n}\left(\tfrac{1}{2}c\cot\frac{\pi}{n}\right)^2$$

$$= \frac{Mc^2}{8n} \cdot \cot^2\frac{\pi}{n} = B_1 \qquad \because\ OL = \frac{c}{2}\cot\frac{\pi}{n}$$

and P.I. of the triangle OBC about OX and OY

$= O = F_1$ $\qquad$ ($\because$ ΔOBC is symmetrical about OX)

Let OP be a line inclined at an angle α to OX, then M.I. of ΔOBC about OP

$$= A_1\cos^2\alpha + B_1\sin^2\alpha - 2F_1\sin 2\alpha$$

$$= \left(\frac{M}{24n}c^2\right)\cos^2\alpha + \left(\frac{Mc^2}{8n}\cot^2\frac{\pi}{n}\right)\sin^2\alpha \qquad ...(1)$$

The M.I. of the other triangles about OP are obtained by replacing α by $\alpha + 2\pi/n, \alpha + 4\pi/n, ...$ in (1), successively, then M.I. of the polygon about OP

$$= \frac{Mc^2}{24n}\left[\cos^2\alpha + \cos^2(\alpha + 2\pi/n) + \cos^2(\alpha + 4\pi/n) + ...\ n \text{ terms}\right]$$
$$+ \frac{Mc^2}{8n}\cot^2\frac{\pi}{n}\left[\sin^2\alpha + \sin^2\left(\alpha + \frac{2\pi}{n}\right) + \sin^2(\alpha + 4\pi/n) + ...\ n \text{ terms}\right]$$
$$= \frac{Mc^2}{24n} \cdot \frac{1}{2}\left[\{1 + \cos 2\alpha\} + \left\{1 + \cos\left(2\alpha + \frac{4\pi}{n}\right)\right\} + ...\ n \text{ terms}\right]$$
$$+ \frac{Mc^2}{8n} \cdot \cot^2\frac{\pi}{n} \cdot \frac{1}{2}\left[\{1 - \cos 2\alpha\} + \left\{1 - \cos\left(2\alpha + \frac{4\pi}{n}\right)\right\} + ...\ n \text{ terms}\right]$$

$$= \frac{Mc^2}{48n}[n + S] + \frac{Mc^2}{16n}\cot^2\frac{\pi}{n}[n - S] \qquad ...(2)$$

where $S = \cos 2\alpha + \cos(2\alpha + 4\pi/n) + \cos(2\alpha + 6\pi/n) + ...\ n$ terms

$$= \frac{\cos\{2\alpha + (n-1)2\pi/n\} \cdot \sin 2\pi}{\sin(2\pi/n)} = 0.$$

$\therefore$ M.I. of the polygon about OP

$$= \frac{Mc^2}{48n} \cdot n + \frac{Mc^2}{16n} \cdot \left(\cot^2\frac{\pi}{n}\right) \cdot n$$

$$= \frac{Mc^2}{48}\left[\frac{\sin^2(\pi/n) + 3\cos^2(\pi/n)}{\sin^2(\pi/n)}\right]$$

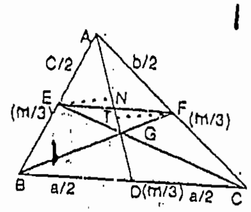
$$= \frac{Mc^2}{48}\left[\frac{(1-\cos)(2\pi/n)+3(1+\cos)(2\pi/n)}{1-\cos(2\pi/n)}\right]$$

$$= \frac{Mc^2}{24}\cdot\frac{2+\cos(2\pi/n)}{1-\cos(2\pi/n)}.$$

Ex. 44. *Show that there is a momental ellipse at the centre of inertia of a uniform triangle which touches the sides of the triangle at the middle points.*

Sol. Let ABC be a triangle of mass M. Let G be its C.G. and D, E, F the middle points of its sides.

Now the momental ellipse at the centre of inertia G will pass through D, E and F if the moments of inertia of the triangle ABC about GD, GE and GF are equal to $\frac{Mk^4}{GD^2}, \frac{Mk^4}{GE^2}$ and $\frac{Mk^4}{GF^2}$ respectively.



Let the ΔABC be replaced by three particles each of mass $\frac{1}{3}M$ placed at the middle points D, E, F.

Then M.I. of the triangle ABC about AD

$= (M/3) \cdot EN^2 + (M/3)\, FT^2 = \frac{1}{3}M\,[(\frac{1}{2}c \sin BAD)^2 + (\frac{1}{2}b \sin CAD)^2]$

$= \frac{1}{12}M\,[c^2 \sin^2 BAD + b^2 \sin^2 CAD]$ ...(1)

But in triangles BAD and CAD, we have

$$\frac{\sin BAD}{a/2} = \frac{\sin B}{AD} \text{ and } \frac{\sin CAD}{a/2} = \frac{\sin C}{AD}$$

$\therefore \sin BAD = \frac{a}{2}\cdot\frac{\sin B}{AD}$ and $\sin CAD = \frac{a}{2}\cdot\frac{\sin C}{AD}$.

$\therefore$ from (1), we have

M.I. of the ΔABC about AD

$$= \frac{1}{12}M\left[\tfrac{1}{4}a^2c^2\sin^2 B + \tfrac{1}{4}a^2b^2\sin^2 C\right]\cdot\frac{1}{AD^2}$$

$$= \frac{1}{12}M(\Delta^2+\Delta^2)\frac{1}{AD^2} = \left(\frac{M\Delta^2}{6}\right)\cdot\frac{1}{AD^2}$$

$$= \left(\frac{M\Delta^2}{54}\right)\cdot\frac{1}{GD^2} \qquad \because GD = \tfrac{1}{3}AD$$

Similarly M.I. of the triangle about $GE = \left(\frac{M\Delta^2}{54}\right)\cdot\frac{1}{GE^2}$

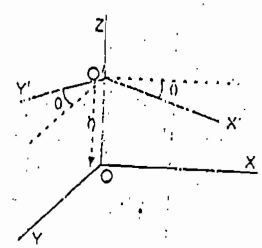
and about $GF = \left(\frac{M\Delta^2}{54}\right)\cdot\frac{1}{GF^2}$.

Thus the momental ellipse at G will pass through P, Q and R. Also GD is the diameter of the ellipse and bisect EF; $\therefore$ the tangent at P will be parallel to EF which is parallel to BC. Hence BC is tangent to the momental ellipse at P. Similarly the sides CA and AB are tangents to the momental ellipse at E and F respectively.

§ 1.25. Principal Axes.

To find whether a given straight line is at any point of its length a principal axis of a material system. And if the line is a principal axis, then to determine the other two principal axes.

Let the given straight line OZ be taken as the axis of z and a point O on it as the origin. Let the two perpendicular lines OX and OY, perpendicular to OZ be taken as the axes of x and y respectively.



Now let the line OZ be the principal axis of the system at O', where $OO' = h$. Let $O'X'$ inclined at an angle θ to a line parallel to OX and $O'Y'$ be the other two principal axes.

Consider a particle of mass m at the point P of the material system. Let (x, y, z) and (x', y', z') be the coordinates of the point P with reference to the two sets of axes OX, OY, OZ and $O'X', O'Y', O'Z'$ respectively, then we have

$x' = x\cos\theta + y\sin\theta,\ y' = -x\sin\theta + y\cos\theta,\ z' = z - h$

We know that the necessary and sufficient conditions for the axes $O'X', O'Y', O'Z'$ to be the principal axes of the system are that the products of inertia of the system with reference to these axes taken two at a time vanish *i.e.*

$\Sigma my'z' = 0,\ \Sigma mz'x' = 0$ and $\Sigma mx'y' = 0$.

We have, $\Sigma my'z' = \Sigma m(-x\sin\theta + y\cos\theta)(z-h)$

$= (\Sigma myz)\cos\theta - (\Sigma mxz)\sin\theta + h(\Sigma mx)\sin\theta - h(\Sigma my)\cos\theta$

$= D\cos\theta - E\sin\theta + Mh(\bar{x}\sin\theta - \bar{y}\cos\theta)$ ...(1)

$\because \bar{x} = \frac{\Sigma mx}{\Sigma m} = \frac{\Sigma mx}{M},\ \bar{y} = \frac{\Sigma my}{M}$.

$\Sigma mz'x' = \Sigma m(z-h)(x\cos\theta + y\sin\theta)$

$= (\Sigma yz)\sin\theta + (\Sigma mzx)\cos\theta - h(\Sigma mx)\cos\theta - h(\Sigma my)\sin\theta$

$= D\sin\theta + E\cos\theta - Mh(\bar{x}\cos\theta + \bar{y}\sin\theta)$ ...(2)

and $\Sigma mx'y' = \Sigma m(x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta)$

$= [(\Sigma my^2) - (\Sigma mx^2)]\sin\theta\cos\theta + (\Sigma mxy)(\cos^2\theta - \sin^2\theta)$

$= \frac{1}{2}\{\Sigma m(y^2+z^2) - \Sigma m(x^2+y^2)\}\sin 2\theta + (\Sigma mxy)\cos 2\theta$

$= \frac{1}{2}(A - B)\sin 2\theta + F\cos 2\theta$ ...(3)

Now $\Sigma mx'y' = 0$, if $\frac{1}{2}(A-B)\sin 2\theta + F\cos 2\theta = 0$

or $\tan 2\theta = \frac{2F}{B-A}$ or $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2F}{B-A}\right)$ ...(4)

Also $\Sigma my'z' = 0$, and $\Sigma mz'x' = 0$, if

$D\cos\theta - E\sin\theta + Mh(\bar{x}\sin\theta - \bar{y}\cos\theta) = 0$

and $D\sin\theta + E\cos\theta - Mh(\bar{x}\cos\theta + \bar{y}\sin\theta) = 0$

$\therefore Mh = \frac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta}; \frac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$

Thus $Mh = \frac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \frac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$

$= \frac{(E\sin\theta - D\cos\theta)\sin\theta + (D\sin\theta + E\cos\theta)\cos\theta}{(\bar{x}\sin\theta - \bar{y}\cos\theta)\sin\theta + (\bar{x}\cos\theta + \bar{y}\sin\theta)\cos\theta} = \frac{E}{\bar{x}}$

Also $Mh = \frac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \frac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$

$= \frac{(E\sin\theta - D\cos\theta)(-\cos\theta) + (D\sin\theta + E\cos\theta)\sin\theta}{(\bar{x}\sin\theta - \bar{y}\cos\theta)(-\cos\theta) + (\bar{x}\cos\theta + \bar{y}\sin\theta)\sin\theta} = \frac{D}{\bar{y}}$

$\therefore Mh = \frac{E}{\bar{x}} = \frac{D}{\bar{y}}$. ...(5)

Thus the condition that the axis OZ may be the principal axis of the system at some point of its length is that

$\frac{E}{\bar{x}} = \frac{D}{\bar{y}}$ ...(6)

And if condition (6) is satisfied then the point O' where the line OZ is the principal axis is given by

$OO' = h = \frac{E}{M\bar{x}} = \frac{D}{M\bar{y}}$ ...(7)

Cor. 1. *If an axis passes through the C.G. of a body and is a principal axis at any point of its lenght, then it is a principal axis at all points of its length.*

Let z axis be a principal axis at O, then $D = E = 0$. $\therefore$ from (7), we get $h = 0$. Which implies that there is no such other point as O'. But if z-axis is a principal axis at O and passes through the C.G. of the body then $\bar{x} = 0, \bar{y} = 0$ and $D = E = 0$, and from (7), we see that h becomes indeterminate.

Hence if an axis passes through the C.G. of a body and is a principal axis at any point of its length, then it is a principal axis at all points of its length.

Cor. 2. *Through each point in the plane of a lamina, there exist a pair of principal axes of the lamina.*

Let a line through any point O of the lamina and perpendicular to its plane be taken as the axis of z. In this case $\bar{z}$ (z coordinate of the C. G. of the body) $= 0$. $\therefore D = 0 = E$. Thus eq. (6) is satisfied for every point O in the plane of the lamina. Also from (7), $h = 0$.

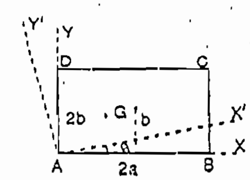
Thus z-axis (the line perpendicular to the plane of the lamina) is a principal axis of the lamina at the point O where it intersects the lamina and the other-two principal axes will be the axes through O in the place of the lamina.

EXAMPLES

Ex. 45. (a). *The lengths AB and AD of the sides of a rectangle ABCD are 2a and 2b: show that the inclination to AB of one of the principal axes at A is* $\frac{1}{2}\tan^{-1}\frac{3ab}{2(a^2 - b^2)}$.

(b) *Find the principal axes at a corner of a square.*

Sol. (a) Let AB and AD be taken as the axes of x and y respectively and z axis, a line through the corner A and perpendicular to the plane of the rectangle.



Then A = M.I. of the rectangle about AB

= M.I. of the rectangle about the axis parallel to AB through C.G. 'G'

+ M.I. of whole mass M at G about AB.

$= \frac{1}{3}Mb^2 + Mb^2 = \frac{4}{3}Mb^2$.

Similarly B = M.I. of the rectangle about AD

$= \frac{1}{3}Ma^2 + Ma^2 = \frac{4}{3}Ma^2$.

and F = P.I. of the rectangle about AB and AD

= P.I. of the rectangle about axes parallel to AX, AY through C.G. 'G'

+ P.I. of whole mass M at G about AB and AD

$= 0 + M.a.b = Mab$

If the principal axis at A is inclind at an angle θ to AB, then

$$\tan 2\theta = \frac{2F}{B-A} = \frac{2\,Mab}{\frac{4}{3}M(a^2-b^2)} = \frac{3ab}{2(a^2-b^2)}$$

$$\therefore \theta = \tfrac{1}{2}\tan^{-1}\frac{3ab}{2(a^2-b^2)}.$$

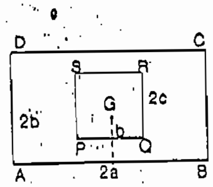
(b) Proceed as in (a). Here $2b = 2a$

$$\therefore \theta = \tfrac{1}{2}\tan^{-1}\infty = \frac{\pi}{4}.$$

Ex. 46. *A uniform rectangular plate whose sides are of lengths 2a, 2b has a portion cut out in the form of a square whose centre is the centre of the rectangle and whose mass is half the mass of the plate. Show that the axes of greatest and least M.I. at a corner of the rectangle make angles θ, $\frac{1}{2}\pi+\theta$ with a side, where*

$$\tan 2\theta = \frac{6}{5}\cdot\frac{ab}{a^2-b^2}.$$

Sol. Let M be the mass of the rectangle $ABCD$ of sides $AB = 2a$, $AD = 2b$ and let $2c$ be the side of the square $PQRS$ cut out from it with its centre at the centre of the rectangle such that the mass of square $= \frac{1}{2}M$.



A = M.I. of the remaining portion about AB

= M.I. of the rectangle about AB − M.I. of the square about AB

$= (\frac{1}{3}Mb^2 + Mb^2) - [\frac{1}{3}(\frac{1}{2}M)c^2 + (\frac{1}{2}M)b^2] = \frac{1}{6}M(5b^2 - c^2)$

Similarly

B = M.I. of the remaining portion about $AD = \frac{1}{6}M(5a^2 - c^2)$,

F = P.I. of the remaining portion about AB and AD

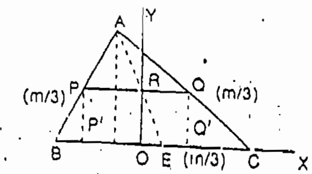
$= (0 + Mab) - (0 + \frac{1}{2}Mab) = \frac{1}{2}Mab.$

$\therefore$ If the principal axes in the plane of the rectangle at O make angles θ and $\frac{1}{2}\pi + \theta$ to the sides AB, then

$$\tan 2\theta = \frac{2F}{B-A} = \frac{Mab}{\frac{1}{6}M(5a^2-5b^2)} = \frac{6}{5}\cdot\frac{ab}{a^2-b^2}.$$

Ex. 47. *ABC is a triangular area and AD is perpendicular to BC and AE is a median, O is the middle point of DE, show that BC is a principal axis of the triangle at O.*

Sol. Let O be the middle point of DE where AD and AE are the perpendiculars from A on BC and the median respectively. Let the lines OX and OY along and perpendicular to BC be taken as the axes of reference.



Let P and Q be the middle points of AB and AC respectively the PQ is parallel to BC and is bisected at the point R where the median AE meets OY.

If m is the mass of the ΔABC then it can be replaced by three particles each of mass m/3 at the middle points m/3 at the middle points E.P. Q of the sides of the triangle.

P.I. of the ΔABC about OX and OY

= P.I. of masses m/3 each at E, P and Q about OX and OY

$= \frac{m}{3}\cdot OE.\,0 + \frac{m}{3}\cdot OQ'.\,QQ' + \frac{m}{3}(-OP').\,PP'$

$= (m/3)\,\frac{1}{2}PQ\,(QQ' - PP')$ $\quad\because OP' = OQ' = \frac{1}{2}PQ$

$= 0.$ $\quad\because PP' = QQ'.$

Thus the P.I. of the triangle vanishes about BC and perpendicular to BC at O. Hence BC is the principal axis of the triangle ABC at O.

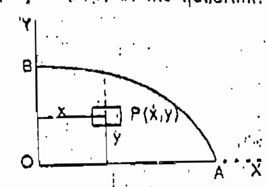
Ex. 48. *Show that at the centre of a quadrant of an ellipse, the principle axis in its plane are inclined at an angle $\frac{1}{2}\tan^{-1}\left(\frac{4}{\pi}\cdot\frac{ab}{a^2-b^2}\right)$ to the axis.*

[Meerut TDC 92, 93(BP)]

Sol. Let OAB be the quadrant of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let $\delta x\,\delta y$ be an elementary area at the point $P(x, y)$ of the quadrant. Then A = M.I. of the quadrant about OX



$$= \int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{(a^2-x^2)}} \rho y^2\,dx\,dy$$

$$= \frac{\rho b^3}{3a^3}\int_0^a (a^2 - x^2)^{3/2}\,dx \quad (\text{Put } x = a\sin\theta)$$

$$= \frac{\rho b^3}{3a^3}a^4\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{\rho b^3 a^4}{3a^3}\cdot\frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{1}{2})}{2\,\Gamma(3)}$$

$= \frac{1}{16}\rho\pi b^3 a = \frac{1}{4} Mb^2$. $\qquad \because M$ (mass of quadrant) $= \frac{\rho}{4}\pi ab$.

B = M.I. of the quadrant about OY

$= \int_{x=0}^{a}\int_{y=0}^{(b/a)\sqrt{(a^2-x^2)}} \rho x^2\, dx\, dy = \rho\frac{b}{a}\int_0^a x^2\sqrt{(a^2-x^2)}\, dx$ $\qquad$ (Put $x = a\sin\theta$)

$= \frac{1}{4} Ma^2$.

F = P.I. of the quadrant about OX and OY

$= \int_{x=0}^{a}\int_{y=0}^{(b/a)\sqrt{(a^2-x^2)}} \rho xy\, dx\, dy = \frac{1}{2}\rho\frac{b^2}{a^2}\int_0^a x(a^2-x^2)\, dx = \frac{Mab}{2\pi}$

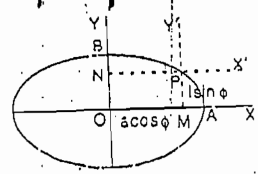
If the principal axes are inclined at an angle θ to OX and OY, then

$\tan 2\theta = \frac{2F}{B-A} = \frac{4ab}{\pi(a^2-b^2)}$, $\quad \therefore \quad \theta = \frac{1}{2}\tan^{-1}\left(\frac{4}{\pi}\cdot\frac{ab}{a^2-b^2}\right)$.

Ex. 49. *Find the principal axes of an elliptic area at any point of its bounding arc.*

Sol. Let $P(a\cos\phi, b\sin\phi)$ be a point on the arc of an elliptic area bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Consider PX' and PY' axes parallel to the axes of the ellipse. Then

A = M.I. of the elliptic area about PX'

$= \frac{1}{4}Mb^2 + M(PM)^2$

$= M(\frac{1}{4}b^2 + b^2\sin^2\phi)$.

B = M.I. of the elliptic area about PY'

$= \frac{1}{4}Ma^2 + M(PN)^2 = M(\frac{1}{4}a^2 + a^2\cos^2\theta)$.

and F = P.I. of the elliptic area about PX' and PY'

$= 0 + M.PM.PN = M\,ab\cos\phi\sin\phi$.

$\therefore$ If the principal axes at P make an angle θ with OX and OY then

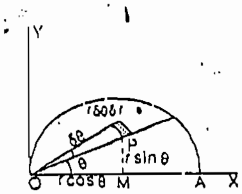
$$\tan 2\theta = \frac{2F}{B-A} = \frac{2M\,ab\cos\phi\sin\phi}{M(\frac{1}{4}a^2+a^2\cos^2\phi) - M(\frac{1}{4}b^2+b^2\sin^2\phi)}$$

$$\therefore \theta = \frac{1}{2}\tan^{-1}\left[\frac{8\,ab\tan\phi}{(a^2-b^2)\sec^2\phi + 4a^2 - 4b^2\tan^2\phi}\right]$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{8\,ab\tan\phi}{(5a^2-b^2)+(a^2-5b^2)\tan^2\phi}\right].$$

Ex. 50. *Show that at an extremity of the bounding diameter of a semi-circular lamina the principal axis makes an angle $\frac{1}{2}\tan^{-1}(8/3\pi)$ to the diameter.*

Sol. Let the axis of x and y be taken along the diameter OA and perpendicular to OA at O in the palne of the lamina.



$\therefore$ Equation of the semi-circular lamina is $r = 2a\cos\theta$.

Let $\rho r\delta\theta\delta r$ be the mass of an elementary area at P.

$\therefore$ A = M.I. of the lamina about OX

$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}(r\sin\theta)^2 . \rho r d\theta dr$

$= \frac{1}{4}(2a)^4\rho\int_0^{\pi/2}\sin^2\theta\cos^4\theta\, d\theta$

$= 4\rho a^4\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{2\Gamma(4)} = \frac{1}{8}\pi\rho a^4$

B = M.I. of the lamina about OY

$= \int_0^{\pi/2}\int_0^{2a\cos\theta}(r\cos\theta)^2\rho r d\theta dr = \frac{1}{4}(2a)^4\rho\int_0^{\pi/2}\cos^6\theta\, d\theta$

$= 4\rho a^4 \cdot \frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{2\Gamma(4)} = \frac{5}{8}\pi\rho a^4$.

and F = P.I. of the lamina about OX and OY

$= \int_0^{\pi/2}\int_0^{2a\cos\theta}(r\cos\theta).(r\sin\theta).\rho r d\theta dr$

$= \frac{1}{4}\rho(2a)^4\int_0^{\pi/2}\cos^5\theta\sin\theta\, d\theta = \frac{2}{3}\rho a^4$.

$\therefore$ If the principal axis make an angle θ' to OX at O then

$\tan 2\theta' = \frac{2F}{B-A} = \frac{8}{3\pi}$

$\therefore \theta' = \frac{1}{2}\tan^{-1}\left(\frac{8}{3\pi}\right)$.

Ex. 51. *Show that the principal axes at the node of a half-loop of the lemniscate $r^2 = a^2\cos 2\theta$ are inclined to the initial line at angles*

$$\frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2}+\frac{1}{2}\tan^{-1}\frac{1}{2}.$$ [Meerut TDC 93, 93(P)]

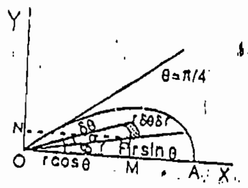
Sol. The equation of the lemniscate is

$r^2 = a^2 \cos 2\theta$.

Consider an element of area $r\,\delta\theta\,\delta r$ at $P(r, \theta)$.

δm = Mass of the elementary area $= \rho r\,\delta\theta\,\delta r$.

$\therefore$ A = M.I. of half loop of the lemniscate about OX



$$\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}} PM^2 \,.\, \rho r d\theta dr = \int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}} r^2 \sin^2\theta \,.\, \rho r d\theta dr$$

$$= \rho \int_0^{\pi/4}\left[\frac{1}{4} r^4\right]_0^{a\sqrt{(\cos 2\theta)}} \sin^2\theta\, d\theta = \tfrac{1}{4}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta . \sin^2\theta\, d\theta$$

$$= \frac{1}{8}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta\,(1 - \cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\rho a^4 \int_0^{\pi/2} (\cos^2 t - \cos^3 t)\, dt \qquad \text{Putting } 2\theta = t$$

$$= \frac{1}{16}\rho a^4 \left[\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{2\Gamma(2)} - \frac{\Gamma(2)\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{5}{2})}\right]$$

$$= \frac{1}{16}\rho a^4 \left(\frac{\pi}{4} - \frac{2}{3}\right) = \frac{\rho a^4}{192}(3\pi - 8)$$

B = M.I. of half loop of the lemniscate about OY

$$= \int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} PN^2 \,.\, \rho r d\theta dr = \int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2 \cos^2\theta \,.\, \rho r d\theta dr$$

$$= \tfrac{1}{4}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta \,.\, \cos^2\theta\, d\theta = \tfrac{1}{8}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta\,(1 + \cos 2\theta)\, d\theta$$

$$= \frac{\rho a^4}{192}(3\pi + 8), \text{ (As above)}$$

and F = P.I. of half loop of the lemniscate about OX, OY

$$= \int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} PM \,.\, PN \,.\, \rho r d\theta dr$$

$$= \int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r \sin\theta \,.\, r\cos\theta \,.\, \rho r d\theta dr$$

$$= \tfrac{1}{4}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta \,.\, \cos\theta \sin\theta\, d\theta = \tfrac{1}{8}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta \,.\, \sin 2\theta\, d\theta$$

$$= \tfrac{1}{8}\rho a^4 \left[-\tfrac{1}{6}\cos^3 2\theta\right]_0^{\pi/4} = \frac{1}{48}\rho a^4.$$

$\therefore$ If the principal axis at O make an angle ϕ to OX then

$$\phi = \tfrac{1}{2}\tan^{-1}\frac{2F}{B-A} = \tfrac{1}{2}\tan^{-1}\left\{\frac{8}{(3\pi+8)-(3\pi-8)}\right\} = \tfrac{1}{2}\tan^{-1}\tfrac{1}{2}$$

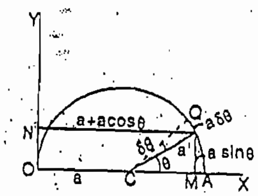
The other principal axis being at right angles to this principal axis will be inclined to OX at angle $\pi/2 + \frac{1}{2}\tan^{-1}\frac{1}{2}$.

Ex. 52. *A wire is in the form of a semi-circle of radius a. Show that at an end of its diameter the principal axes in its plane are inclined to the diameter at angles*

$$\frac{1}{2}\tan^{-1}\frac{4}{\pi} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{4}{\pi}.$$

Sol. Let C be the centre and OA the diameter of a semi-circular wire of radius a. Let the axis OX and OY be taken along and perpendicular to the diameter OA.

Consider an elementary arc $a\delta\theta$ at P, then its mass, $\delta m = \rho a \delta\theta$.



where $\rho = \dfrac{M}{(\pi a)}$

$\therefore$ A = M.I. of the wire about OX

$$= \int_0^{\pi} PM^2 \,.\, \rho a d\theta = \int_0^{\pi} a^2 \sin^2\theta \,.\, \rho a d\theta = \tfrac{1}{2}\rho a^3 \int_0^{\pi} (1 - \cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\rho a^3 \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi} = \tfrac{1}{2}\rho\pi a^3 = \tfrac{1}{2}Ma^2.$$

B = M.I. of the wire about OY

$$= \int_0^{\pi} PN^2 \,.\, \rho a d\theta = \int_0^{\pi} (a + a\cos\theta)^2 \,.\, \rho a d\theta$$

$$= \rho a^3 \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \rho a^3 \int_0^{\pi} \{1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\}\, d\theta$$

$$= \tfrac{1}{2}\rho a^3 \int_0^{\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta$$

$= \frac{1}{2} \rho a^3 \left[3\theta + 4 \sin\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi}$

$= \frac{3}{2}\pi \rho a^3 = \frac{3}{4} Ma^2.$

and F = P.I. of the wire about OX and OY

$= \int_0^{\pi} PM \cdot PN \cdot \rho a d\theta = \int_0^{\pi} a \sin\theta \cdot (a + a \cos\theta) \cdot \rho a d\theta$

$= \rho a^3 \int_0^{\pi} (\sin\theta + \frac{1}{2} \sin 2\theta)\, d\theta = \rho a^3 \left[-\cos\theta - \frac{1}{4} \cos 2\theta \right]_0^{\pi}$

$= 2\rho a^3 = \frac{2}{\pi} Ma^2.$

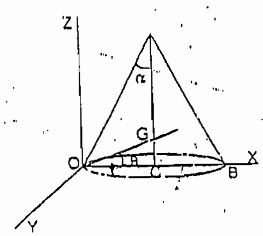
$\therefore$ If the principal axis at O make an angle θ to OX, then

$$\theta = \frac{1}{2} \tan^{-1} \frac{2F}{B-A} = \frac{1}{2} \tan^{-1} \left\{ \frac{\frac{4}{\pi} Ma^2}{(\frac{3}{4} - \frac{1}{2}) Ma^2} \right\} = \frac{1}{2} \tan^{-1} \frac{4}{\pi}.$$

The other principal axis being at right angles to this principal axis will be inclined to OX at angle $\frac{\pi}{2} + \frac{1}{2} \tan^{-1} \frac{4}{\pi}$.

Ex. 53. *Find the principal axes of a right circular cone at a point on the circumference of the base, and show that one of them will pass through its C.G. if the vertical angle of the cone is $2 \tan^{-1} \frac{1}{2}$.*

Sol. Let O be a point on the circumference of the base of a right circular cone of mass M, height h and semi-vertical angle α. Take the axis OX along the diameter OB of the base, axis OY perpendicular to OB and in the plane of the base and axis OZ perpendicular to the base of the cone.



Then from Ex. 36 on page 51, we have

A = M.I. of the cone about OX

$= \frac{M}{20}(3a^2 + 2h^2)$

B = M.I. of the cone about

$OY = \frac{M}{20}(23a^2 + 2h^2)$

C = M.I. of the cone about $OZ = \frac{13}{10} Ma^2$

D = P.I. about $OY, OZ = 0$,
E = P.I. about $OZ, OX = \frac{1}{4} Mah$, and
F = P.I. about $OX, OY = 0$.
Here $D = 0$ and $F = 0$, therefore the axis OY will be the principal axis at O. Other two principal axes will be in the xz plane. If one of these principal axes is inclined at an angle θ to OX in xz plane, then

$$\tan 2\theta = \frac{2E}{C-A} = \frac{\frac{1}{2} Mah}{\frac{13}{10} Ma^2 - \frac{M}{20}(3a^2 + 2h^2)} = \frac{10ah}{23a^2 - 2h^2} \quad ...(1)$$

The other principal axis will be perpendicular to this principal axis in xz plane.
2nd Part. If one of the principal axis pass through the C.G. 'G' of the cone, then

$$\tan\theta = \frac{CG}{OC} = \frac{h}{4a}.$$

$$\therefore \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{8ah}{16a^2 - h^2}. \quad ...(2)$$

$\therefore$ From (1) and (2), we have

$$\frac{10ah}{23a^2 - 2h^2} = \frac{8ah}{16a^2 - h^2}.$$

or $5(16a^2 - h^2) = 4(23a^2 - 2h^2)$
or $3h^2 = 12a^2$ or $h = 2a$.

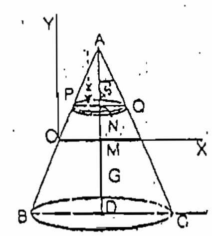
$\therefore \tan\alpha = \frac{OC}{AC} = \frac{a}{h} = \frac{1}{2}.$

i.e. verticle angle of the cone $= 2\alpha = 2 \tan^{-1} \frac{1}{2}$.

Ex. 54. *If the vertical angle of the cone is 90° the point at which a generator is a principal axis divides the generator in the ratio 3 : 7.*

Sol. Let h be the height of a cone of vertical angle 90°.

Let the generator AB be the principal axis of the cone at the point O. Consider the section of the cone through the generator AB and the axis AD. Take OX and OY, the axis of x and axis of y, perpendicular to AD and parallel to AD respectively in this section and OZ the z-axis perpendicular to this section of the cone.



Since the cone is symmetrical about OZ,
$\therefore D = O = E$.

$\therefore$ OZ is a principal axis at O. The other two principal axes at O are the generator AB and the line through O and perpendicular to generator AB in the above section of the cone.

Consider an elementary circular disc of width δx at a distance x from the vertex A and perpendicular to the axis AD, i.e. $AN = x$.

$\therefore$ Radius of the disc $= PN = x \tan 45° = x$.

Mass of the elementary disc, $\delta m = \rho\pi x^2 \delta x$.

M.I. of this disc about $OX = \frac{1}{4} PN^2 \delta m + MN^2 \delta m$

$= (\frac{1}{4} x^2 + (AM - x)^2) \rho\pi x^2 \delta x.$

$\therefore$ A = M.I. of the cone about OX

$= \int_0^h (\frac{1}{4}x^2 + (AM - x)^2) \rho\pi x^2 \, dx$

$= \pi\rho \int_0^h (\frac{5}{4}x^4 - 2AM \cdot x^3 + AM^2 \cdot x^2) \, dx$

$= \pi\rho \, [\frac{5}{4} \cdot \frac{1}{5}h^5 - 2AM \cdot \frac{1}{4}h^4 + AM^2 \cdot \frac{1}{3}h^3]$

$= \frac{1}{12} \pi\rho h^3 (3h^2 - 6h \cdot AM + 4AM^2).$

Also M.I. of the elementary disc about OY

$= \frac{1}{2} PN^2 \delta m + OM^2 \delta m = (\frac{1}{2} x^2 + AM^2) \pi\rho x^2 \delta x, \quad \because OM = AM$

$\therefore$ B = M.I. of the cone about OY

$= \int_0^h (\frac{1}{2}x^2 + AM^2) \pi\rho x^2 \, dx = \pi\rho \int_0^h (\frac{1}{2}x^4 + AM^2 x^2) \, dx$

$= \pi\rho \left(\frac{1}{10} h^4 + AM^2 \frac{1}{3} h^3 \right) = \frac{1}{30} \pi\rho h^3 (3h^2 + 10 AM^2).$

Since the principal axis AB make an angle $AOX = 45°$ to OX,

$\therefore$ From $\tan 2\theta = \frac{2F}{B - A}$, we have

$\tan 90° = \frac{2F}{B - A}$ or $\infty = \frac{2F}{B - A}$ or $B - A = 0$ or $A = B$.

$\therefore \frac{1}{12} \pi\rho h^3 (3h^2 - 6hAM + 4AM^2) = \frac{1}{30} \pi\rho h^3 (3h^2 + 10AM^2).$

or $\frac{1}{12} (3h^2 - 6hAM) = \frac{1}{30} \cdot 3h^2$

or $5(3h^2 - 6hAM) = 6h^2$ or $9h^2 = 30hAM$ or $AM = \frac{3}{10} h$.

From similar triangles AOM and ABD,

$\frac{AO}{AB} = \frac{AM}{AD} = \frac{AM}{h} = \frac{3}{10}$;

$\therefore AO = \frac{3}{10} AB$ and $OB = AB - AO = AB - \frac{3}{10} AB = \frac{7}{10} AB.$

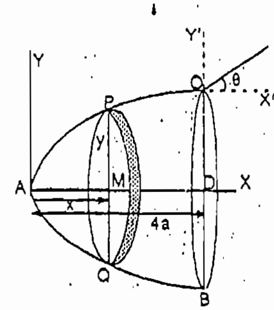
$\therefore \frac{AO}{OB} = \frac{3}{7}.$

Ex. 55. *The length of the axis of a solid parabola of revolution is equal to the latus-rectum of the generating parabola. Prove that one principal axis at a point in the circular rim meets the axis of revolution at an angle* $\frac{1}{2} \tan^{-1} \frac{2}{3}$.

Sol. Let the length of L.R. of the parabola be $4a$.

$\therefore$ Length of the axis $AD = 4a$, and equation of the parabola is

$y^2 = 4ax.$ ...(1)



Let O be a point in the circular rim and OX', OY' the axes parallel to AX and AY.

If the principal axis at O is inclined at an angle θ to OX' (i.e. to the axis of revolution AX), then

$\theta = \frac{1}{2} \tan^{-1} \frac{2F}{B - A}$ ...(1)

Consider an elementary strip PQ of width δx at a distance x from A and perpendicular to AX, then its mass

$\delta m = \rho\pi PM^2 \delta x = \rho\pi y^2 \delta x,$

where (x, y) are coordinates of the point P.

M.I. of this elementary disc about OX'

$= \frac{1}{4} PM^2 \delta m + OD^2 \delta m$

$= (\frac{1}{4} y^2 + OD^2) \rho\pi y^2 \delta x$

$= (\frac{1}{4} y^2 + (4a)^2) \rho\pi y^2 \delta x$

$(\because$ at O, $x = AD = 4a$, $y = OD$ $\therefore OD^2 = 4a \cdot 4a$ or $OD = 4a)$

$= (2ax + 16a^2) 4\rho\pi a x \delta x$ $\quad y^2 = 4ax.$

A = M.I. of the solid about OX'

$= \int_0^{4a} (2ax + 16a^2) 4\rho\pi \cdot ax dx$

$$= 4\pi\rho a\left[2a\frac{x^3}{3} + 16a^2\frac{x^2}{2}\right]_0^{4a}$$

$$= \frac{64 \times 32}{3}\rho\pi a^5.$$

Also M.I. of the elementary disc about OY'

$= \frac{1}{4} PM^2\delta m + MD^2\delta m + [\frac{1}{4}y^2 + (4a-x)^2]\, \rho\pi y^2\delta x$

$= [ax + (4a-x)^2]\, 4\pi\rho a x\delta x = (16a^2 - 7ax + x^2)\, 4\pi\rho a x\delta x$

$\therefore B =$ M.I. of the solid about OY'

$$= \int_0^{4a} (16a^2 - 7ax + x^2)\, 4\pi\rho a x\,dx$$

$$= 4\pi\rho a\left[8a^2x^2 - \frac{7a}{3}x^3 + \frac{1}{4}x^4\right]_0^{4a} = \frac{1}{3} \times 64 \times 8\pi\rho a^5.$$

And P.I. of the elementary disc about OX', OY'

$= O + OD \cdot MD \cdot \delta m = 4a \cdot (4a - x)\, \rho\pi y^2\delta x$

$= 4a \cdot (4a-x)\, \rho\pi \cdot 4ax\delta x.$

$\therefore F =$ P.I. of the solid about OX', OY'

$$= \int_0^{4a} \rho\pi\, 16a^2 (4a - x)\, x\,dx$$

$$= 16\rho\pi a^2\left[2ax^2 - \frac{1}{3}x^3\right]_0^{4a} = \frac{1}{3} \times 16 \times 32\rho\pi a^5.$$

$\therefore$ From (1), we have

$$\theta = \frac{1}{2}\tan^{-1}\frac{\frac{2}{3} \times 16 \times 32\rho\pi a^5}{(\frac{1}{3} \times 64 \times 8 - \frac{1}{3} \times 64 \times 32)\,\rho\pi a^5} = \frac{1}{2}\tan^{-1}(\tfrac{2}{3}) \text{ numerically.}$$

Ex. 56. *A uniform lamina is bounded by a parabolic arc, of latus rectum $4a$, and a double ordinate at a distance b from the vertex. If $b = \frac{1}{3}a(7 + 4\sqrt{7})$, show that two of the principal axes at the end of a latus rectum are the tangent and normal there.*

Sol. Let the equation of the parabola be

$y^2 = 4ax$ ...(1)

$\therefore$ Coordinates of the end L of L.R. LL' are $(a, 2a)$.

Differentiating (1) we, get $\dfrac{dy}{dx} = \dfrac{2a}{y}$.

$\therefore$ At $L(a, 2a)$, $\dfrac{dy}{dx} = \dfrac{2a}{2a} = 1$.

$\therefore$ Equation of the tangent LT at L is

$y - 2a = 1 \cdot (x - a)$ or $y - x - a = 0$ ...(2)

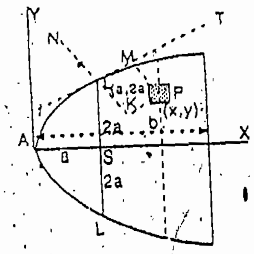
and the equation of the normal LN at L is

$y - 2a = -\frac{1}{1}(x - a)$ ...(3)

or $y + x - 3a = 0$.

Consider an element $\delta x\delta y$ at the point $P(x, y)$ of the lamina, then

PM = length of perpendicular from P on tangent LT given by (2)



$$\frac{y - x - a}{\sqrt{(1+1)}} = \frac{y - x - a}{\sqrt{2}}$$

and PK = length of perpendicular from P on the normal LN given by (3)

$$= \frac{y + x - 3a}{\sqrt{2}}.$$

P.I. of the element about LT and LN

$$PM \cdot PK \cdot \delta m = \left(\frac{y - x - a}{\sqrt{2}}\right)\left(\frac{y + x - 3a}{\sqrt{2}}\right)\rho\delta x\delta y$$

If the tangent and normal at L are the principal axes, then the P.I. of the lamina about these will be zero.

i.e. P.I. of the lamina about LT and LN

$$= \int_{x=0}^{b}\int_{y=-2\sqrt{(ax)}}^{2\sqrt{(ax)}}\left(\frac{y - x - a}{\sqrt{2}}\right)\left(\frac{y + x - 3a}{\sqrt{2}}\right)\cdot\rho\,dx\,dy = 0$$

or $$\frac{\rho}{2}\int_0^b\int_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}} (y^2 - 4ay + (3a^2 + 2ax - x^2))\,dx\,dy = 0$$

or $$\int_0^b\left\{\frac{1}{3}y^3 - 2ay^2 + (3a^2 + 2ax - x^2)\,y\right\}_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}} dx = 0$$

or $$2\int_0^b\left\{\frac{8}{3}ax\sqrt{(ax)} + 2(3a^2 + 2ax - x^2)\sqrt{(ax)}\right\}dx = 0$$

or $$\int_0^b (\tfrac{8}{3}a^{3/2}x^{3/2} + 6a^{5/2}x^{1/2} + 4a^{3/2}x^{3/2} - 2a^{1/2}x^{5/2})\,dx = 0$$

or $$\left[\frac{16}{15}a^{3/2}b^{5/2} + 4a^{5/2}b^{3/2} + \tfrac{8}{5}a^{3/2}b^{5/2} - \tfrac{4}{7}a^{1/2}b^{7/2}\right] = 0$$

or $\frac{16}{15}ab + 4a^2 + \frac{8}{5}ab - \frac{4}{7}b^2 = 0$.

or $b^2 - \frac{14}{3}ab - 7a^2 = 0$

$$\text{or } b = \frac{\frac{14}{3}a \pm \sqrt{\left\{\frac{196}{9}a^2 + 28a^2\right\}}}{2} = \frac{1}{2}\left(\frac{14}{3} \pm \frac{8}{3}\sqrt{7}\right)a$$

or $b = \frac{a}{3}(7 + 4\sqrt{7})$. Leaving – ve sign, as b can not be negative.

Hence if $b = \frac{a}{3}(7 + 4\sqrt{7})$,

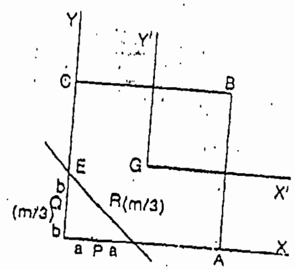
then the principal axes at L are the tangents and normal there.

Ex. 57. *A uniform square lamina is bounded by the axes of x and y and the lines $x = 2c, y = 2c$, and a corner is cut off by the line $x/a + y/b = 2$. Show that the principal axes at the centre of the square are inclined to the axis of x at angles given by*

$$\tan 2\theta = \frac{ab - 2(a+b)c + 3c^2}{(c-b)(a+b-2c)}$$

Sol. Let $OABC$ be the square lamina of mass M bounded by the axes and the lines $x = 2c, y = 2c$.

The line $\frac{x}{a} + \frac{y}{b} = 2$ i.e. $\frac{x}{2a} + \frac{y}{2b} = 1$, cut off intercepts $OD = 2a$ and $OE = 2b$ on the axes. Let m be the mass of the triangular lamina ODE cut off from the square. The triangle ODE can be replaced by three particles each of mass $m/3$ at the middle points P, Q, R of its sides.



Consider the lines GX', GY' through G and parallel the sides of the square as the new axes of reference. With reference to these new axes the coordinates of P are $[-(c-a), -c]$, Q are $[-c, -(c-b)]$, R are $[-(c-a), -(c-b)]$.

$\therefore A =$ M.I. of the remaining area about GX'

= M.I. of square $OABC$ about GX' – M.I. of ΔODE about GX'

= M.I. of square $OABC$ about GX' —(M.I. of three particles each of mass $m/3$ at P, Q and R)

$$= \frac{1}{3}Mc^2 - \frac{m}{3}\left[c^2 + (c-b)^2 + (c-b)^2\right]$$

Similarly

$B =$ M.I. of the remaining area about GY'

$$= \frac{1}{3}Mc^2 - \frac{m}{3}\left[(c-a)^2 + c^2)^2 + (c-a)^2\right]$$

and $F =$ P.I. of the remaining area about GX', GY'

= P.I. of the square $OABC$ about GX', GY'

–(P.I. of three particles each of mass $m/3$ at P, Q and R)

$$= 0 - \frac{m}{3}\left[(c-a)c + c(c-b) + (c-a)(c-b)\right].$$

$$= -\frac{m}{3}\left[(ab - 2(a+b)c + 3c^2\right].$$

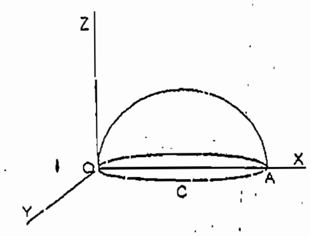
$\therefore$ If the principal axis at the centre G is inclined at an angle θ to the axis of x, then

$$\tan 2\theta = \frac{2F}{B-A} = \frac{-(m/3)\left[ab - 2c(a+b) + 3c^2\right]}{(m/3)\left[2(c-b)^2 - 2(c-a)^2\right]}$$

$$= \frac{ab - 2c(a+b) + 3c^2}{(a-b)(a+b-2c)}.$$

Ex. 58. *Show that one of the principal axes at a point on the circular rim of the solid hemisphere, is inclined at an angle $\tan^{-1}\frac{1}{3}$ to the radius through the point.*

Sol. Let C be the centre and OA the diameter of the circular rim of a hemisphere of radius a and mass M. Take OX and OY the axis of x and y along and perpendicular to OA in the plane of the circular rim of the hemisphere, and OZ the z-axis perpendicular to this plane. As in Ex. 35 on page 49 we have



$A =$ M.I. of the hemisphere about $OX = \frac{2}{5}Ma^2$, $B = \frac{7}{5}Ma^2$, $C = \frac{7}{5}Ma^2$

$D =$ P.I. about $OY, OZ = 0$, $E = \frac{3}{4}Ma^2$ and $F = 0$.

Since $D = O = F$. $\therefore$ y-axis OY is the principal axis at the point O and the other two principal axes at O lie in xz plane. If one of these principal axes make an angle θ to OX, then

$$\tan 2\theta = \frac{2E}{C-A} = \frac{\frac{3}{4}Ma^2}{(\frac{7}{5} - \frac{2}{5})Ma^2} = \frac{3}{4}$$

$$\text{or } \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{3}{4}$$

or $3\tan^2\theta + 8\tan\theta - 3 = 0$ or $(3\tan\theta - 1)(\tan\theta + 3) = 0$

$\therefore$ $\tan\theta = \frac{1}{3}$ or $\theta = \tan^{-1}(\frac{1}{3})$ $\because$ $\tan\theta = -3 \Rightarrow \theta > \pi/2$

which is inadmissible.

Ex. 59. *Show that one of the principal axes at any point on the edge of the circular base of a thin hemispherical shell is inclined at an angle $\pi/8$ to the radius through the point.*

Sol. Let OA be the diameter of the circular base of a thin hemispherical shell of radius a and mass M. Take OX, OY, OZ the axes of x, y and z as in the last Ex. 58.

As in Ex. 34 on page 48, we have

$A = \frac{2}{3} Ma^2, B = \frac{5}{3} Ma^2, C = \frac{5}{3} Ma^2, D = 0, E = \frac{1}{2} Ma^2$ and $F = 0$.

Since $D = 0 = F$, $\therefore$ OY is the principal axis at O and the other two principal axes at O will lie in xz plane. If one of these principal axes make an angle θ to OX, then

$$\theta = \frac{1}{2} \tan^{-1} \frac{2E}{C-A} = \frac{1}{2} \tan^{-1} \frac{Ma^2}{(\frac{5}{3} - \frac{2}{3}) Ma^2} = \frac{1}{2} \tan^{-1} 1 = \frac{\pi}{8}$$

§ 1.26. Principal Moments :

Moments of inertia of a body about its principal axes at any point are called its principal moments at that point.

The equation of the ellipsoid at any point is given by

$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = Mk^4$. ...(1)

Taking the principal axes as the coordinate axes equation (1) reduces to the form

$$A'x^2 + B'y^2 + C'z^2 = Mk^4$$

Where A', B', C' are the principal moments and are the values of λ in the cubic equation

$$\begin{vmatrix} A-\lambda & H & G \\ H & B-\lambda & F \\ G & F & C-\lambda \end{vmatrix} = 0.$$

This cubic equation in λ is called the reduction cubic.

EXAMPLES

Ex. 60. *If A and B be the moments of inertia of a uniform lamina about perpendicular axes OX and OY, lying in its plane, and F be the product of inertia of the lamina about these lines, show that the principal moments at O are equal to*

$$\tfrac{1}{2}[A + B \pm \sqrt{\{(A-B)^2 + 4F^2\}}]$$

Sol. Here we consider the uniform lamina, so there will be momental ellipse at O whose equation is given by

$Ax^2 + By^2 - 2Fxy = \text{Constant}$ ...(1)

Taking the principal axes as the coordinate axes, equation (1) reduce to the form

$A'x^2 + B'y^2 = \text{Constant}.$ ...(2)

Equating the invariants* of (1) and (2) we have

$A' + B' = A + B$ ...(3)

and $A'B' = AB - F^2$ ...(4)

$\therefore$ $A' - B' = \sqrt{\{(A'+B')^2 - 4A'B'\}} = \sqrt{\{(A+B)^2 - 4(AB - F^2)\}}$

or $A' - B' = \sqrt{\{(A-B)^2 + 4F^2\}}$ ...(5)

Adding and subtracting (3) and (5) we have

$A' = \frac{1}{2}[A + B + \sqrt{\{(A-B)^2 + 4F^2\}}]$

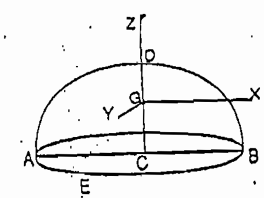
and $B' = \frac{1}{2}[A + B - \sqrt{\{(A-B)^2 + 4F^2\}}]$

i.e. the principal moments at O are equal to

$\frac{1}{2}[A + B \pm \sqrt{(A-B)^2 + 4F^2}]$

Ex. 61. *Show that for a thin hemispherical solid of radius a and mass M, the principal moments of inertia at the centre of gravity are* $\frac{83}{320} Ma^2, \frac{83}{320} Ma^2, \frac{2}{5} Ma^2$.

Sol. Let G be the centre of gravity of a hemispherical solid of radius a and mass M. If C is the centre and CD the central radius of the hemisphere, then $CG = 3a/8$.



Take GX and GY the axes through G and parallel to the plane base be taken as the axis of x and y respectively and GZ the central radius as the z-axis then

$A = \text{M.I.}$ about $GX = \text{M.I.}$ about $AB - M.CG^2$

$$= \frac{2}{5} Ma^2 - M\left(\frac{3}{8}a\right)^2 = \frac{83}{320} Ma^2,$$

$$B = \text{M.I. about } GY = \frac{2}{5} Ma^2 - M\left(\frac{3a}{8}\right)^2 = \frac{83}{320} Ma^2,$$

* Invariants : If by change axes without the change of origin $ax^2 + by^2 + 2hxy$ transform to $a'x^2 + b'y^2 + 2h'xy$ then $a' + b' = a + b$ and $a'b' - h'^2 = ab - h^2$. Thus $a + b$ and $ab - h^2$ are called the invariant of the system.

$C =$ M.I. about $CZ = \frac{2}{3} Ma^2$.

Now coordinates of C are $(0, 0, -3a/8)$.

$D =$ P.I. about GY, GZ

$=$ P.I. about parallel lines $CB, CE -$ P.I. of M at C about GY, GZ

$= 0 - M.0.(-3a/8) = 0.$

Similarly, $E = 0, F = 0, D = 0, E = F$.

$\therefore$ GX, GY, GZ are the principal axes at G.

Hence $\frac{83}{320} Ma^2, \frac{83}{320} Ma^2, \frac{2}{5} Ma^2$ are the principal moments.

Ex. 62. *Show that for a thin hemispherical shell of radius a and mass M, the principal moments of inertia at the centre of gravity are* $\frac{5}{12} Ma^2, \frac{5}{12} Ma^2, \frac{2}{3} Ma^2$.

Sol. (Refer figure of Ex. 61).

Let G be the C.G. of the hemispherical shell of radius a and mass M. Here C.G. $= a/2$. Taking the axes of x, y, z as in Ex. 61, coordinates of C are $(0, 0, -a/2)$.

$$\therefore A = \frac{2}{3} Ma^2 - M.CG^2 = \frac{2}{3} Ma^2 - M\left(\frac{a}{2}\right)^2 = \frac{5}{12} Ma^2.$$

Similarly, $B = \frac{5}{12} Ma^2, C = \frac{2}{3} Ma^2$ and $D = 0 = E = F$.

$\because$ $D = 0 = E = F$, $\therefore$ the lines GX, GY, GZ are the principal axes at G. Thus the principal moments at G are

$\frac{5}{12} Ma^2, \frac{5}{12} Ma^2, \frac{2}{3} Ma^2$.

Ex. 63. *A uniform solid circular cone of semi-vertical angle α and height h is cut in half by a plane through its axis. Show that the principal moments of inertia at the vertex for one of the halves are* $\frac{3}{5} Mh^2 (1 + \frac{1}{4} \tan^2 \alpha)$ *and* $\frac{3}{10} Mh^2 \left(1 + \frac{3}{4} \tan^2 \alpha\right)$

$$\pm \frac{3}{10} Mh^2 \sqrt{\left[\left(1 - \frac{1}{4} \tan^2 \alpha\right)^2 + \left(\frac{64}{9\pi^3}\right) \tan^2 \alpha\right]}.$$

Sol. Let $OACBDO$ be the half cone of mass M, $ACBD$ its semi-circular base and OAB its triangular face. Take the z-axis OZ along OC, y-axis OY perpendicular to OC in the plane of the triangular face and x-axis OX perpendicular to this triangular face.

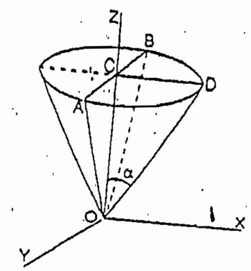
Since half cone is symmetrical about zx plane which is perpendicular to OY.

$\therefore$ $D =$ P.I. about $OY, OZ = 0$ and $F =$ P.I. about $OX, OY = 0$.

$\therefore$ OY is the principal axis at O.

$M =$ Mass of the half cone

$= \frac{1}{2} (\frac{1}{3} \rho \pi h^3 \tan^2 \alpha)$.



$B =$ Principal moment about OY

$$= \frac{1}{2}\left[\frac{1}{20} \rho \pi h^5 (\tan^2 \alpha + 4) \tan^2 \alpha\right]$$

(see Ex. 23 on page 34)

$= \frac{3}{20} Mh^2 (4 + \tan^2 \alpha)$

$= \frac{3}{5} Mh^2 \left(1 + \frac{1}{4} \tan^2 \alpha\right)$

$A =$ M.I. about $OX =$ M.I. about $OY = \frac{3}{5} Mh^2 \left(1 + \frac{1}{4} \tan^2 \alpha\right)$

$C =$ M.I. about $OZ = \frac{1}{2}\left[\frac{1}{10} \rho \pi h^5 \tan^4 \alpha\right] = \frac{3}{10} Mh^2 \tan^2 \alpha$

$E =$ P.I. about OX, OZ (see Ex. 9 on page 21)

$$= 2 \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\alpha} \int_{r=0}^{h \sec \theta} \rho r d\theta dr \,.\, r \sin \theta d\phi \,.\, r \cos \theta \, r \sin \theta \cos \phi$$

$$= 2\rho \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\alpha} \frac{1}{5} h^5 \sec^5 \theta \,.\, \sin^2 \theta \cos \theta \cos \phi d\phi \, d\theta$$

$$= \frac{2\rho}{5} h^5 \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\alpha} \tan^2 \theta \sec^2 \theta \cos \phi \, d\phi \, d\theta$$

$$= \frac{2\rho}{5} h^5 \int_{\phi=0}^{\pi/2} \left[\frac{1}{3} \tan^3 \theta\right]_0^{\alpha} \cos \phi d\phi = \frac{2\rho}{15} h^5 \tan^3 \alpha \,.\, [\sin \phi]_0^{\pi/2}$$

$$= \frac{4}{5\pi} Mh^2 \tan \alpha.$$

If the principal axis (other than OY) make an angle θ to OZ, then

$$\tan 2\theta = \frac{2E}{A - C} = \frac{(8/5\pi) Mh^2 \tan \alpha}{\frac{3}{5} Mh^2 (1 + \frac{1}{4} \tan^2 \alpha) - \frac{3}{10} Mh^2 \tan^2 \alpha} = \frac{(8/3\pi) \tan \alpha}{1 - \frac{1}{4} \tan^2 \alpha}$$

$$\therefore \sin 2\theta = \frac{(8/3\pi) \tan \alpha}{\sqrt{[(64/9\pi^2) \tan^2 \alpha + (1 - \frac{1}{4} \tan^2 \alpha)^2]}}$$

and $\cos 2\theta = \dfrac{1-(1/4)\tan^2\alpha}{\sqrt{[(64/9\pi^2)\tan^2\alpha + (1-\frac{1}{4}\tan^2\alpha)^2]}}$.

Hence the other principal moment

$= C\cos^2\theta + A\sin^2\theta - 2E\sin\theta\cos\theta$

$= \frac{1}{2}C(1+\cos 2\theta) + \frac{1}{2}A(1-\cos 2\theta) - E\sin 2\theta$

$= \frac{3}{20}Mh^2\tan^2\alpha(1+\cos 2\theta) + \frac{3}{10}Mh^2(1+\frac{1}{4}\tan^2\alpha)(1-\cos 2\theta)$

$- \frac{4}{5\pi}Mh^2\tan\alpha\sin 2\theta$

$= \frac{3}{10}Mh^2(1+\frac{1}{4}\tan^2\alpha) - \frac{3}{10}Mh^2(1-\frac{1}{4}\tan^2\alpha)\cos 2\theta$

$- \frac{4}{5\pi}Mh^2\tan\alpha\sin 2\theta$

$= \frac{3}{10}Mh^2(1+\frac{1}{4}\tan^2\alpha) - \frac{3}{10}Mh^2(1-\frac{1}{4}\tan^2\alpha)$.

$$\frac{(1-\frac{1}{4}\tan^2\alpha)}{\sqrt{[(64/9\pi^2)\tan^2\alpha + (1-\frac{1}{4}\tan^2\alpha)^2]}}$$

$$-\frac{4}{5\pi}Mh^2\tan\alpha\,.\,\frac{(8/3\pi)\tan\alpha}{\sqrt{[(64/9\pi^2)\tan^2\alpha + (1-\frac{1}{4}\tan^2\alpha)^2]}}$$

$$= \frac{3}{10}Mh^2(1+\tfrac{1}{4}\tan^2\alpha) - \frac{3}{10}Mh^2\frac{(1-\frac{1}{4}\tan^2\alpha)^2 + (64/9\pi^2)\tan^2\alpha}{\sqrt{[(64/9\pi^2)\tan^2\alpha + (1-\frac{1}{4}\tan^2\alpha)^2]}}$$

$$= \frac{3}{10}Mh^2(1+\tfrac{1}{4}\tan^2\alpha) - \frac{3}{10}Mh^2\sqrt{[(1-\tfrac{1}{4}\tan^2\alpha)^2 + (64/9\pi^2)\tan^2\alpha]}.$$

Replacing θ by $\theta + \pi/2$, the other principal moment is

$= C\sin^2\theta + A\cos^2\theta + 2E\cos\theta\sin\theta$

$= \frac{3}{10}Mh^2(1+\frac{1}{4}\tan^2\alpha) + \frac{3}{10}Mh^2\sqrt{[(1-\frac{1}{4}\tan^2\alpha) + (64/9\pi^2)\tan^2\alpha]}$.

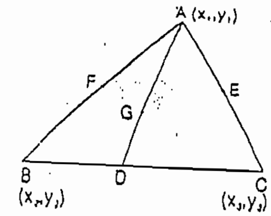
Ex. 64. *Prove that the principal radii of gyration at the C.G. of a triangle are the roots of the equation*

$$x^4 - \frac{a^2+b^2+c^2}{36}x^2 + \frac{\Delta^2}{108} = 0$$

where Δ is the area of the triangle.

Sol. Let ABC be the triangle of mass M. Taking the centre of gravity of the triangle G as the origin and the principal axes through G as axes. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, be the coordinates of the vertices A, B, C respectively.

Since C.G. 'G' is taken as origin and



$\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$

are the coordinates of C.G.,

$\therefore\ x_1 + x_2 + x_3 = 0$

and $y_1 + y_2 + y_3 = 0$ ...(1)

Thus $(x_1 + x_2 + x_3)^2 = 0$

or $x_1^2 + x_2^2 + x_3^2 = -2(x_1x_2 + x_2x_3 + x_3x_1)$ ...(2)

Similarly $y_1^2 + y_2^2 + y_3^2 = -2(y_1y_2 + y_2y_3 + y_3y_1)$ ...(3)

Now, $BC^2 = a^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2$

$CA^2 = b^2 = (x_1 - x_3)^2 + (y_1 - y_3)^2$

and $AB^2 = c^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\therefore\ a^2 + b^2 + c^2 = 2(x_1^2 + x_2^2 + x_3^2) + 2(y_1^2 + y_2^2 + y_3^2)$

$- 2(x_1x_2 + x_2x_3 + x_3x_1) - 2(y_1y_2 + y_2y_3 + y_3y_1)$

$= 2(x_1^2 + x_2^2 + x_3^2) + 2(y_1^2 + y_2^2 + y_3^2) + (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2)$

or $a^2 + b^2 + c^2 = 3(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2)$ ...(4)

The triangle ABC may be replaced by three particles each of mass $M/3$ placed at the middle points D, E, F of the sides whose coordinates are $\left(\dfrac{x_2+x_3}{2}, \dfrac{y_2+y_3}{2}\right), \left(\dfrac{x_3+x_1}{2}, \dfrac{y_3+y_1}{2}\right)$ and $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$ respectively.

$\therefore A$ = Principal moment about x axis.

$= \frac{1}{3}M\left(\dfrac{x_2+x_3}{2}\right)^2 + \frac{1}{3}M\left(\dfrac{x_3+x_1}{2}\right)^2 + \frac{1}{3}M\left(\dfrac{x_1+x_2}{2}\right)^2$

$= \dfrac{M}{12}[2(x_1^2 + x_2^2 + x_3^2) + 2(x_1x_2 + x_2x_3 + x_3x_1)]$

$= \frac{1}{12}M(x_1^2 + x_2^2 + x_3^2)$ Using (2)

Similarly B = Principal moment about y-axis

$= \frac{1}{12}M(y_1^2 + y_2^2 + y_3^2)$

$\therefore\ A + B = \frac{1}{12}M(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2)$

or $A + B = \frac{1}{36}M(a^2 + b^2 + c^2)$ Using (4)

Since x, y axes through G are principal axes.

$\therefore$ P.I. about x, y axes $= 0$

or $\frac{M}{3}\left(\frac{x_2+x_3}{2}\right)\left(\frac{y_2+y_3}{2}\right)+\frac{M}{3}\left(\frac{x_3+x_1}{2}\right)\left(\frac{y_3+y_1}{2}\right)$
$+\frac{M}{3}\left(\frac{x_1+x_2}{2}\right)\left(\frac{y_1+y_2}{2}\right)=0$

or $(x_2+x_3)(y_2+y_3)+(x^3+x^1)(y_3+y_1)+(x_1+x_2)(y_1+y_2)=0$

or $(-x_1)(-y_1)+(-x_2)(-y_2)+(-x_3)(-y_3)=0$ Using (1)

or $x_1y_1+x_2y_2+x_3y_3=0$ ...(5)

Also $AB=\frac{1}{144}M^2(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2)$

$=\frac{1}{144}M^2[(x_1y_1+x_2y_2+x_3y_3)^2+(x_1y_2-x_2y_1)^2+(x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2]$

NNow $\Delta =$ area of the triangle ABC

$=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$

or $2\Delta = x_1(y_2+y_1+y_2)+x_2(-y_1-y_2-y_1)+(-x_1-x_2)(y_1-y_2)$ Using (1)

$=3(x_1y_2-x_2y_1)$ or $x_1y_2-x_2y_1=\frac{2}{3}\Delta$.

Similarly, $x_2y_3-x_3y_2=\frac{2}{3}\Delta$ and $x_3y_1-x_1y_3=\frac{2}{3}\Delta$.

$\therefore AB=\frac{1}{144}M^2[0+(\frac{2}{3}\Delta)^2+(\frac{2}{3}\Delta)^2+(\frac{2}{3}\Delta)^2]=\frac{M^2\Delta^2}{108}$ ...(6)

If k_1 and k_2 are the principal radii of gyration, then $A=Mk_1^2$ and $B=Mk_2^2$

$\therefore k_1^2+k_2^2=\frac{1}{M}(A+B)=\frac{1}{36}(a^2+b^2+c^2)$, [from (4)]

and $k_1^2 . k_2^2=\frac{AB}{M^2}=\frac{\Delta^2}{108}$. [from (6)]

$\therefore k_1^2$ and k_2^2 are the roots of the equation

$x^4-(k_1^2+k_2^2)x^2+(k_1^2 . k_2^2)=0$

or $x^4-\frac{1}{36}(a^2+b^2+c^2)+\frac{1}{108}\Delta^2=0.$

Ex. 65. *Three rods AB, BC, CD, each of mass m and length 2a are such that each is perpendicular to the other two. Show that the principal moments of inertia at the centre of mass are ma^2, $\frac{11}{3}ma^2$ and $4ma^2$.*

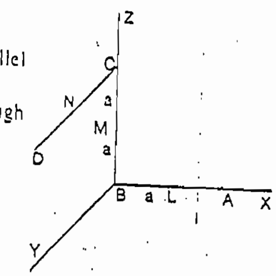
Sol. Let BY be a line parallel to CD. Taking BA, BY, BC as the axes of x, y, z respectively, the coordinates of middle points L, M, N of rods AB, BC, CD are $(a, 0, 0)$, $(0, 0, a)$ and $(0, a, 2a)$ respectively.

If $(\bar{x}, \bar{y}, \bar{z})$ are the coordinates of the C.G. 'G' of the rods AB, BC, CD each of mass m, then

$\bar{x}=\frac{m.a+m.0+m.0}{m+m+m}=\frac{1}{3}a,\ \bar{y}=\frac{m.0+m.0+m.a}{m+m+m}=\frac{1}{3}a$

and $\bar{z}=\frac{m.0+m.a+m.2a}{m+m+m}=a$,

i.e. coordinates of G are $(\frac{1}{3}a, \frac{1}{3}a, a)$.



Let GX', GY', GZ' be the axes parallel to BA, BY and BC.

In reference to these axes through G the coordinates of

L are $(a-a/3, 0-a/3, 0-a)$

i.e $(\frac{2}{3}a, -\frac{1}{3}a, -a)$,

M are $(0-\frac{1}{3}a, 0-\frac{1}{3}a, a-a)$

i.e. $(-\frac{1}{3}a, -\frac{1}{3}a, 0)$,

and N are $(0-\frac{1}{3}a, a-\frac{1}{3}a, 2a-a)$ i.e. $(-\frac{1}{3}a, \frac{2}{3}a, a)$

$\therefore A_1 =$ M.I. of the three rods about GX'

$=$ M.I. of AB + M.I. of BC + M.I. of CD about GX'

$=[m\{(-\frac{1}{3}a)^2+(-a)^2\}]$
$+[\frac{1}{3}ma^2+m\{(-\frac{1}{3}a)^2+0^2\}]+[\frac{1}{3}ma^2+(\frac{2}{3}a)^2+a^2]]=\frac{11}{3}ma^2$,

$B_1=$ M.I. of the three rods about GY'

$=[\frac{1}{3}ma^2+m\{(-a)^2+(\frac{2}{3}a)^2\}+[\frac{1}{3}ma^2+m\{0+(-\frac{1}{3}a)^2\}]$
$+m\{a^2+(-\frac{1}{3}a)^2\}=\frac{10}{3}ma^2$

$C_1=$ M.I. of the three rods about GZ'

$=[\frac{1}{3}ma^2+m\{(\frac{2}{3}a)^2+(-\frac{1}{3}a)^2\}]+[m\{(-\frac{1}{3}a)^2+(-\frac{1}{3}a)^2\}]$
$+[\frac{1}{3}ma^2+m\{(-\frac{1}{3}a)^2+(\frac{2}{3}a)^2\}]=2ma^2$,

$D_1=$ P.I. about GY', $GZ'=\Sigma my_1z_1$

$=m(-\frac{1}{3}a)(-a)+m(-\frac{1}{3}a).0+m(\frac{2}{3}a)(a)=ma^2$

$E_1=$ P.I. about GZ'. $GX'=\Sigma mz_1x_1$

$=m(-a)(\frac{2}{3}a)+m(0)(-\frac{1}{3}a)+m.a(-\frac{1}{3}a)=-ma^2$,

and $F_1=$ P.I. about GX', $GY'=\Sigma mx_1y_1$

$=m(\frac{2}{3}a)(-\frac{1}{3}a)+m(-\frac{1}{3}a)(-\frac{1}{3}a)+m(-\frac{1}{3}a)(\frac{2}{3}a)=-\frac{1}{3}ma^2$.

Hence the momental ellipsoid at G is

$A_1x^2+B_1y^2+C_1z^2-2D_1yz-2E_1zx-2F_1xy=3mk^4$

or $\frac{11}{3}ma^2x^2+\frac{10}{3}ma^2y^2+2ma^2z^2-2ma^2yz+2ma^2zx+\frac{2}{3}ma^2xy=3mk^4$

or $\frac{1}{3}ma^2[11x^2+10y^2+6z^2-6yz+6zx+2xy]=3mk^4$. ...(1)

Reducing $10x^2 + 10y^2 + 6z^2 - 6yz + 6zx + 2xy$ by means of the discriminating cubic $\lambda^3 - (a+b+c)\lambda^2 + (ab+bc+ca-f^2-g^2-h^2)\lambda - (abc + 2fgh - af^2 - bg^2 - ch^2) = 0$, we have

$\lambda^3 - 26\lambda^2 + 201\lambda - 396 = 0$

or $(\lambda - 3)(\lambda - 11)(\lambda - 12) = 0 \therefore \lambda = 3, 11, 12.$

Hence the equation of the momental ellipsoid (1) referred to the principal axes through G takes the form

$\frac{1}{3}ma^2(3x^2 + 11y^2 + 12z^2) = 3mk^4$

or $ma^2x^2 + \frac{11}{3}ma^2y^2 + 4ma^2z^2 = 3mk^4$.

Hence the principal moments at the centre of inertia are ma^2, $\frac{11}{3}ma^2$ and $4ma^2$.

EXERCISE ON CHAPTER I

1. Show that the moment of inertia of the part of the area of parabola cut off by any ordinate at a distance x form the vertex is $(3/7)\,Mx^2$ about the tangent at the vertex, and $(1/5)\,My^2$ about the principal diameter where y is the ordinate corresponding to x [Hint. See Ex. 4 on page 15].

2. The principal axes at the centre of gravity being the axes of reference, prove that the momental ellipsoid at the point (p, q, r) is
$$(A/M + q^2 + r^2)x^2 + (B/M + r^2 + p^2)y^2 + (C/M + p^2 + q^2)z^2 - 2qryz - 2rpzx - 2pqxy = \text{constant},$$
when referred to its centre of gravity as origin.

3. Show that a uniform rod, of mass m, is kinetically equivalent to three particles rigidly connected and situated one at each end of the rod and one at its middle point, the masses of the particles being $m/6, m/6, 2m/3$.

4. Show that any lamina is dynamically equivalent to the three particles; each one-third of the mass of the lamina; placed at the corners of a maximum triangle inscribed in the ellipse, whose equation referred to the principal axes at the centre of inertia is $x^2/B + y^2/A = 2$, where mA and mB are the principal moments of inertia about OX and OY, and m is the mass.

5. Show that there is momental ellipse at an angular point of a triangular area which touches the oposite side at its middle point and bisects the adjacent sides

6. Find the principal axes at a corner point of solid cube [Hint. In Ex. 32 on page 47, $D = E = F = 0$. $\therefore$ OG is one principal axis at O. Other two principal axes pass through O and at right angles to OG].

7. Two particles, each of mass m are placed at the extremities of the minor axis of an elliptic area of mass M. Prove that principal axes at any point of the circumference of the ellipse will be the tangent and normal to the ellipse, if
$$\frac{m}{M} = \frac{5}{8} \cdot \frac{e^2}{1 - 2e^2}.$$

8. A uniform lamina bounded by the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ has an elliptic hole (semi-axes c, d) in it, whose major axis lies in the line $x = y$, the centre being at a distance r from origin, prove that if one of the principal axes at the point (x, y) makes angle θ with x-axis, then
$$\tan\theta = \frac{8abxy - cd\{4(x\sqrt{2} - r)(y\sqrt{2} - r) - (c^2 - d^2)\}}{ab\{(x^2 - y^2 + a^2 - b^2)\} - cd\{2(x\sqrt{2} - r)^2 - 2(y\sqrt{2} - r)^2\}}$$

9. The principal axes at the centre of gravity being the axes of reference, obtain the equation of the ellipsoid at the point (p, q, r) and show that the principal moments of inertia at this point are roots of
$$\begin{vmatrix} (I - A)/M - q^2 - r^2 & pq & rp \\ pq & (I - B)/M - r^2 - p^2 & qr \\ rp & qr & (I - C)/M - p^2 - q^2 \end{vmatrix} = 0$$
where I, M, A, B have there usual meanings.

10. Find the M.I. of a quadrant of the elliptic are $x^2/a^2 + y^2/b^2 = 1$ of mass M about line through its centre and perpendicular to its plane, the density at any point is proportional to xy. [Meerut TDC 90 (S)]

11. Find the M.I. of the solid generation by the revolution of the parabola $y^2 = 4ax$ about the x-axis from $x = 0$ to $x = h$ about x-axis [Meerut TDC 92(P)] (Hint. See Ex. 7]

12. Find the M.I. of an ellipsoid about the axis of z. [Meerut TDC 94, 94(P)]
$$\left[Ans.\ \frac{1}{5}M(a^2 \times b^2)\right]$$

13. Find the M.I. of the cardoid $r = a(1 + \cos\theta)$ of density ρ, about the initial line. [Meerut TDC 90]

2 D' Alembert's Principle

§ 2.1. Motion of a Particle.

The motion of a particle is determined by Newton's second law of motion, which states that '*the rate of change of momentum in any direction is proportional to the applied force in that direction*'. From this law, we deduce the formula $P = mf$, where f is the acceleration of the particle of mass m in the direction of the applied force P.

If (x, y, z) be the coordinates of a moving particle of mass m, at any time t and X, Y, Z be the components of the forces parallel to the axes, then by Newton's second law of motion the equations of motion of the particle are

$$m\ddot{x} = X,\ m\ddot{y} = Y,\ m\ddot{z} = Z.$$

§ 2.2. Motion of a Rigid Body.

A rigid body is an assemblage of particles rigidly connected together such that the distance between any two constituent particles does not change on account of the effect of forces.

For a rigid body we assume that

(i) the action between its two particles act along the straight line joining them.

(ii) the action an reaction between the two particles are equal and opposite.

In considering the motion of a rigid body, we write the equation of motion of the particles of the body according to the equations in § 2.1. But here the external forces acting on a particle of the body include, together with the applied force, the unknown inner forces acting due to the action of the rest of the body on it.

D' Alembert proposed a method which enables us to obtain all the necessary equations without writing down the equations of motion of all particles and without considering the unknown inner forces. This important principle is based on the forllowing rule which is a natural consequence of Newton's third law of motion.

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

§ 2.3. Definitions.

Impressed forces. (Meerut 95(BP))

The external forces acting on a body are called 'impressed forces'. For example, the weight of the body is the impressed force on the body.

In case a body is tied to a string then the tension in the string is also an impressed force on the body.

Effective forces. (Meerut 95(BP))

The effective force on a particle is defined as the product of its mass m and its acceleration f. If a particle of mass m is situated at the point (x, y, z) at time t, then the effective forces on this particle at this time t are $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$ parallel to the axes.

§ 2.4. D' Alembert's Principle.

The reversed effective forces at each point of the body and the impressed (external) forces on the system are in equilibrium.

[Meerut 85, 86, 87, 88, 90, TDC 91, 91(P), 92, 92(P), 93, 93(P), 94, 94(P), 95(P)]

Let (x, y, z) be the coordinates of a particle of mass m, of a rigid body which is in motion, at any time t. If f is the resultant of component accelerations $\ddot{x}, \ddot{y}, \ddot{z}$ then the effective force on the particle is mf. Let F denote the resultant of the impressed forces and R the resultant of the internal forces (mutual actions) on the particle. Then by Newton's second law of forces, mf is the resultant of F and R. Thus $-mf$ (reversed effective force), F and R are in equilibrium. This holds good for every particle of the body.

Thus $\Sigma(-mf)$, ΣF and ΣR are in equilibrium, the summation extending to all the particles of the body.

But the internal actions and reactions of different particles of a body are in equilibrium i.e. $\Sigma R = 0$, therefore $\Sigma(-mf)$ and ΣF are in equilibrium.

Hence the reversed effective forces acting at each particle of the body and the impressed (external) forces on the system are in equilibrium.

Vector Method :

Consider a rigid body in motion. At time t, let $\mathbf{r}$ be the position vector of a particle of mass m and $\mathbf{F}$ and $\mathbf{R}$ the external and internal forces respectively acting on it.

$\therefore$ By newton's second law

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} + \mathbf{R}$$

or

$$\mathbf{F} + \mathbf{R} - m\frac{d^2\mathbf{r}}{dt^2} = 0$$

i.e. the forces $\mathbf{F}$, $\mathbf{R}$, $-m\dfrac{d^2\mathbf{r}}{dt^2}$ acting on a particle of mass m are in equilibrium.

Now applying the same argument of every particle of the rigid body, the forces $\Sigma\mathbf{F}$, $\Sigma\mathbf{R}$ and $\Sigma\left(-m\dfrac{d^2\mathbf{r}}{dt^2}\right)$ are in equilibrium, where the summation extends to all particles.

But, the internal forces acting on the body form pairs of equal and opposite forces $\therefore\ \Sigma R = 0$.

Thus the forces ΣF and $\Sigma\left(-m\dfrac{d^2\mathbf{r}}{dt^2}\right)$ are in equilibrium,

i.e.
$$\Sigma \mathbf{F} + \Sigma\left(-m\frac{d^2\mathbf{r}}{dt^2}\right) = 0$$

Hence the reversed effective forces acting at each particle of the body and the impressed (external) forces on the system are in equilibrium.

Note. The above D' Alembert's principle reduces the problem of dynamics to the problem of statics. Thus we mark all the external forces of the system and mark the effective forces in opposite directions and then solve this problem as a problem of statics by equating to zero the resolved parts of all these forces in two mutually perpendicular directions and taking moments about suitable points.

§ 2.5. General Equations of motion of a body.

To deduce the general equations of motion of a rigid body from D' Alembert's principle.

[Meerut 89 ; TDC 92, 92 (P), 94, 94 (P), 95 (P), 96 ;]

Let X, Y, Z be the components, parallel to the axes, of the external force acting on a particle of mass m whose coordinates are (x, y, z) at time t, referred to any set of rectangular axes. Then reversed effective forces parallel to the axes on the particle m are $-m\ddot{x}, -m\ddot{y}, -m\ddot{z}$. Thus the resultant of external forces and the reversed effective forces acting on the particle m parallel to the axes are $X - m\ddot{x}$, $Y - m\ddot{y}$ and $Z - m\ddot{z}$ respectively.

By D' Alembert's principle the forces whose components are $X - m\ddot{x}$, $Y - m\ddot{y}$, $Z - m\ddot{z}$ acting at the particle m at (x, y, z) together with similar forces acting at each other particle of the body, form a system in equilibrium.

Hence, as in statics the six conditions of equilibrium are

$\Sigma (X - m\ddot{x}) = 0$, $\Sigma (Y - m\ddot{y}) = 0$, $\Sigma (Z - m\ddot{z}) = 0$

$\Sigma \{y (Z - m\ddot{z}) - z (Y - m\ddot{y})\} = 0$, $\Sigma \{z (X - m\ddot{x}) - x (Z - m\ddot{z})\} = 0$

and $\Sigma \{x (Y - m\ddot{y}) - y (X - m\ddot{x})\} = 0$,

where the summation is extended to all the particles of the body.

These six equations can be written as

$\Sigma m\ddot{x} = \Sigma X$ ...(1) $\qquad \Sigma m\ddot{y} = \Sigma Y$ ...(2)

$\Sigma m\ddot{z} = \Sigma Z$ ...(3) $\qquad \Sigma m (y\ddot{z} - z\ddot{y}) = \Sigma (yZ - zY)$ ...(4)

$\Sigma m (z\ddot{x} - x\ddot{z}) = \Sigma (zX - xZ)$ ...(5)

and $\Sigma m (x\ddot{y} - y\ddot{x}) = \Sigma (xY - yX)$ ...(6)

These equations (1) to (6) are the general equations of motion of a body.

Equations (1), (2), (3) state that *the sums of the components, parallel to the coordinate axes, of the effective forces is respectively equal to the sums of the components parallel to the same axes of the external (impressed) forces.*

Equations (4), (5), (6) state that *the sums of the moments about the axes of coordinates of the effective forces are respectively equal to the sums of the moments about the same axes of the external (impressed) forces*

The equations (1), (2) and (3) can be written as $\dfrac{d}{dt}(\Sigma m\dot{x}) = \Sigma X$,

$\dfrac{d}{dt}(\Sigma m\dot{y}) = \Sigma Y$ and $\dfrac{d}{dt}(\Sigma m\dot{z}) = \Sigma Z$,

Which shows that *the rate of change of linear momentum of the system in any direction is equal to the total external force in that direction.*

The equations (4), (5) and (6) can be written as

$\dfrac{d}{dt}\{\Sigma m (y\dot{z} - z\dot{y})\} = \Sigma (yZ - zY)$, $\dfrac{d}{dt}\{\Sigma m (z\dot{x} - x\dot{z})\} = \Sigma (zX - xZ)$

and $\dfrac{d}{dt}\{\Sigma m (x\dot{y} - y\dot{x})\} = \Sigma (xY - yX)$

Which shows that *the rate of change of angular momentum (moment of momentum) about any given axis is equal to the total moment of all the external forces about the axis.*

Vector Method : Consider a rigid body in motion. At time t let $\mathbf{r}$ be the position vector of a particle of mass m and $\mathbf{F}$ the external force acting on it.

Then by D' Alembert's principle, we have

$$\Sigma \mathbf{F} + \Sigma\left(-m\frac{d^2\mathbf{r}}{dt^2}\right) = 0$$

or
$$\Sigma m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}. \qquad ...(1)$$

Taking cross product by $\mathbf{r}$, we have

$$\Sigma m\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \Sigma \mathbf{r} \times \mathbf{F} \qquad ...(2)$$

Equations (1) and (2) are in general vector equations of motion of a rigid body.

Deduction of general equations of motion in scaler form.

To deduce the general equations of motion of a rigid body, we substitute the following in (1), (2).

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \text{ and } \mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$$

where (x, y, z) are the cartesian coordinates of the particle m and X, Y, Z are the components of force $\mathbf{F}$ parallel to the axes respectively.

Substituting in (1) and (2), we get

$$\Sigma m(\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k})=\Sigma(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})$$

and $\Sigma\{m(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\times(\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k})\}=\Sigma(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\times(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})$ ...(3)

or $\Sigma m\{(y\ddot{z}-z\ddot{y})\mathbf{i}+(z\ddot{x}-x\ddot{z})\mathbf{j}+(x\ddot{y}-y\ddot{x})\mathbf{k}\}$

$$=\Sigma\{(yZ-zY)\mathbf{i}+(zX-xZ)\mathbf{j}+(xY-yX)\mathbf{k}\} \quad ...(4)$$

Equating coefficients of i, j, k on the two sides of equations (3) and (4), we get the six equations of motion of the rigid body in cartesian form.

§ 2.6. Linear Momentum.

The linear momentum in a given direction is equal to the product of the whole mass of the body and the resolved part of the velocity of its centre of gravity in that direction.

Let $(\bar{x},\bar{y},\bar{z})$ be the coordinates of the centre of gravity of a body of mass M, then we have

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}=\frac{\Sigma mx}{M} \quad \because \Sigma m=M.$$

$\therefore$ $\Sigma mx=M\bar{x}$. Similarly, $\Sigma my=M\bar{y}$ and $\Sigma mz=M\bar{z}$.

Differentiating these relation w.r.t. 't', we get

$\Sigma m\dot{x}=M\dot{\bar{x}}$, $\Sigma m\dot{y}=M\dot{\bar{y}}$, and $\Sigma m\dot{z}=M\dot{\bar{z}}$.

Hence the result.

§ 2.7. Motion of the Centre of Inertia.

To show that the centre of inertia of a body moves as if all the mass of the body were collected at it and if all the external forces acting on the body were acting on it in directions parallel to those in which they act.

If $(\bar{x},\bar{y},\bar{z})$ be the coordinates of the centre of inertia of a body of mass M, then as in § 2.6, we have

$$\Sigma mx=M\bar{x},\ \Sigma my=M\bar{y},\ \Sigma mz=M\bar{z}.$$

Differentiating twice w.r.t. 't', we get

$$\Sigma m\ddot{x}=M\ddot{\bar{x}},\ \Sigma m\ddot{y}=M\ddot{\bar{y}} \text{ and } \Sigma m\ddot{z}=M\ddot{\bar{z}}. \quad ...(1)$$

But from the general equations of motion of a body, we get (see § 2.5)

$$\Sigma m\ddot{x}=\Sigma X,\ \Sigma m\ddot{y}=\Sigma Y \text{ and } \Sigma m\ddot{z}=\Sigma Z. \quad ...(2)$$

$\therefore$ From (1) and (2), we get

$$M\ddot{\bar{x}}=\Sigma X,\ M\ddot{\bar{y}}=\Sigma Y \text{ and } M\ddot{\bar{z}}=\Sigma Z.$$

These are the equations of motion of a particle of mass M placed at the centre of inertia of the body, and acted on by forces ΣX, ΣY, ΣZ parallel to the original directions of the forces acting on the different points of the body.

This proves the theorem.

Vector method. Consider a rigid body in motion. At time t let $\mathbf{r}$ be the position vector of a particle m of the body and $\mathbf{F}$ the external force acting on it. Then the equation of motion of the body is

$$\Sigma m\frac{d^2\mathbf{r}}{dt^2}=\Sigma\mathbf{F}. \quad ...(1)$$

If $\bar{\mathbf{r}}$ is the position vector of the centre of inertia of the body, then we have

$$\bar{\mathbf{r}}=\frac{\Sigma m\mathbf{r}}{\Sigma m}=\frac{\Sigma m\mathbf{r}}{M} \text{ or } \Sigma m\mathbf{r}=M\bar{\mathbf{r}}$$

$$\therefore\ \Sigma m\frac{d^2\mathbf{r}}{dt^2}=M\frac{d^2\bar{\mathbf{r}}}{dt^2}. \quad ...(2)$$

From (1) and (2), we have

$$M\frac{d^2\bar{\mathbf{r}}}{dt^2}=\Sigma\mathbf{F}. \quad ...(3)$$

Which is the vector form of the equation of motion of a particle of mass M placed at the centre of inertia of the body and acted upon by the external forces $\Sigma\mathbf{F}$.

Deduction of the equations of motion of the centre of inertia in scalar form.

Substituting $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $\mathbf{F}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ in (3) and equating the coefficients of $\mathbf{i},\mathbf{j},\mathbf{k}$ from the two sides we can get the equations of motion of the centre of inertia in scalar form.

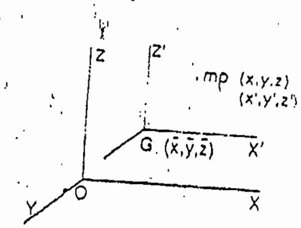
Note. The proposition discussed in § 2.7 is called *the principle of conservation of motion of translation*. From this it follows that the motion of C.G. is independent of rotation.

§ 2.8. Motion Relative to the Centre of Inertia.

To show that the motion of a body about its centre of inertia is the same as it would be if the centre of inertia were fixed and the same forces acted on the body.

[Meerut 91 (S) ; TDC 94(R), 96(P)]

Let $(\bar{x},\bar{y},\bar{z})$ be the coordinates of the centre of gravity (centre of inertia) G of the body referred to the rectangular axes OX, OY, OZ through a fixed point O. Let GX', GY', GZ' be the axes through G parallel to the axes OX, OY, OZ respectively.



If (x, y, z) and (x', y', z') are the coordinates of a particle of mass m at P referred to the coordinate axes OX, OY, OZ and

parallel axes GX', GY', GZ' respectively, then

$x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$.

$\therefore \ddot{x} = \ddot{\bar{x}} + \ddot{x}'$, $\ddot{y} = \ddot{\bar{y}} + \ddot{y}'$, $\ddot{z} = \ddot{\bar{z}} + \ddot{z}'$.

Now consider the equation $\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma(yZ - zY)$, which becomes

$$\Sigma m\{(\bar{y} + y')(\ddot{\bar{z}} + \ddot{z}') - (\bar{z} + z')(\ddot{\bar{y}} + \ddot{y}')\}$$
$$= \Sigma\{(\bar{y} + y')Z - (\bar{z} + z')Y\}$$

or
$$\Sigma m(y'\ddot{z}' - z'\ddot{y}') + \bar{y}\ddot{\bar{z}}\,\Sigma m + \bar{y}\,\Sigma m\ddot{z}' + \ddot{\bar{z}}\,\Sigma my'$$
$$- \bar{z}\ddot{\bar{y}}\,\Sigma m - \bar{z}\,\Sigma m\ddot{y}' - \ddot{\bar{y}}\,\Sigma mz'$$
$$= \Sigma(y'Z - z'Y) + \bar{y}\,\Sigma Z - \bar{z}\,\Sigma Y. \quad \ldots(1)$$

Now referred to GX', GY', GZ' as axes the coordinates of G are $(0, 0, 0)$.

$$\therefore \frac{\Sigma mx'}{\Sigma m} = 0 \text{ or } \Sigma mx' = 0.$$

Similarly, $\Sigma my' = 0$, $\Sigma mz' = 0$.

$\therefore \Sigma m\ddot{x}' = 0$, $\Sigma m\ddot{y}' = 0$, $\Sigma m\ddot{z}' = 0$.

Also from § 2.7, we have $M\ddot{\bar{x}} = \Sigma X$, $M\ddot{\bar{y}} = \Sigma Y$, $M\ddot{\bar{z}} = \Sigma Z$.

Thus, from eqn. (1), we get

$$\Sigma m(y'\ddot{z}' - z'\ddot{y}') + \bar{y}\ddot{\bar{z}}M - \bar{z}\ddot{\bar{y}}M = \Sigma(y'Z - z'Y) + \bar{y}\,\Sigma Z - \bar{z}\,\Sigma Y$$

or $\Sigma m(y'\ddot{z}' - z'\ddot{y}') + \bar{y}\,\Sigma Z - \bar{z}\,\Sigma Y = \Sigma(y'Z - z'Y) + \bar{y}\,\Sigma Z - \bar{z}\,\Sigma Y$

or
$$\Sigma m(y'\ddot{z}' - z'\ddot{y}') = \Sigma(y'Z - z'Y).$$

Similarly, we get the other two equations as

$$\Sigma m(z'\ddot{x}' - x'\ddot{z}') = \Sigma(z'X - x'Z)$$

and
$$\Sigma m(x'\ddot{y}' - y'\ddot{x}') = \Sigma(x'Y - y'X).$$

But these equations are the same as would have been obtained if we had regarded the centre of gravity as fixed point.

Hence the proposition.

Vector method. Consider a rigid body in motion. At time t, let $\bar{r}$ be the position vector of the centre of inertia G of a rigid body of mass M. Let m be the mass of a particle of the body and r its position vector referred to the fixed origin O and r' its position vector referred to the centre of inertia $\bar{r}$.

$$\therefore r = \bar{r} + r', \text{ so that } \frac{d^2r}{dt^2} = \frac{d^2\bar{r}}{dt^2} + \frac{d^2r'}{dt^2}$$

The moment vector equation of the rigid body is

$$\Sigma mr \times \frac{d^2r}{dt^2} = \Sigma r \times F,$$

or
$$\Sigma\left\{m(\bar{r} + r') \times \left(\frac{d^2\bar{r}}{dt^2} + \frac{d^2r'}{dt^2}\right)\right\} = \Sigma\{(\bar{r} + r') \times F\}$$

or
$$\Sigma mr' \times \frac{d^2r'}{dt^2} + \bar{r} \times \frac{d^2\bar{r}}{dt^2}\,\Sigma m + \bar{r}\,\Sigma m\frac{d^2r'}{dt^2} + \frac{d^2\bar{r}}{dt^2}\,\Sigma mr'$$

$$= \bar{r}\,\Sigma F + \Sigma r' \times F. \quad \ldots(1)$$

Now position vector of the centre of inertia G of the body referred to G as origin is O.

$$\therefore \frac{\Sigma mr'}{\Sigma m} = 0, \text{ i.e. } \Sigma mr' = 0, \text{ so that } \Sigma m\frac{d^2r'}{dt^2} = 0.$$

Also the equation of motion of the centre of inertia is

$$M\frac{d^2\bar{r}}{dt^2} = \Sigma F.$$

$\therefore$ From eqn. (1), we have

$$\Sigma mr' \times \frac{d^2r'}{dt^2} + \bar{r} \times \left(\frac{d^2\bar{r}}{dt^2}\,.\,M\right) + 0 + 0 = \bar{r} \times \Sigma F + \Sigma r' \times F.$$

or
$$\Sigma mr' \times \frac{d^2r'}{dt^2} + \bar{r} \times \Sigma F = \bar{r} \times \Sigma F + \Sigma r' \times F$$

or
$$\Sigma mr' \times \frac{d^2r'}{dt^2} = \Sigma r' \times F. \quad \ldots(2)$$

Which is the vector equation of motion of a rigid body when the centre of inertia is regarded as a fixed point.

Deduction of the corresponding equations in scalar form.

If (x, y, z) and (x', y', z') are the cartesian coordinates of the particles m referred to the rectangular axes through the fixed point O and the parallel axes through the centre of inertia G respectively, then we have

$$r = xi + yj + zk \text{ and } r' = x'i + y'j + z'k.$$

Let $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of G referred to the axes through O, then

$$\bar{r} = \bar{x}i + \bar{y}j + \bar{z}k.$$

Also if X, Y, Z are the components of external force F parallel to the axes, then

$$F = Xi + Yj + Zk.$$

Substituting in (2), we have

$$\Sigma m\{(x'i + y'j + z'k) \times (\ddot{x}'i + \ddot{y}'j + \ddot{z}'k)\}$$
$$= \Sigma\{(x'i + y'j + z'k) \times (Xi + Yj + Zk)\}$$

or
$$\Sigma m\{(y'\ddot{z}' - z'\ddot{y}')i + (z'\ddot{x}' - x'\ddot{z}')j + (x'\ddot{y}' - y'\ddot{x}')k\}$$
$$= \Sigma\{(y'Z - z'Y)i + (z'X - x'Z)j + (x'Y - y'X)k\}.$$

Equating the coefficients of i, j, k from the two sides we shall get the equations of motion of the body in scalar form referred to the centre of inertia as fixed point.

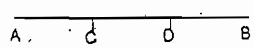
Note 1. The proposition discussed in § 2.8 is called the *principle of conservation of motion of rotation*. From this it follows that the motion round the centre of inertia is independent of its motion of translation.

Note 2. The two propositions discussed in § 2.7 and 2.3 together prove the principle of the independence of the motion of translation and rotation.

EXAMPLES

Ex. 1. *A rod revolving on a smooth horizontal plane about one end, which is fixed, breaks into two parts, what is the subsequent motion of the two parts.*

Sol. Let the rod AB revolving about the end A on a smooth horizontal plane break into two parts AC and CB. Clearly the part AC will continue to rotate about A with the same angular velocity.



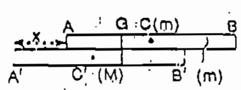
The part CB at the instant of breaking acquires the same angular velocity and its centre of gravity D has a linear velocity. Hence this part CB will fly off along the tangent line (the direction of linear velocity) at D to the circle with A as centre and AD as radius. Also, since the motion of a body about its centre of inertia is the same as if the centre of inertia was fixed and the same forces acted on the body, the part CB will continue rotating about D with the same angular velocity.

Hence the part CB will move along the tangent at D to the circle with A as centre and AD as radius with the velocity acquired by its centre of gravity at the instant of breaking and this part will also go on rotating about D with the same angular velocity.

Ex. 2. *A rough uniform board, of mass m and length 2a, rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Find the distance through which the board moves in this time.*

[Meerut TDC 91(P), 91(S), 92(P), 93(P), 93(BP), 95(P);]

Sol. Here the external forces are (i), the weights of the board and the man acting vertically downwards and (ii) the reaction of the horizontal plane acting vertically upwards. Thus there are no external forces in the horizontal direction, therefore by D' Alembert's principle, the C.G. of the system will remain at rest. As a matter of fact as the man moves forward, the board slips backwards, keeping the poistion of C.G. of the system unchanged.



Let AB be the position of the board when the man of mass M is at A.

$\therefore$ Distance of C.G. of the system from A (towards B)

$$= \frac{M.0 + m.AC}{M+m} = \frac{M.0 + m.a}{M+m} = \frac{ma}{M+m} = x_1 \text{ (say)}, \quad (\because AG = BG = a)$$

Let $A'B'$ be the positioin of the board when the man reaches the other end B of the board. If the board slips through a distance $AA' = x$ (backwards) during the time the man walks from A to B, then in this position the distance of C.G. of the system from A (towards B)

$$= \frac{M.AB' + m.AC'}{M+m} = \frac{M.(2a-x) + m(a-x)}{M+m} = x_2 \text{ (say)}$$

Since the position of the C.G., 'G' of the system remains unchanged

$\therefore x_1 = x_2$.

or $\dfrac{ma}{M+m} = \dfrac{M(2a-x) + m(a-x)}{M+m}$

or $ma = 2aM + ma - (M+m)x$ or $x = 2aM/(m+M)$

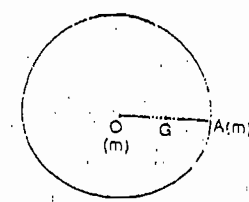
Which is the required distance.

Ex. 3. *A circular board is placed on a smooth horizontal plane and a boy runs round the edge of it at a uniform rate, what is the motion of the board.*

Sol. Let M be the mass and O the centre of the board. If initially the boy is at the point A on the edge of the board then the C.G. 'G' of the system will be on the radius OA; such that

$$OG = \frac{M.0 + m.a}{M+m} = \frac{ma}{M+m}.$$

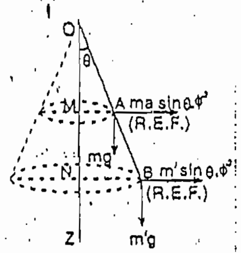
Since the external forces, weight of the board and the boy act vertically downwards and the reaction of the smooth horizontal plane act vertically upwards, therefore there is no external force in the horizontal direction during the motion. Thus by D' Alembert's principle the C.G. 'G' of the system will remain at rest. Hence as the boy runs round the edge of the board with uniform speed, the centre O of the board will describe a circle of radius $OG = ma/(M+m)$ round the centre at G.



Ex. 4. *Find the motion of the rod OAB, with two masses m and m' attached to it at A and B respectively, when it moves round the vertical as a conical pendulum with uniform angular velocity, the angle θ which the rod makes with the vertical being constant.*

Sol. Let OAB be the rod with two masses m and m' attached at A and B respectively such that $OA = a$ and $OB = b$. When the rod OAB moves round the vertical as a conical pendulem with uniform angular velocity,

making constant angle θ with the vertical the masses m and m' move in circles on horizontal planes with radii $a\sin\theta$ and $b\sin\theta$ and centres at M and N respectively. The motion about the vertical being with uniform angular velocity, the effective forces are entirely in wards. Let ϕ be the angle that the plane through OAB makes with a fixed vertical plane through OZ, then the only effective forces on the particles



are $ma\sin\theta\,\dot\phi^2$ and $m'b\sin\theta\,\dot\phi^2$ along AM and BN respectively.

By D' Alembert's principle the external forces, weights mg, $m'g$ and the reaction at O, and the reversed effective forces $ma\sin\theta\,\dot\phi^2$ along MA and $m'b\sin\theta\,\dot\phi^2$ along NB will keep the rod in equilibrium.

To avoid reaction at O, taking moment about the point O, we get

$$ma\sin\theta\,\dot\phi^2 \,.\, OM + m'b\sin\theta\,\dot\phi^2\,.\,ON - mg\,.\,MA - m'g\,.\,NB = 0$$

or $(ma\sin\theta\,.\,a\cos\theta + m'b\sin\theta\,.\,b\cos\theta)\,\dot\phi^2 = g\,(ma\sin\theta + m'b\sin\theta)$

or $\dot\phi^2 = \dfrac{(ma+m'b)\,g}{(ma^2+m'b^2)\cos\theta}$ $\quad(\because \sin\theta \neq 0)$

Which will determine the motion of the rod.

5(c) Ex. 5. *A uniform rod OA, of length $2a$, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is either zero or $\cos^{-1}(3g/4a\omega^2)$.*

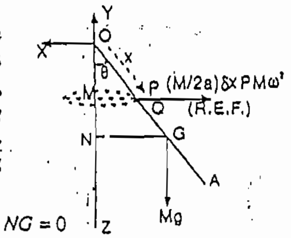
[Meerut TDC 92, 94(P), 95(BP) ; Rohilkhand 83]

Sol. Let the rod OA of length $2a$ and mass M revolve with uniform angular velocity ω about the vertical OZ through O, making a constant angle α to OZ. Let $PQ = \delta x$ be an element of the rod at a distance x from O. The mass of the element PQ is $\dfrac{M}{2a}\delta x$.

This element PQ will make a circle in the horizontal plane with radius PM $(= x\sin\alpha)$ and centre at M. Since the rod revolve with uniform angular velocity, the only effective force on this element is $\dfrac{M}{2a}\delta x\,.\,PM\,.\,\omega^2$ along PM.

Thus the reversed effective force on the element PQ is

$\dfrac{M}{2a}\delta x\,.\,x\sin\alpha\,.\,\omega^2$ along MP.



Now by D' Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight mg and rection at O are in equilibrium. To avoid reaction at O, taking moment about O, we get

$$\sum\left(\frac{M}{2a}\delta x\,.\,\omega^2\,.\,\sin\alpha\right).\,OM - Mg\,.\,NG = 0$$

or $\displaystyle\int_0^{2a}\frac{M}{2a}\omega^2x^2\sin\alpha\cos\alpha\,dx$

$-Mg\,.\,a\sin\alpha = 0, \quad (\because OM = x\cos\alpha)$

or $\dfrac{M}{2a}\omega^2\,.\left\{\frac{1}{3}(2a)^3\right\}.\sin\alpha\cos\alpha - Mg\,a\sin\alpha = 0$

or $Mg\,a\sin\alpha\left(\dfrac{4a}{3g}\omega^2\cos\alpha - 1\right) = 0$

$\therefore$ either $\sin\alpha = 0$ i.e. $\alpha = 0$

or $\dfrac{4a}{3g}\omega^2\cos\alpha - 1 = 0$, i.e. $\cos\alpha = \dfrac{3g}{4a\omega^2}$

Hence, the rod is inclined at an angle zero or $\cos^{-1}\left(\dfrac{3g}{4a\omega^2}\right)$

Note. If $\omega^2 < \dfrac{3g}{4a}$, then $\cos\alpha > 1$, $\therefore$ in this case $\cos\alpha = \dfrac{3g}{4a\omega^2}$ gives an impossible value of α i.e. when $\omega^2 < \dfrac{3g}{4a}$, then $\alpha = 0$ is the only possible value of α.

Ex. 6 *A rod, of length $2a$, revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α, show that $\omega^2 = 3g/(4a\cos\alpha)$.*

Prove also that direction of reaction at the hinge makes with the vertical an angle $\tan^{-1}\left(\frac{3}{4}\tan\alpha\right)$

Sol. Refer figure of last Ex. 5.

Proceeding as in last Ex. 5, we get

$$\cos\alpha = \frac{3g}{4a\omega^2}, \quad i.e.\ \omega^2 = \frac{3g}{4a\cos\alpha} \quad ...(1)$$

Second Part :

If X and Y are the horizontal and vertical components of the reaction at the hinge O, as shown in the figure, then resolving the forces horizontally and vertically we get

$$X = \Sigma \frac{M}{2a}\,\delta x.\,PM.\omega^2 = \int_0^{2a} \frac{M}{2a}\,\omega^2 x \sin\alpha\, dx \qquad (\because PM = x\sin\alpha)$$

$$= \frac{M}{2a}\,\omega^2 \left\{\tfrac{1}{2}(2a)^2\right\}\sin\alpha = Ma\omega^2\sin\alpha$$

and $Y = Mg$.

If the reaction at O make an angle θ with the vertical, then

$$\tan\theta = \frac{X}{Y} = \frac{Ma\omega^2\sin\alpha}{Mg} = \frac{a}{g}\left(\frac{3g}{4a\cos\alpha}\right)\sin\alpha \qquad \text{[substituting from (1)]}$$

or $$\theta = \tan^{-1}\left(\tfrac{3}{4}\tan\alpha\right)$$

Ex. 7. *Two uniform spheres, each of mass M and radius a, are firmly fixed to the ends of two uniform thin rods, each of mass m and length l, and the other ends of the rods are freely hinged to a point O. The whole system revolves as in the Governor of a steam Engine, about a vertical line through O with the angular velocity ω. Show that when the motion is steady, the rods are inclined to the vertical at an angle θ, given by the equation*

$$\cos\theta = \frac{g}{\omega^2}\cdot\frac{M(l+a)+\frac{1}{2}ml}{M(l+a)^2+\frac{1}{3}ml^2}$$

Sol. Let OA, OB be two rods, each of length l and mass M attached freely to a point O. Let C and D be the centres of two spheres each of mass M and radius a attached to the other ends of the two rods. When the motion is steady let θ be the inclination of the rods to the vertical.

Consider the motion of one of the spheres, say the sphere with centre at C. Let δx be an element PQ of the rod at P such that $OP = x$, then mass of the element is $(m/l)\,\delta x$.

The reversed effective force at the element δx at P is

$$\frac{m}{l}\,\delta x.\omega^2.PM = \frac{m}{l}\,\delta x\,\omega^2 x\sin\theta \text{ alone } MP.$$

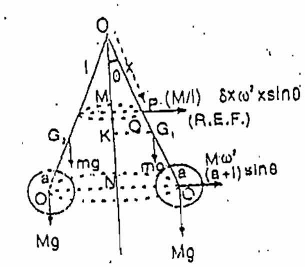
And the reversed effective force on the sphere is

$$M\omega^2 CN = M\omega^2(a+l)\sin\theta \text{ along } CN.$$

The external forces on the rod OA and sphere with centre at C are the weights mg and the Mg and reaction at O.

To avoid reaction at O, taking moment about O, we get

$$\Sigma\frac{m}{l}\,\delta x\,\omega^2 x\sin\theta.OM + M\omega^2(a+l)\sin\theta\,.\,ON$$



$$-mg\,.\,KG_1 - Mg\,.\,NC = 0$$

or $$\int_0^l \frac{m}{l}\,\omega^2 x^2\sin\theta\cos\theta\,dx + M\omega^2(a+l)^2\sin\theta\cos\theta$$

$$-mg\,\frac{1}{2}\sin\theta - Mg(a+l)\sin\theta = 0$$

$$[\omega^2.\{\tfrac{1}{3}ml^2 + M(a+l)^2\}.\cos\theta - g\{\tfrac{1}{2}ml + M(a+l)\}]\sin\theta = 0$$

$\therefore$ Either $\sin\theta = 0$, *i.e.* $\theta = 0$ which is inadmissible.

$$\therefore\ \omega^2\{\tfrac{1}{3}ml^2 + M(a+l)^2\}\cos\theta - g\{\tfrac{1}{2}ml + M(a+l)\} = 0$$

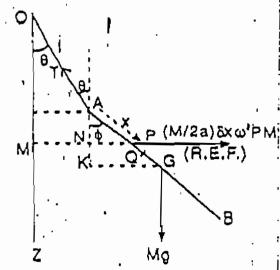
or $$\cos\theta = \frac{g}{\omega^2}\cdot\frac{M(a+l)+\frac{1}{2}ml}{M(a+l)^2+\frac{1}{3}ml^2}$$

Ex. 8. *A rod of length $2a$, is suspended by a string of length l, attached to one end, if the string and rod revolve about the vertical with uniform angular velocity, and their inclinations to the vertical be θ and ϕ respectively, show that*

$$\frac{3l}{a} = \frac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}.$$

[Meerut, 86, 88, 90; T.D.C. 91, 93(P), 94, 94(P), 96, Kanpur 81, 83; Raj. 83]

Sol. Let the rod AB of length $2a$ and mass m be suspended by a string OA of length l. Let θ and ϕ be the inclinations of the string and the rod to the vertical respectively.



Consider an element PQ ($=\delta x$) of the rod at a distance x from A, then mass of this element is $(M/2a)\,\delta x$.

As the rod revolve with uniform angular velocity ω, about the vertical OZ, the element δx will describe a circle of radius PM in the horizontal plane.

The reversed effective force on element δx is

$\frac{M}{2a}\delta x.\omega^2 . PM = \frac{M}{2a}\delta x.\omega^2 . (l \sin\theta + x\sin\phi)$, along MP.

The external forces acting on the rod are (i) tension T at A along AO, and (ii) its weight Mg acting vertically down wards at its middle point G.

Resolving horizontally and vertically the forces acting on the rod, we get

$$T\sin\theta = \Sigma\frac{m}{2a}\delta x\,\omega^2(l\sin\theta + x\sin\phi)$$

or $$T\sin\theta = \frac{M}{2a}\omega^2\int_0^{2a}(l\sin\theta + x\sin\phi)\,dx$$

or $$T\sin\theta = \frac{M}{2a}\omega^2\left[lx\sin\theta + \tfrac{1}{2}x^2\sin\phi\right]$$

or $$T\sin\theta = M\omega^2(l\sin\theta + a\sin\phi). \quad ...(1)$$

and $$T\cos\theta = Mg. \quad ...(2)$$

Now taking moment about A of all the forces acting on the rod AB, we get

$$-Mg\,.\,KG + \Sigma\frac{M}{2a}\delta x\omega^2(l\sin\theta + x\sin\phi)\,.\,AN = 0$$

or $$Mga\sin\phi = \frac{M\omega^2}{2a}\int_0^{2a}(l\sin\theta + x\sin\phi)\,x\cos\phi\,d\phi$$

$$= \frac{M}{2a}\omega^2\left[\frac{1}{2}lx^2\sin\theta + \frac{1}{3}x^3\sin\phi\right]_0^{2a}\cos\phi$$

$$= a M\omega^2\left(l\sin\theta + \frac{4a}{3}\sin\phi\right).\cos\theta$$

or $$g\tan\phi = \tfrac{1}{3}\omega^2(3l\sin\theta + 4a\sin\phi). \quad ...(3)$$

Dividing (1) by (2), we get

$$\tan\theta = \frac{\omega^2}{g}(l\sin\theta + a\sin\phi)$$

or $$\omega^2 = g\tan\theta/(l\sin\theta + a\sin\phi).$$

Substituting in (3), we get

$$g\tan\phi = \frac{1}{3}\frac{g\tan\theta\,(3l\sin\theta + 4a\sin\phi)}{(l\sin\theta + a\sin\phi)}$$

or $$3\tan\phi\,(l\sin\theta + a\sin\phi) = \tan\theta\,(3l\sin\theta + 4a\sin\phi)$$

or $$3l\sin\theta\,(\tan\phi - \tan\theta) = \sin\phi\,(4a\tan\theta - 3a\tan\phi)$$

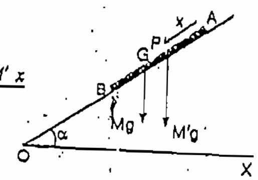
or $$\frac{3l}{a} = \frac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}$$

Ex. 9. *A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M', starting from the upper end, walks down the plank so that it does not move, show that he gets to the other end in time*

$$\sqrt{\left\{\frac{2M'a}{(M+M')g\sin\alpha}\right\}},$$ *where a is the length of the plane.*

[Meerut, 84, 85, 87, 89, TDC 94(R), 97; Kanupr 82;]

Sol. Let the plank AB of mass M and length a rest along the line of greatest slope of a smooth plane inclined at an angle α to the horizon. A man of mass M' starts moving down the plank from the upper end A. Let the man move down the plank through a distance $AP = x$ in time t. Since the plank does not move, therefore if $\bar{x}$ is the distance of the C. G. of the plank and the man from A in this position, then



$$\bar{x} = \frac{M\,.\,AG + M'\,.\,AP}{M+M'} = \frac{M\,.\,(a/2) + M'x}{M+M'}$$

Differentiating twice w. r. t. 't', we get

$$\ddot{\bar{x}} = \frac{M'}{M+M'}\ddot{x}, \quad ...(1)$$

Now the total weight $(M+M')g$ will act vertically downwards at the C. G. of the system.

$\therefore$ The equation of motion of the C. G. of the system is given by

$$(M+M')\ddot{\bar{x}} = (M+M')\,g\sin\alpha. \qquad \ldots(2)$$

$\therefore$ From (1) and (2), we get

$$M'\ddot{x} = (M+M')\,g\sin\alpha.$$

Integrating, we get $M'\dot{x} = (M+M')\,g\sin\alpha\,.\,t + c_1$.

But initially when $t=0,\ \dot{x}=0 \quad \therefore\ c_1 = 0$.

$\therefore\ M'\dot{x} = (M+M')\,g\sin\alpha\,.\,t$.

Integrating again, we get $M'x = (M+M')\,g\sin\alpha\,.\,\frac{1}{2}t^2 + c_2$.

Initially when $t=0,\ x=0. \quad \therefore\ c_2 = 0$.

$\therefore\ M'x = (M+M')\,g\sin\alpha\,.\,\frac{1}{2}t^2$.

or
$$t = \sqrt{\left\{\frac{2M'x}{(M+M')\,g\sin\alpha}\right\}}$$

Putting $x = AB = a$, the time to reach the other end B of the plank is given by

$$t = \sqrt{\left\{\frac{2M'x}{(M+M')\,g\sin\alpha}\right\}}$$

§ 2.9 Impulse of a Force.

The impulse of a force acting on a particle in any interval of time is defined to be the change in momentum produced.

Thus due to a force F, if the velocity of a particle of mass m changes from v_1 to v_2 in time t, then the impulse I is given by

$$I = mv_2 - mv_1 = m(v_2 - v_1)$$

$$= m\int_{t_1}^{t_2} dv = \int_{t_1}^{t_2} m\frac{dv}{dt}\,dt$$

$$= \int_{t_1}^{t_2} F\,.\,dt \text{ since } F = m\frac{dv}{dt}$$

Thus the impulse of the force F is the time integral of the force.

Now let the force F increase indefinitely and the interval $(t_2 - t_1)$ decrease to a very small quantity such that the time integral $\int_{t_1}^{t_2} F\,dt$ remains finite. Such a force is called impulsive force.

Note. The impulsive force can be measured by the change in momentum produced.

§ 2.10 An Important Rule.

The effect of an impulse on a body remains the same even if all the finite forces acting simultaneously on it are neglected.

let I be the impulse due to an impulsive force F which acts for a time T. If f is the finite force acting simultaneously on the body, then

$$m(v_2 - v_1) = \int_0^T F\,dt + \int_0^T f\,dt = I + fT$$

Since $fT \to 0$ as $T \to 0 \quad \therefore\ I = m(v_2 - v_1)$,

Which shows that the finite force f acting on the body may be neglected in forming the equations.

§ 2.11 General Equations of Motion under Impulsive Forces.

To determine the general equations of motion of a system acted on by a number of impulses at a time. [Meerut TDC 96 (BP)]

Let u, v, w and u', v', w', be the velocities parallel to the axes respectively before and after the action of impulsive forces on the particle of mass m. If X', Y', Z', are the resolved parts of the total impulse on m parallel to the axes, then

$$\Sigma m(u' - u) = \Sigma\int_0^\tau X\,dt = \Sigma X'$$

or $\quad \Sigma mu' - \Sigma mu = \Sigma X' \qquad \ldots(1)$

Similarly $\quad \Sigma mv' - \Sigma mv = \Sigma Y' \qquad \ldots(2)$

and $\quad \Sigma mw' - \Sigma mw = \Sigma Z' \qquad \ldots(3)$

i.e. the change in momentum parallel to any of the axes is equal to the total impulse of the external forces parallel to the corresponding axis.

Hence the change in momentum parallel to any of the axes of the whole mass M, supposed collected at the centre of inertia and moving with it, is equal to the impulse of the external force parallel to the corresponding axis.

Again we have the equation

$$\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma m(yZ - zY)$$

or
$$\frac{d}{dt}\Sigma m(y\dot{z} - z\dot{y}) = \Sigma m(yZ - zY)$$

Integrating this, we have

$$\left[\Sigma m(y\dot{z} - z\dot{y})\right]_0^\tau = \Sigma\left[y\int_0^\tau Z\,dt - z\int_0^\tau Y\,dt\right]$$

Since the time interval τ is so small that the body has not moved during this interval, we may take x, y, z, as constants. Thus the above equation becomes

$$\Sigma m\{y(w' - w) - z(v' - v)\} = \Sigma(yZ' - zY')$$

or
$$\Sigma m(yw' - zv') - \Sigma m(yw - zv) = \Sigma(yZ' - zY') \qquad \ldots(4)$$

Similarly,

$$\Sigma m(xv' - yu') - \Sigma m(xv - yu) = \Sigma(xY' - yX') \quad ...(5)$$

and

$$\Sigma m(zu' - xw') - \Sigma m(zu - xw) = \Sigma(zX' - xZ') \quad ...(6)$$

Hence the change in the moment of momentum about any of the axes is equal to the moment about that axis of the impulses of the external forces.

Vector method. Let I and I′ be the resultant external and internal impulses acting on the particle of mass m at P. Also let the velocity of m change from v_1 to v_2 then.

∵ Impulses = change in momentum

$$\therefore \; \mathbf{I} + \mathbf{I}' = m(\mathbf{v}_2 - \mathbf{v}_1) \quad ...(1)$$

or

$$\Sigma \mathbf{I} + \Sigma \mathbf{I}' = \Sigma m\mathbf{v}_2 - \Sigma m\mathbf{v}_1$$

But $\Sigma \mathbf{I}' = 0$, by Newton's third law

∴ we get, $\Sigma \mathbf{I} = \Sigma m\mathbf{v}_2 - \Sigma m\mathbf{v}_1$

i. e. the total external impulse applied to the system of particles is equal to the change of linear momentum produced.

Now let $\vec{OP} = \mathbf{r}$; then from (1), we get

$$\Sigma \mathbf{r} \times (\mathbf{I} + \mathbf{I}') = \Sigma \mathbf{r} \times m(\mathbf{v}_2 - \mathbf{v}_1)$$

or

$$\Sigma \mathbf{r} \times \mathbf{I} = \Sigma \mathbf{r} \times m\mathbf{v}_2 - \Sigma \mathbf{r} \times m\mathbf{v}_1 \quad (\text{Since } \Sigma \mathbf{r} \times \mathbf{I}' = 0)$$

i. e. the total vector sum of the moments of the external impulses about any point O is equal to the increase in the angular momentum produced about the same point.

EXAMPLES

Ex. 10. *Two persons are situated on a perfectly smooth horizontal plane at a distance a from each other. One of the persons, of mass M throws a ball of mass m towards the other which reaches him in time t, prove that the first person will begin to slide along the plane with velocity ma/(Mt).*

Sol. Let I be the impulse between the ball and the first person. If the first person throws a ball with the velocity u and begins to slide along the plane with velocity v,

then since, impulse = change in momentum

$$\therefore \; I = M(v - 0) \quad \text{(for the first person)}$$

and

$$I = m(u - 0) \quad \text{(for the ball)}$$

$$\therefore \; mu = Mv \quad ...(1)$$

Since the ball reaches the second person in time t,

$$\therefore \; a = ut \quad ...(2)$$

From (2), $u = a/t$. Substituting in (1), we get

$$v = \frac{ma}{Mt}$$

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Ex. 11. *A cannon of mass M, resting on a rough horizontal plane of coefficient of friction μ, is fired with such a charge that the relative velocity of the ball and cannon at the moment when it leaves the cannon is u. Show that the cannon will recoil a distance*

$$\left(\frac{mu}{M+m}\right)^2 \cdot \frac{1}{2\mu g},$$

along the plane, m being the mass of the ball.

[Meerut TDC 94(R)]

Sol. Let I be the impulse between the cannon and the ball. If v is the velocity of the ball and V the velocity of cannon in opposite direction, then the relative velocity of the ball and cannon at the moment the ball leaves the cannon is

$$v + V = u \text{ (given)} \quad ...(1)$$

Also since, impulse = change in momentum

$$\therefore \; I = m(v - 0) \quad \text{(for the ball)}$$

and

$$I = M(V - 0) \quad \text{(for the cannon)}$$

$$\therefore \; mv = MV \text{ or } v = \frac{MV}{m} \quad ...(2)$$

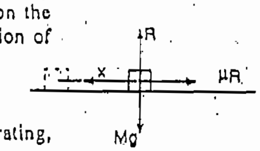
Substituting from (2), in (1), we get

$$\frac{MV}{m} + V = u \text{ or } V(M+m) = mu$$

or

$$V = mu/(M+m) \quad ...(3)$$

If the cannon moves through a distance x in the direction opposite to the direction of motion of the ball in time t, then on the rough plane, for the cannon the equation of motion is



$$M\ddot{x} = -\mu R = -\mu Mg$$

or

$$\ddot{x} = -\mu g$$

Multiplying both sides by $2\dot{x}$ and integrating, we get

$$\dot{x}^2 = -2\mu gx + C$$

But initially when $x = 0$, $\dot{x} = V$ (Starting velocity of the cannon)

$$\therefore \; C = V^2$$

$$\therefore \; \dot{x}^2 = V^2 - 2\mu gx.$$

When the cannon comes to rest $\dot{x} = 0$,

$$\therefore \; 0 = V^2 - 2\mu gx$$

or

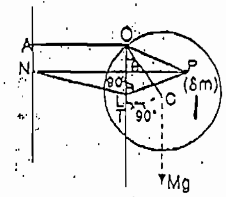
$$x = \frac{V^2}{2\mu g} = \left(\frac{mu}{M+m}\right)^2 \cdot \frac{1}{2\mu g} \quad \text{[Substituting from (3)]}$$

which is the required distance.

MISCELLANEOUS EXAMPLES

Ex. 12. A thin circular disc of mass M and radius a, can turn freely about a thin axis OA, which is perpendicular to its plane and passes through a point O of its circumference. The axis OA is compelled to move in a horizontal plane with angular velocity ω about its end A. Show that the inclination θ to the vertical of the radius of the disc through O is $\cos^{-1}(g/a\omega^2)$, unless $\omega^2 < (g/a)$ and then θ is zero.

Sol. Let C be the centre of the thin circular disc of mass M which can turn about a thin horizontal axis OA perpendicular to its plane and passing through a point O of its circumference. When the axis OA turns horizontally round A, the disc will be raised in its own vertical plane. Let the radius OC turn through an angle θ to the vertical in time t.



Consider an element of mass δm of the disc at P. Let PN and PL be the perpendiculars from P on the verticals through A and O respectively. Then obviously the ΔPNL will be in the horizontal plane and NL will be parallel

cribe a circle of radius PN with constant angular velocity ω about the vertical through A. Thus the reversed affective force on the element δm at P along NP is $\delta m . NP . \omega^2$.

But $\vec{NP} = \vec{NL} + \vec{LP}$.

$\therefore\ \delta m\, \omega^2\, \vec{NP} = \delta m \omega^2\, \vec{NL} + \delta m\, \omega^2\, \vec{LP}$,

Thus the reversed effective force $\delta m\omega^2\, NP$ along NP is equivalent to forces $\delta m\omega^2\, NL$ along NL and $\delta m\omega^2\, LP$ along LP. The external forces on the disc are its weight Mg acting verticaly downwards at it centre C and the reaction at O.

By D' Alembert's principle, reversed effective forces and the external forces keep the system in equilibrium.

To avoid reaction at O, we take the moment about the axis OA.

The forces $\delta m\omega^2\, NL$ along NL acts parallel to OA, hence its moment about OA vanishes.

$\therefore$ Taking moment of all the forces about OA, we have

$$Mg.CT = \Sigma\, \delta m\omega^2\, LP\,.\,OL + 0$$

or $Mg\, a \sin\theta = \omega^2\, \Sigma\, \delta m\, LP\,.\,OL$

$= \omega^2$.(P.I. of the disc about OL and the horizontal line through O)

$= \omega^2$.(P.I. of the disc about the parallel lines through C.G. 'C' + P.I. of whole mass M at C.G. 'C' about the horizontal and vertical lines through O)

$= \omega^2 (O + M.CT.OT) = \omega^2\, Ma \sin\theta . a\cos\theta$

or $\sin\theta\,(g - a\omega^2 \cos\theta) = 0$

which gives, either $\sin\theta = 0$ *i.e.* $\theta = 0$,

or $g - a\omega^2\cos\theta = 0$, *i.e.* $\cos\theta = g/a\omega^2$ or $\theta = \cos^{-1}(g/a\omega^2)$.

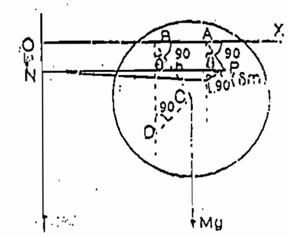
If $\omega^2 < (g/a)$, $\cos\theta > 1$, which is not possible and hence in this case $\theta = 0$ is the only possible value.

Ex. 13. A thin heavy disc can turn freely about an axis in its own plane, and this axis revolves horizontally with a uniform angular velocity ω about a fixed point on itself. Show that the inclination θ of the plane of the disc to the vertical is given by $\cos\theta = (gh/k^2\omega^2)$ where h is the distance of the centre of inertia of the disc from the axis and k is the radius of gyration of the disc about the axis. If $\omega^2 < gh/k^2$, prove that the plane of the disc is vertical.

Sol. Let C be the centre of a thin heavy disc of mass M which can turn about an axis OX in its own plane. When the axis revolves horizontally with a uniform angular velocity ω about a fixed point O on itself, the disc will turn about OX. Let θ be the inclination of the plane of the disc to the vertical at time t. If CB is the perpendicular from C on OX, then

$$CB = h \text{ (given)}$$

Consider an element of mass δm of the disc at P. Let PN be the perpendiculars from P on the vertical through O and PA perpendicular on OX. Let PL be the perpendicular from P on the vertical through A, then obviously the ΔPNL will be in a horizontal plane and NL will be parallel to OA. Also $\angle PAL = \theta = \angle CBD$, where CD is perpendicular from C on the vertical through B.



Now P will describe a circle of radius PN with constant angular velocity ω about the vertical through the fixed point O.

Thus the reversed effective force on the element δm at P along NP is $\delta m \cdot NP \cdot \omega^2$.

But $\vec{NP} = \vec{NL} + \vec{LP}$.

$\therefore \delta m \omega^2 \vec{NP} = \delta m \omega^2 \vec{NL} + \delta m \omega^2 \vec{LP}$.

Thus the reversed effective force $\delta m \omega^2 NL$ along NP is equivalent to the forces $\delta m \omega^2 NL$ along NL and $\delta m \omega^2 LP$ along LP. The external forces on the disc are its weight Mg acting vertically down wards at its centre C and the reaction at the axis OX.

By D' Alembert's principle, reversed effective forces and the external forces keep the system in equilibrium.

To avoid reaction on the axis OX, we take the moment about the axis OX.

The force $\delta m \omega^2 NL$ along NL is parallel to OX, hence its moment about OX vanishes.

Therefore taking moment of all forces about OX, we have

$$Mg.DC = \Sigma\, \delta m \omega^2 LP.AL + O$$

or $$Mgh \sin\theta = \omega^2\, \Sigma\, \delta m \,.\, AP \sin\theta \,.\, AP \cos\theta.$$

$$= \omega^2 \sin\theta \cos\theta\, \Sigma\, \delta m\, AP^2$$

$$= \omega^2 \sin\theta \cos\theta \quad \text{(M.I. of the disc about } OX\text{)}$$

$$= \omega^2 \sin\theta \cos\theta \,.\, Mk^2$$

or $$\sin\theta\,(gh - \omega^2 k^2 \cos\theta) = 0,$$

which gives either $\sin\theta = 0$, *i.e.* $\theta = 0$, or $gh - \omega^2 k^2 \cos\theta = 0$, i.e. $\cos\theta = (gh/k^2\omega^2)$.

Now when $\omega^2 < gh/k^2$, $\cos\theta > 1$, which is not possible and hence in this case $\theta = 0$ is the only possible value, i.e. when $\omega^2 < (gh/k^2)$, the plane of the disc is vertical.

EXERCISE ON CHAPTER II

1. State D' Alembert's principle and apply it to prove that the motions of translation and rotation of a rigid body can be regarded as independent of each other. [Hint. See § 2.7 and § 2.8]. [Raj. 80]
2. A light rod OAB can turn freely in a vertical plane about a smooth fixed hinge at O; two heavy particles of masses m and m' are attached to the rod at A and B oscillate with it. Find the motion.
3. A plank of mass M', and length $2a$ is initially at rest along a line of greatest slope of a smooth plane inclined at angle α to the horizon and a man, of mass M starting from the upper end walks down the plank, so that it does not move, show that he gets to the other end in time.

$$\left\{\frac{4Ma}{(M+M')\,g \sin\alpha}\right\}^{1/2}$$

[Hint. See Ex. 9 on page 105].

FLUID DYNAMICS

1

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KINEMATICS
(Equations of Continuity)

1.1. Definitions and Basic Concepts

1. **Hydrodynamics :** Hydrodynamics is that branch of mathematics which deals with the motion of fluids or that of bodies in fluids.

2. **Fluid :** By fluid we mean a substance which is capable of flowing. Actual fluids are divided into two categories : (i) liquids, (ii) gases. We regard liquids as incompressible fluids for all practical purposes and gases as compressible fluids. Actual fluids have five physical properties : density, volume, temperature, pressure and viscosity.

3. **Shearing stress :** Two types of forces act on a fluid element. One of them is *body force* and the other is *surface force*. The body force is proportional to the mass of the body on which it acts while the surface force acts on the boundary of the body and so it is proportional to the surface area.

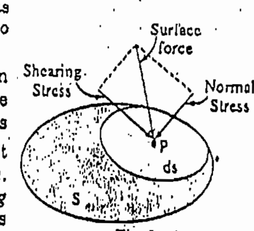


Fig. 1.

Suppose F is a surface force acting on an elementary surface area dS at the point P of surface S. Let F_1 and F_2 be resolved parts of F in the directions of tangent and normal at P. The normal force per unit area is called *normal stress* and is also called *pressure*. The tangential force per unit area is called *shearing stress*. Hence F_1 is a kind of shearing stress and F_2 is a normal stress.

4. **Perfect Fluid :** A fluid is said to be perfect if it does not exert any shearing stress, however small. the following have the same meaning : perfect, frictionless, inviscous, nonviscous and ideal.

From the definition of shearing stress and body force it is clear that body force per unit area at every point of surface of a perfect fluid acts along the normal to the surface at that point.

5. **Difference between Perfect fluid and Real fluid :** Actual fluid or real fluid is viscous and compressible. The main difference between real fluid and perfect fluid is that stress across any plane surface of perfect fluid is always normal to the surface, while it is not true in case of real fluid. In case of viscous fluid, both shearing stress and normal stress exist.

6. **Viscosity :** Viscosity is that property of real fluid as a result of which they offer some resistance to shearing, *i.e.*, sliding movement of one particle past or near

another particle. Viscosity is also known as internal friction of fluid. All known fluids have this property in varying degree. Viscosity of glycerine and oil is large in comparison to viscosity of water or gases.

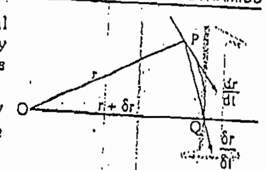


Fig. 2

7. Velocity : Let a fluid particle be at P at any time t s. $\vec{OP} = \mathbf{r}$ and at time $t + \delta t$, let it be at Q, where $\vec{OQ} = \mathbf{r} + \delta\mathbf{r}$.

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \mathbf{r} + \delta\mathbf{r} - \mathbf{r} = \delta\mathbf{r}.$$

Thus δt seconds produce increment $\vec{PQ} = \delta\mathbf{r}$ in $\mathbf{r}$. If $\delta t \to 0$, $\delta\mathbf{r} \to 0$, then $Q \to P$ and the chord PQ coincides with the tangent at P to the curve.

We define
$$\frac{d\mathbf{r}}{dt} = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta\mathbf{r}}{\delta t}$$

The vector $\frac{d\mathbf{r}}{dt}$ is defined as velocity $\mathbf{q}$ of the particle at P.

Thus
$$\mathbf{q} = \frac{d\mathbf{r}}{dt} = \mathbf{f}(\mathbf{r}, t).$$

8. Flux (Flow) across any surface

The rate of flow, *i.e.*, flux across any surface S is defined as the integral $\int_S \rho\,(\mathbf{q} \cdot \mathbf{n})\, ds$.

We also define

Flux = density . normal velocity . area of the surface.

$\mathbf{n}$ being unit outward normal vector of any point P.

1.2. The fluid motion may be studied by two different methods

(1) Lagrangian method, (2) Eulerian method.

1. Lagrangian method : In this method, any particle of the fluid is selected and its motion is studied. Hence we determine the history of every fluid particle.

Let a fluid particle be initially at the point (a, b, c). After a lapse of time t, let the same fluid particle be at (x, y, z). It is obvious that x, y, z are functions of t. But since particles which have initially different positions occupy different positions after the motion is allowed, hence the coordinates of final position, *i.e.*, (x, y, z) depend on (a, b, c) also. Thus

$$x = f_1(a, b, c, t), \qquad y = f_2(a, b, c, t), \qquad z = f_3(a, b, c, t)$$

If the motion is everywhere continuous, then f_1, f_2, f_3 are continuous functions so that we can assume that first and second order partial derivatives w.r.t. a, b, c, t exist. Components of acceleration of a fluid particle are $\ddot{x}, \ddot{y}, \ddot{z}$, where

$$\ddot{x} = \frac{\partial^2 f_1}{\partial t^2}, \quad \ddot{y} = \frac{\partial^2 f_2}{\partial t^2}, \quad \ddot{z} = \frac{\partial^2 f_2}{\partial t^2}$$

2. Eulerian method : In this method, any point fixed in the space occupied by a fluid is selected and we observe the change in the state of the fluid as the fluid passes through this point. Since the point is fixed and so x, y, z are independent variables. Hence the symbols $\dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}$ have no meaning.

Remark : In Eulerian method we study the motion of every fluid particle at a fixed point, whereas in Lagrangian method we study the motion of every fluid particle at every point. Hence Euler's method corresponds to Local time-rate of change and Lagrangian method corresponds to Individual time rate of change.

Ex. *Explain the difference between Eulerian and Lagrangian methods in hydrodynamics.*

1.3. Local and individual time rate of change

Consider a fluid motion associated with scalar point function $\phi(\mathbf{r}, t)$. Keeping the point $P(\mathbf{r})$ fixed, the change in ϕ is

$$\phi(\mathbf{r}, t + \delta t) - \phi(\mathbf{r}, t)$$

and its rate of change is

$$\lim_{\delta t \to 0} \frac{\phi(\mathbf{r}, t + \delta t) - \phi(\mathbf{r}, t)}{\delta t} = \frac{\partial \phi}{\partial t}$$

Since $P(\mathbf{r})$ is fixed hence $\frac{\partial \phi}{\partial t}$ is called *local time rate of change.*

Keeping the particle fixed, change in ϕ is

$$\phi(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - \phi(\mathbf{r}, t)$$

and its rate of change is

$$\lim_{\delta t \to 0} \frac{\phi(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - \phi(\mathbf{r}, t)}{\delta t} = \frac{d\phi}{dt}$$

$\frac{d\phi}{dt}$ is called *individual time rate of change.*

Since
$$\phi = \phi(\mathbf{r}, t) = \phi(x, y, z, t)$$

$$\therefore \quad d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz + \frac{\partial \phi}{\partial t} dt.$$

Dividing by dt,

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dt} + \frac{\partial \phi}{\partial t} = u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + w \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial t}$$

$$\left[\text{For} \quad \frac{d\mathbf{r}}{dt} = \mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt}.\right]$$

or,
$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \left[\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi\right] \cdot (\mathbf{i}u + \mathbf{j}v + \mathbf{k}w)$$

$$= \frac{\partial \phi}{\partial t} + (\nabla\phi) \cdot \mathbf{q}$$

or
$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \mathbf{q} \cdot \nabla\phi.$$

This is the relation between the two time rates.

Note : Similarly, for a vector function, it can be proved that

$$\frac{d\mathbf{f}}{dt} = \frac{\partial \mathbf{f}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{f}.$$

1.4. Acceleration

To explain the method of differentiation following the fluid and to obtain an expression for acceleration.

Consider a scalar function $\phi(\mathbf{r}, t)$ associated with fluid motion. Then $\phi(\mathbf{r}, t) = \phi(x, y, z, t)$.

$$\therefore \quad d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz + \frac{\partial \phi}{\partial t} dt.$$

Dividing by dt and taking

$$\dot{x} = \frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w,$$

we obtain $$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x} u + \frac{\partial \phi}{\partial y} v + \frac{\partial \phi}{\partial z} w + \frac{\partial \phi}{\partial t}.$$

Taking $$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \quad \frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \mathbf{q} \cdot \nabla \phi$$

or $$\frac{d\phi}{dt} = \left[\frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla\right] \phi.$$

This $\Rightarrow$ $$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla$$

The operator $\frac{d}{dt}$ is called '*Differentiation following the fluid*'. *(Agra 2004)*

Sometimes we also write $\frac{D}{Dt}$ in place of $\frac{d}{dt}$. Acceleration $\mathbf{a}$ is defined as total derivative (Material derivative) of $\mathbf{q}$ w.r.t. t. Then

$$\mathbf{a} = \frac{d\mathbf{q}}{dt} = \left[\frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla\right] \mathbf{q} = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}\right) \mathbf{q}.$$

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ from both sides,

$$a_1 = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}\right) u,$$

$$a_2 = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}\right) v,$$

$$a_3 = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}\right) w.$$

where a_1, a_2, a_3 are components of the acceleration along the axis.

1.5. Kinds of Motion

1. Stream line (Laminar) motion : A fluid motion is said to be stream line motion if the tracks of a fluid particle form parts of regular cuves. *(Kanpur 2001)*

2. Turbulent motion : A fluid motion is said to be turbulent if the paths are widely irregular. *(Kanpur 2001)*

3. Steady motion : A fluid motion is said to be steady if the condition at any point in the fluid at any time remains the same for all time. That is to say, a fluid motion is said to be steady if *(Kanpur 2000)*

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial p}{\partial t} = 0, \quad \frac{\partial \mathbf{q}}{\partial t} = 0,$$

where $\rho, p, \mathbf{q}$ denote density, pressure, velocity respectively.

Rotational motion : A fluid motion is said to be rotational if $W = \text{curl } \mathbf{q} \neq 0$ at every time and at every point.

5. Irrotational motion : A fluid motion is said to be irrotational if $W = \text{curl } \mathbf{q} = 0$ at every point and at every time.

1.6. Definitions of some curves

1. Stream line *(Agra 2004, Kanpur 2000)*

A stream line or line of flow is a curve s.t. the tangent at any point if it, at any instant of time, coincides with the direction of the motion of the fluid at that point. It means that direction of tangent and direction of velocity are parallel, *i.e.*, $\mathbf{q}$ is parallel to $d\mathbf{r}$ and so $\mathbf{q} \times d\mathbf{r} = 0$.

This $\Rightarrow$ $$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{or} \quad \frac{dr}{-\frac{\partial \phi}{\partial r}} = \frac{r\, d\theta}{-\frac{1}{r} \frac{\partial \phi}{\partial \theta}} = \frac{r \sin\theta\, d\omega}{-\frac{1}{r \sin\theta} \frac{\partial \phi}{\partial \omega}}$$

These are the required differential equations of a stream line. Stream lines form doubly infinite set at any time t. Here

$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}.$$

2. Stream tube : The stream lines drawn though each point of a closed curve enclose a tubular surface in the fluid which is called stream tube or tube of flow. A tube of flow of infinitesimal cross section is called *stream filament.*

3. Path lines *(Kanpur 2000)*

A path line is a curve which a particular fluid particle describes during its motion. The differential equations of path lines are

$$\frac{d\mathbf{r}}{dt} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \quad i.e.,$$

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w.$$

Path lines form a triply infinite set.

4. Difference between stream lines and path lines

The tangents to the stream lines give the directions of velocities of fluid particles at various points at a given time, while tangents to the path lines give the directions of velocities of a given fluid particle at various times. That is to say, stream lines show how each fluid particle is moving at a given instant whereas the path lines show how a given fluid particle is moving at each instant. In steady flow, stream lines do not vary with time and coincide with path lines.

Streak lines : A streak line is a line on which lie all those fluid elements that at some earlier instant passed through a particular point in space.

A streak line is defined as the locus of different particles passing through fixed point.

1.7. Velocity potential

Suppose $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is velocity at any point $P(x, y, z)$. Also suppose the expression $u\, dx + v\, dy + w\, dz$ is an exact differential, say $-d\phi$.

Then $$-d\phi = u\, dx + v\, dy + w\, dz$$

or $$-\left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz + \frac{\partial \phi}{\partial t} dt\right) = u\, dx + v\, dy + w\, dz$$

where $\phi = \phi(x, y, z, t)$.

This $\Rightarrow$ $u = -\dfrac{\partial \phi}{\partial x}$, $v = -\dfrac{\partial \phi}{\partial y}$, $w = -\dfrac{\partial \phi}{\partial z}$, $0 = -\dfrac{\partial \phi}{\partial t}$.

Hence $\quad \mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

$$= -\left(\mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z} \right) = -\nabla \phi.$$

Integration of $-\dfrac{\partial \phi}{\partial t} = 0$ declares that $\phi = 0 + f_1(x, y, z)$, where $f_1(x, y, z)$ is a constant of integration. This equation also declares that $\phi = \phi(x, y, z)$.

$\mathbf{q} = -\nabla\phi$ is the required relation. Here ϕ is defined as *velocity potential* or *velocity function*. The negative sign in the equation $\mathbf{q} = -\nabla\phi$ is a convention. It ensures that flow takes from higher to lower potentials.

Theorem 1. To show that surfaces exist which cut stream lines orthogonally if the velocity potential exists.

Proof : The differential equations of stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \ldots (1)$$

The surfaces which cut (1) orthogonally are given by

$$u\,dx + v\,dy + w\,dz = 0. \qquad \ldots (2)$$

The necessary and sufficient condition for the existence of (2) is that (2) must admit a solution of the type

$$f(x, y, z) = c$$

c being constant of integration. ... (3)

The necessary and sufficient condition for the existence of (3) is that

$$u\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + v\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + w\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0. \qquad \ldots (4)$$

If we show that (4) is satisfied whenever velocity potential exists, i.e., when $u = -\dfrac{\partial \phi}{\partial x}$, $v = -\dfrac{\partial \phi}{\partial y}$, $w = -\dfrac{\partial \phi}{\partial z}$, the result will follows.

$$\text{L.H.S. of (4)} = u\left(-\frac{\partial^2 \phi}{\partial z\,\partial y} + \frac{\partial^2 \phi}{\partial y\,\partial z}\right) + v\left(-\frac{\partial^2 \phi}{\partial x\,\partial z} + \frac{\partial^2 \phi}{\partial z\,\partial x}\right) + w\left(-\frac{\partial^2 \phi}{\partial x\,\partial y} + \frac{\partial^2 \phi}{\partial x\,\partial y}\right)$$

$$= u \,.\, 0 + v \,.\, 0 + w \,.\, 0 = 0 = \text{R.H.S. of (4)}$$

Hence (4) is satisfied.

1.8. Some definitions

1. Vorticity Vector : If $\mathbf{q}$ be the velocity vector, then $\mathbf{W} = \text{curl } \mathbf{q}$ or rot $\mathbf{q}$ is called vorticity vector. The mathematicians Lamb, Milne-Thompson, Rutherford, Goldstein etc. follow the definition $\mathbf{W} = \text{curl } \mathbf{q}$ whereas Birkhoff, Robertson etc. follow the definition $\mathbf{W} = \frac{1}{2}\,\text{curl } \mathbf{q}$. If we write $\mathbf{W} = \mathbf{W}(\xi, \eta, \zeta)$ then $\mathbf{W} = \text{curl } \mathbf{q}$ gives

$$\xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ on both sides,

$$\xi = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \eta = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

A fluid motion is said to be irrotational if $\xi = 0, \eta = 0, \zeta = 0$ otherwise rotational.

2. Vortex line : Vortex line is a curve such that tangent at each point of it at any instant of time is in the direction of vorticity vector at that point. It means that $\mathbf{W}$ is parallel to $d\mathbf{r}$. This $\Rightarrow$ $\mathbf{W} \times d\mathbf{r} = 0 \Rightarrow \dfrac{dx}{\xi} = \dfrac{dy}{\eta} = \dfrac{dz}{\zeta}$, these are the differential equations of a vortex line.

3. Vortex tube : The vortex lines drawn through each point of a closed curve enclose a tubular space in the fluid known as vortex tube. A vortex tube of infinitesimal cross section is called a *vortex filament* or simply *vortex*. *(Agra 2004)*

4. Beltranic flow : A fluid motion is said to be Beltranic flow if $\mathbf{q}$ is parallel to $\mathbf{W}$; i.e., if $\mathbf{q} \times \mathbf{W} = 0$. In this case $\mathbf{q}$ is called Beltranic vector.

1.9. Boundary surface

The contact between the fluid and the surface will be maintained if the fluid and surface have the same velocity along the normal to the surface. Let P be an arbitrary point on the boundary surface $F(\mathbf{r}, t) = 0$ where the fluid velocity is $\mathbf{q}$ and velocity of the surface is $\mathbf{u}$. Since normal component of velocity of fluid = normal component of the velocity of surface.

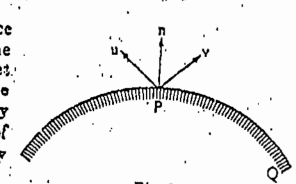


Fig. 3

This $\Rightarrow$ $\quad \mathbf{q} \,.\, \mathbf{n} = \mathbf{u} \,.\, \mathbf{n}$ ... (1)

Since ∇F is normal to the surface $F(\mathbf{r}, t) = 0$. Hence $\mathbf{n}$ and ∇F both are parallel vectors. Now (1) takes the form

$$\mathbf{q} \,.\, \nabla F = \mathbf{u} \,.\, \nabla F. \qquad \ldots (2)$$

Let $P(\mathbf{r}, t)$ move to a point $Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t)$ in time δt. Since Q also lies on $F(\mathbf{r}, t) = 0$. Hence $F(\mathbf{r} + \delta\mathbf{r}, t + \delta t) = 0$. By Taylor's theorem, namely

$$f(x + h, y + k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) + \ldots\ldots$$

We get $\quad F(\mathbf{r}, t) + \left(\delta\mathbf{r} \,.\, \nabla F + \delta t \dfrac{\partial F}{\partial t} \right) = 0. \qquad$ Also $F(\mathbf{r}, t) = 0$.

Hence $\quad \dfrac{\delta \mathbf{r}}{\delta t} \,.\, \nabla F + \dfrac{\partial F}{\partial t} = 0 \quad$ or $\quad \dfrac{\partial F}{\partial t} = -\dfrac{d\mathbf{r}}{dt} \,.\, \nabla F = -\mathbf{u} \,.\, \nabla F.$

For $\mathbf{u}$ is the velocity of the surface.

Now (2) becomes

$$\mathbf{q} \,.\, \nabla F = -\frac{\partial F}{\partial t} \quad \text{or} \quad \left[\frac{\partial}{\partial t} + \mathbf{q} \,.\, \nabla \right] F = 0 \quad \text{or} \quad \frac{dF}{dt} = 0$$

or equivalently $\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)F=0.$

This is the required condition for the surface to be a possible form of boundary surface. If the surface is a rigid surface, then the condition becomes

$$u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0.$$

Remark : Normal component of velocity for the boundary

$$=\mathbf{q}\cdot\mathbf{n}=\mathbf{q}\cdot\frac{\nabla F}{|\nabla F|}=-\frac{\partial F/\partial t}{|\nabla F|}\quad \because \frac{\partial F}{\partial t}+\mathbf{q}\cdot\nabla F=0$$

$$=-\frac{\partial F/\partial t}{(F_x^2+F_y^2+F_z^2)^{1/2}}\text{ where } F_x=\frac{\partial F}{\partial x}\text{ etc.}$$

1.10. Equation of continuity

The rate of generation of mass within a given volume must be balanced by an equal net outward flow of mass from the volume. This amounts to saying that matter is neither created nor destroyed.

1.11. Equation of Continuity by Euler's method

Or. Determine equation of continuity by vector approach for a non-homogeneous incompressible fluid. *(Kanpur 2005, Meerut 2003)*

Consider a fixed surface S, enclosing a volume V in the region occupied by a moving fluid. Let $\mathbf{n}$ be a unit outward normal vector drawn on the surface element dS, where fluid velocity is $\mathbf{q}$ and fluid density is ρ. Inward normal velocity is $-\mathbf{n}.\mathbf{q}$. Mass of the fluid entering across the surface S in unit time is

$$\int_S \rho(-\mathbf{n}.\mathbf{q})\,dS=-\int_S \mathbf{n}\cdot\rho\,\mathbf{q}\,dS=-\int_V \nabla\cdot(\rho\mathbf{q})\,dV \quad \ldots(1)$$

The mass of the fluid within the volume V is $\int_V \rho\,dV$.

Rate of generation of the fluid within the volume is

$$\frac{\partial}{\partial t}\int_V \rho\,dV=\int_V \frac{\partial\rho}{\partial t}\,dV. \quad \ldots(2)$$

[For $\rho\frac{\partial}{\partial t}(dV)=\rho\, d\left(\frac{\partial V}{\partial t}\right)=\rho\cdot 0=0$, as volume is constant w.r.t. time]. Here local time rate of change has been taken because the surface is stationary. Equation of continuity gives

$$\int_V \frac{\partial\rho}{\partial t}\,dV=-\int_V \nabla\cdot(\rho\mathbf{q})\,dv \quad \text{[on equating (1) to (2)]}$$

or $$\int_V\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{q})\right]dv=0$$

Since S is arbitrary and so V is arbitrary. Hence integrand of the last integral vanishes.

$$\therefore\quad \frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{q})=0. \quad \ldots(3)$$

This is Eulerian equation of continuity.

By (3), $$\frac{\partial\rho}{\partial t}+\mathbf{q}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{q}=0$$

or $$\left[\frac{\partial}{\partial t}+\mathbf{q}\cdot\nabla\right]\rho+\rho\nabla\cdot\mathbf{q}=0$$

or $$\frac{d\rho}{dt}+\rho\,\nabla\cdot\mathbf{q}=0. \quad \ldots(4)$$

This is an alternate form of (3).

[Equation (3) is also called equation of mass of conservation].

Deductions : (i) To prove $\frac{d}{dt}(\log\rho)+\nabla.\mathbf{q}=0.$

Dividing (4) by ρ and writing

$$\frac{1}{\rho}\frac{d\rho}{dt}=\frac{d}{dt}(\log\rho),$$

we get the required result.

(ii) To write cartesian form of the equation of continuity. We know

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{q}\cdot\nabla=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

Now (4) is reduced to

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\rho+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0.$$

This is the cartesian form.

(iii) Suppose the fluid is incompressible so that

$$\frac{d\rho}{dt}=0.\quad \text{Then (4)} \Rightarrow \rho\nabla\cdot\mathbf{q}=0 \Rightarrow \nabla.\mathbf{q}=0$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

This is the equation of continuity in this case.

Note : In this case $\mathbf{q}$ is solenoidal vector. For a vector $\mathbf{f}$ is said to be solenoidal vector if $\nabla.\mathbf{f}=0$.

(iv) Let the motion be irrotational and incompressible. Then there exists velocity potential ϕ s.t. $\mathbf{q}=-\nabla\phi$.

Here also $\frac{d\rho}{dt}=0$. Now (4) becomes

$$0+\rho\nabla\cdot(-\nabla\phi)=0 \quad\text{or}\quad \nabla^2\phi=0$$

or $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0.$$

This is the equation of continuity in this case.

Note : This deduction can also be expressed as : *Show that the equation of continuity reduces to Laplace's equation when the liquid is incompressible and irrotational.*

(v) Suppose the motion is symmetrical.

In this case velocity has only one component, say u.

Then we have $\dfrac{d}{dt}=\dfrac{\partial}{\partial t}+u\dfrac{\partial}{\partial x}$ as $\mathbf{q}=u$, $\nabla=\dfrac{\partial}{\partial x}$.

Now (4) becomes

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\rho+\rho\frac{\partial u}{\partial x}=0.$$

(vi) **For steady motion :** In this case $\dfrac{\partial \rho}{\partial t}=0$. Now equation (3) becomes

$$\nabla\cdot(\rho\mathbf{q})=0.$$

or equivalently,
$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0.$$

This is Euler's equation of continuity for steady motion.

Problem. *Write full form for the operator used for differentiation following the fluid motion and give equation of continuity.* *(Agra 2004)*

Solution : $\dfrac{d}{dt}=\dfrac{\partial}{\partial t}+u\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial y}+w\dfrac{\partial}{\partial z}$

$\dfrac{d}{dt}$ = operator of differentiation following fluid motion.

Equation of continuity is

$$\frac{d\rho}{dt}+\rho\,\nabla\cdot\mathbf{q}=0$$

1.12. Equation of continuity by Lagrange's method

Let initially a fluid particle be at (a, b, c) at time $t=t_0$, when its volume is dV_0 and density is ρ_0. After a lapse of time t, let the same fluid particle be at (x, y, z) when its volume is dV and density ρ. Since the mass of fluid element remains invariant during its motion. Hence

$$\rho_0\,dV_0=\rho\,dV \quad \text{or} \quad da\,db\,dc=\rho\,dx\,dy\,dz$$

or
$$\rho_0\,da\,db\,dc=\rho\frac{\partial(x,y,z)}{\partial(a,b,c)}\,da\,db\,dc$$

or
$$\rho J=\rho_0 \quad \ldots(1) \qquad \text{where } J=\frac{\partial(x,y,z)}{\partial(a,b,c)}$$

(1) is the required equation of continuity.

Remark : This article can also be expressed as : *By considering the constancy of mass of a finite volume of the fluid, obtain the equation of continuity.*

1.13. Equivalence between Eulerian and Lagrangian forms of equations of continuity

(Meerut 1993, Garhwal 2001)

Let initially a fluid particle be at (a, b, c) at time $t=t_0$, when its volume is dV_0 and density is ρ_0. After a lapse of time t, let the same fluid particle be at (x, y, z) when its volume is dV and density is ρ. The velocity components in the two systems are connected by the equations :

$$u=\dot{x},\ v=\dot{y},\ w=\dot{z},\quad \mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}.$$

Also $\quad x=x(a,b,c;t),\ y=y(a,b,c,t),\ z=z(a,b,c,t)$

$$\frac{\partial u}{\partial a}=\frac{\partial}{\partial a}\left(\frac{dx}{dt}\right)=\frac{d}{dt}\left\{\frac{\partial x}{\partial a}\right\}.\ \text{Similarly, } \frac{\partial v}{\partial a}=\frac{d}{dt}\left(\frac{\partial y}{\partial a}\right)\text{ etc.}$$

Firstly, we shall determine $\dfrac{dJ}{dt}$.

$$J=\frac{\partial(x,y,z)}{\partial(a,b,c)}=\begin{vmatrix}\frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c}\\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c}\\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c}\end{vmatrix}$$

$$\therefore \quad \frac{dJ}{dt}=\begin{vmatrix}\frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c}\\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c}\\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c}\end{vmatrix}+\begin{vmatrix}\frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c}\\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} & \frac{\partial v}{\partial c}\\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c}\end{vmatrix}+\begin{vmatrix}\frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c}\\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c}\\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c}\end{vmatrix}$$

or
$$\frac{dJ}{dt}=J_1+J_2+J_3 \qquad \ldots(1),\ \text{say}$$

Now J_1 is expressible as

$$J_1=\begin{vmatrix}\frac{\partial u}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a}\\ \frac{\partial u}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b}\\ \frac{\partial u}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial a}\frac{\partial x}{\partial a}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial a}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a}\\ \frac{\partial u}{\partial b}\frac{\partial x}{\partial b}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial b}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b}\\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial c}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c}\end{vmatrix}$$

$$=\frac{\partial u}{\partial x}\frac{\partial(x,y,z)}{\partial(a,b,c)}+\frac{\partial u}{\partial y}\frac{\partial(y,y,z)}{\partial(a,b,c)}+\frac{\partial u}{\partial z}\frac{\partial(z,y,z)}{\partial(a,b,c)}$$

$$=J\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot 0+\frac{\partial u}{\partial z}\cdot 0=J\frac{\partial u}{\partial x}$$

$\therefore \quad J_1=J\dfrac{\partial u}{\partial x}$. [For a determinant vanishes if any two of its columns are identical]

Similarly, $\quad J_2=J\dfrac{\partial v}{\partial y};\quad J_3=J\dfrac{\partial w}{\partial z}$;

Now (1) becomes $\quad \dfrac{dJ}{dt}=J\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}\right)=J\,\nabla\cdot\mathbf{q}$

$$\frac{dJ}{dt} = J\,\nabla.\mathbf{q} \qquad \ldots (2)$$

Step I. Lagrangian equation of continuity

$$\Rightarrow \rho J = \rho_0 \;\Rightarrow\; \frac{d}{dt}(\rho J) = 0 \;\Rightarrow\; \frac{d\rho}{dt} J + \rho \frac{dJ}{dt} = 0$$

$$\Rightarrow J\frac{d\rho}{dt} + \rho J\,\nabla.\mathbf{q} = 0, \text{ by (2)}$$

Dividing by J, $\frac{d\rho}{dt} + \rho\,\nabla.\mathbf{q} = 0.$

$\Rightarrow$ Eulerian equation of continuity.

Step II. Eulerian equation of continuity

$$\Rightarrow \frac{d\rho}{dt} + \rho\nabla.\mathbf{q} = 0 \;\Rightarrow\; \frac{d\rho}{dt} + \rho\frac{1}{J}\frac{dJ}{dt} = 0, \text{ by (2)}$$

$$\Rightarrow J\frac{d\rho}{dt} + \rho\frac{dJ}{dt} = 0 \;\Rightarrow\; \frac{d}{dt}(\rho J) = 0,$$

integrating we get $\rho J = \rho_0$, say.

$\Rightarrow$ Lagrangian equation of continuity.

1.17. Generalised Orthogonal curvilinear co-ordinates

Suppose $f_1(x, y, z) = \alpha_1, f_2(x, y, z) = \alpha_2, f_3(x, y, z) = \alpha_3$, are the three independent orthogonal families of surfaces, where (x, y, z) are cartesian co-ordinates of a point, the surfaces $\alpha_1 =$ const., $\alpha_2 =$ const., $\alpha_3 =$ const. form an orthogonal system in which $(\alpha_1, \alpha_2, \alpha_3)$ may be used as the *orthogonal curvilinear co-ordinates* of a point in the space. The relation between the two co-ordinates (x, y, z) and $(\alpha_1, \alpha_2, \alpha_3)$ can also be expressed by the relations :

$$x = x(\alpha_1, \alpha_2, \alpha_3),\; y = y(\alpha_1, \alpha_2, \alpha_3),\; z = z(\alpha_1, \alpha_2, \alpha_3).$$

$$dx = \frac{\partial x}{\partial \alpha_1} d\alpha_1 + \frac{\partial x}{\partial \alpha_2} d\alpha_2 + \frac{\partial x}{\partial \alpha_3} d\alpha_3$$

$$dy = \frac{\partial y}{\partial \alpha_1} d\alpha_1 + \frac{\partial y}{\partial \alpha_2} d\alpha_2 + \frac{\partial y}{\partial \alpha_3} d\alpha_3$$

$$dz = \frac{\partial z}{\partial \alpha_1} d\alpha_1 + \frac{\partial z}{\partial \alpha_2} d\alpha_2 + \frac{\partial z}{\partial \alpha_3} d\alpha_3$$

Squaring and adding these equations column-wise, we obtain

$$dx^2 + dy^2 + dz^2 = (h_1\, d\alpha_1)^2 + (h_2\, d\alpha_2)^2 + (h_3\, d\alpha_3)^2 + \text{coeff. of } d\alpha_1\, d\alpha_2 + \text{coeff. of } d\alpha_2\, d\alpha_3 + \text{coeff. of } d\alpha_3\, d\alpha_1$$

where $$h_1^2 = \left(\frac{\partial x}{\partial \alpha_1}\right)^2 + \left(\frac{\partial y}{\partial \alpha_1}\right)^2 + \left(\frac{\partial z}{\partial \alpha_1}\right)^2 \text{ etc.}$$

By orthogonal property, the terms containing $d\alpha_1\, d\alpha_2$, $d\alpha_2\, d\alpha_3$, $d\alpha_3\, d\alpha_1$ vanish.

Hence
$$dx^2 + dy^2 + dz^2 = (h_1\, d\alpha_1)^2 + (h_2\, d\alpha_2^2) + (h_3\, d\alpha_3)^2.$$

Using the fact that the line element in cartesian co-ordinates is given by $ds^2 = dx^2 + dy^2 + dz^2$, we get

$$ds^2 = (h_1\, d\alpha_1)^2 + (h_2\, d\alpha_2)^2 + (h_3\, d\alpha_3)^2.$$

1.18. Equation of continuity in generalised orthogonal curvilinear co-ordinates

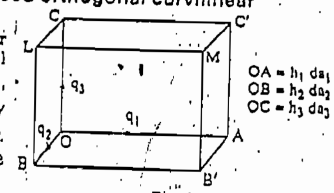


Fig. 4

Let ρ be the fluid density at a curvilinear point $P(\alpha_1, \alpha_2, \alpha_3)$ enclosed by a small parallelopiped with edges of lengths $h_1\, d\alpha_1$, $h_2\, d\alpha_2$, $h_3\, d\alpha_3$. Let q_1, q_2, q_3 be the velocity components along OA, OB, OC respectively. Mass of the fluid that passes in unit time across the face $OBLC$

$$= \text{density . area . normal velocity}$$
$$= \rho\,(h_2\, d\alpha_2 \,.\, h_3\, d\alpha_3)\,.\, q_1$$
$$= \rho\, q_1\, h_2 h_3\, d\alpha_2\, d\alpha_3$$
$$= f(\alpha_1, \alpha_2, \alpha_3), \text{ say.}$$

Mass of the fluid that passes in unit time across the face

$$C'MB'A = f(\alpha_1 + \delta\alpha_1, \alpha_2, \alpha_3)$$
$$= f(\alpha_1, \alpha_2, \alpha_3) + \delta\alpha_1 \,.\, \frac{\partial f}{\partial \alpha_1}.$$

Now the excess of flow in overflow out from the faces $OBLC$ and $MB'AC'$ in unit time

$$= f - \left(f + \delta\alpha_1 \frac{\partial f}{\partial \alpha_1}\right)$$
$$= -\,\delta\alpha_1 \,.\, \frac{\partial f}{\partial \alpha_1}$$
$$= -\,\delta\alpha_1 \,.\, \frac{\partial}{\partial \alpha_1}(\rho q_1 h_2 h_3)\, d\alpha_2 \,.\, d\alpha_3$$
$$= -\frac{\partial}{\partial \alpha_1}(\rho q_1 h_2 h_3)\, d\alpha_1 \,.\, d\alpha_2 \,.\, d\alpha_3$$

Similarly, the excess of flow in over flow out from the faces $CLMC'$ and $OBB'A$; $OCC'A$ and $LMB'B$ are respectively

$$-\frac{\partial}{\partial \alpha_3}(\rho q_3 h_1 h_2)\, d\alpha_1\, d\alpha_2\, d\alpha_3 \text{ and } -\frac{\partial}{\partial \alpha_2}(\rho q_2 h_1 h_3)\, d\alpha_1\, d\alpha_2\, d\alpha_3.$$

Rate of increment in mass of the fluid within the parallelopiped

$$= \frac{\partial}{\partial t}(\rho h_1\, d\alpha_1 \,.\, h_2\, d\alpha_2 \,.\, h_3\, d\alpha_3)$$
$$= \frac{\partial \rho}{\partial t} \,.\, h_1 h_2 h_3\, d\alpha_1 \,.\, d\alpha_2 \,.\, d\alpha_3$$

Equation of continuity says that

Increase in mass = total excess of flow in over flow out

i.e.,
$$\frac{\partial \rho}{\partial t} h_1 h_2 h_3\, da_1\, da_2\, da_3 = -\left[\frac{\partial}{\partial a_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial a_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial a_3}(\rho q_3 h_1 h_2)\right] da_1\, da_2\, da_3$$

or
$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial}{\partial a_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial a_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial a_3}(\rho q_3 h_1 h_2)\right] \cdot \frac{1}{h_1 h_2 h_3} = 0$$

This is the required equation of continuity.

Deductions : (i) Rectangular cartesian co-ordinates :
$$ds^2 = dx^2 + dy^2 + dz^2 = (h_1\, da_1)^2 + (h_2\, da_2)^2 + (h_3\, da_3)^2.$$

Hence $h_1 = h_2 = h_3 = 1,\ a_1 = x,\ a_2 = y,\ a_3 = z.$

In this case the equation of continuity becomes
$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial}{\partial x}(\rho q_1) + \frac{\partial}{\partial y}(\rho q_2) + \frac{\partial}{\partial z}(\rho q_3)\right] = 0$$

(ii) Spherical co-ordinates :

Here $ds^2 = (dr)^2 + (r d\theta)^2 + (r \sin\theta\, d\omega)^2.$

Then $h_1 = 1,\ h_2 = r,\ h_3 = r \sin\theta,\ a_1 = r,\ a_2 = \theta,\ a_3 = \omega.$

In this case the equation of continuity becomes
$$\frac{\partial \rho}{\partial t} + \frac{1}{1 \cdot r \cdot r \sin\theta}\left[\frac{\partial}{\partial r}(\rho q_1 \cdot r \cdot r \sin\theta) + \frac{\partial}{\partial \theta}(\rho q_2 \cdot 1 \cdot r \sin\theta) + \frac{\partial}{\partial \omega}(\rho q_3 \cdot 1 \cdot r)\right] = 0$$

or
$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho q_2 \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \omega}(\rho q_3) = 0$$

(iii) Cylindrical co-ordinates. Here we have
$$ds^2 = (dr)^2 + (r d\theta)^2 + (dz)^2.$$

Then $h_1 = 1,\ h_2 = r,\ h_3 = 1,\ a_1 = r,\ a_2 = \theta,\ a_3 = z.$

The equation of continuity is
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho q_1 r) + \frac{\partial}{\partial \theta}(\rho q_2) + \frac{\partial}{\partial z}(\rho q_3 r)\right] = 0$$

or
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho q_1 r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_2) + \frac{\partial}{\partial z}(\rho q_2) = 0$$

1.19. Equation of continuity in cartesian co-ordinates

(Garhwal 2002, 2003)

Let ρ denote fluid density at $P(x, y, z)$ enclosed by a small parallelopiped with edges of lengths $\delta x, \delta y, \delta z$. Let u, v, w be velocity components along AA', AP, AB respectively. Mass of the fluid that passes in unit time across the face $APCB$

= density . area . normal velocity

$= \rho \cdot \delta y \cdot \delta z \cdot u = f(x, y, z)$, say.

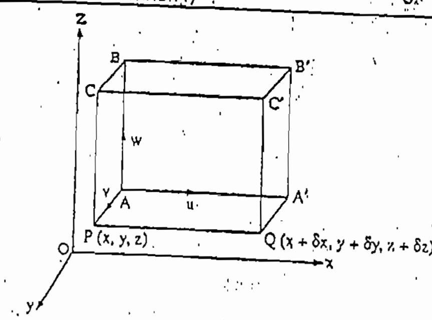


Fig. 5

Mass of the fluid that passes in unit time across the face $QA'B'C'$
$$= f(x + \delta x, y, z) = f + \delta x \cdot \frac{\partial f}{\partial x}$$

Now the excess of flow in flow out from the face $APCB$ and $QA'B'C'$ in unit time
$$= f - \left(f + \delta x \cdot \frac{\partial f}{\partial x}\right) = -\delta x \cdot \frac{\partial f}{\partial x} = -\delta x \cdot \frac{\partial}{\partial x}(\rho u\, \delta y\, \delta z)$$
$$= -\frac{\partial (\rho u)}{\partial x} \cdot \delta x\, \delta y\, \delta z.$$

Similarly, the excess of flow in over flow out from the faces $CC'B'B$, $PQA'A$ and $AA'B'B$, $CC'QP$ is respectively
$$-\frac{\partial}{\partial z}(\rho w) \cdot \delta x\, \delta y\, \delta z \quad \text{and} \quad -\frac{\partial}{\partial y}(\rho v)\, \delta x\, \delta y\, \delta z.$$

Rate of increment in mass of the fluid within the parallelopiped
$$= \frac{\partial}{\partial t}(\rho\, \delta x \cdot \delta y \cdot \delta z) = \frac{\partial \rho}{\partial t} \cdot \delta x \cdot \delta y \cdot \delta z$$

Equation of continuity says that

Increase in mass = total excess of flow in over flow out i.e.,
$$\frac{\partial \rho}{\partial t}\, \delta x\, \delta y\, \delta z = -\delta x\, \delta y\, \delta z\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]$$

or
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \qquad \ldots (1)$$

This is the required equation of continuity.

Deductions : (i) If the fluid is incompressible, then (1) becomes
$$0 + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0$$

or
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

This is the equation of continuity in this case.

(ii) The equation (1) is also expressible as

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\rho+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

(iii) If velocity has one component u, say, then (1) becomes

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}=0.$$

This equation is very important for further study.

1.20. Equation of continuity in spherical polar co-ordinates

To derive the equation of conservation of mass in spherical co-ordinates.

(Kanpur 1992; Garhwal 2004)

Let ρ denote fluid density at a point $P(r, \theta, \omega)$ enclosed by a small parallelopiped with edges of lengths $\delta r, r\,\delta\theta, r\sin\theta\,\delta\omega$. Let u, v, w be velocity components along AA', AP, AB respectively. Mass of the fluid that passes in unit time across the face $APCB$ is density . area . normal velocity

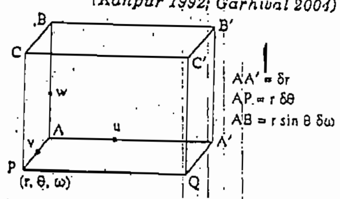


Fig. 6

$$=\rho\,.\,(r\,\delta\theta . r\sin\theta\,\delta\omega)\,.\,u$$

$$=\rho\, r^2 u \sin\theta\,\delta\theta.\delta\omega = f(r, \theta, \omega), \text{ say.}$$

Mass of the fluid that passes in unit time across the face $A'QC'B'$ is

$$f(r+\delta r, \theta, \omega)=f+\delta r . \frac{\partial f}{\partial r}.$$

Now excess of flow in over flow out from the faces $APCB$, $A'QC'B'$ in unit time

$$=f-\left(f+\delta r . \frac{\partial f}{\partial r}\right)=-\delta r . \frac{\partial f}{\partial r}$$

$$=-\delta r . \frac{\partial}{\partial r}(\rho\, r^2 u \sin\theta\,\delta\theta.\delta\omega)$$

$$=-\delta r . \frac{\partial}{\partial r}(\rho u . r\,\delta\theta . r\sin\theta\,\delta\omega).$$

Similarly, the excess of flow in over flow out from the faces $APQA'$, $CC'B'B$ and $AA'B'B$, $PQC'C$ are, respectively

$$-r\sin\theta\,\delta\omega . \frac{\partial}{r\sin\theta\,\partial\omega}(\rho w . r\,\delta\theta . \delta r)$$

and

$$-r\,\delta\theta . \frac{\partial}{r\,\partial\theta}(\rho v . \delta r . r\sin\theta\,\delta\omega)$$

Total excess of flow in over flow out

$$=-\delta r . \frac{\partial}{\partial r}(\rho u . r^2 \sin\theta\,\delta\theta.\delta\omega)-\delta\omega\frac{\partial}{\partial\omega}(\rho w\, r\,\delta\theta\,\delta r)-\delta\theta . \frac{\partial}{\partial\theta}(\rho v\, r\sin\theta\,\delta r\,\delta\omega)$$

$$=-\left[\frac{\partial}{\partial r}(\rho u\, r^2) . \sin\theta+r\frac{\partial}{\partial\theta}(\rho v\sin\theta)+r\frac{\partial}{\partial\omega}(\rho w)\right] . \delta r . \delta\theta . \delta\omega$$

Rate of increment in mass of the fluid within the parallelopiped

$$=\frac{\partial}{\partial t}(\rho\,\delta r . r\,\delta\theta . r\sin\theta\,\delta\omega)$$

$$=\frac{\partial\rho}{\partial t} . r^2\sin\theta\,\delta r . \delta\theta . \delta\omega.$$

By equation of continuity

$$\frac{\partial\rho}{\partial t}r^2\sin\theta\,\delta r.\delta\theta\,\delta\omega=-\left[\frac{\partial}{\partial r}(\rho u\, r^2) . \sin\theta+r\frac{\partial}{\partial\theta}(\rho v\sin\theta)+r\frac{\partial(\rho w)}{\partial\omega}\right]\delta r\,\delta\theta\,\delta\omega$$

Simplifying this we get

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u\, r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\omega}(\rho w)=0.$$

This is the required equation of continuity.

Problem 1. *Each particle of a mass of liquid moves in a plane through axis of z; find the equation of continuity.*

Solution : Prove as in above Article 1.20 that

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\omega}(\rho w)=0$$

Fluid particles move along the axis of z and hence $w=0$.

Equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)=0.$$

Problem 2. *Homogeneous liquid moves so that the path of any particle P lies in the plane POX, where OX is fixed axis.*

Prove that if $OP=r$, $\angle POX=\theta$, $\mu=\cos\theta$, the equation of continuity is

$$\frac{\partial}{\partial r}(r^2 q_r)-\frac{\partial}{\partial\mu}(r q_\theta\sin\theta)=0,$$

where q_r, q_θ are the components of velocity along and prependicular to OP in the plane POX.

Solution : Here motion lies in xy-plane.

Hence $w=0$. Prove as a Article 1.20 that

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial(\rho w)}{\partial\omega}=0$$

Put $w=0$, $\rho=$ const. so that $\frac{\partial\rho}{\partial t}=0$.

we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u\, r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)=0$$

$$\frac{\partial}{\partial r}(ur^2)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(rv\sin\theta)=0 \qquad \dots(1)$$

But $\mu=\cos\theta \Rightarrow d\mu=-\sin\theta\,d\theta$

$$\Rightarrow \frac{\partial}{\partial\mu}=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

Also $u=q_r, v=q_\theta$. With these values (1) becomes

$$\frac{\partial}{\partial r}(r^2 q_r) - \frac{\partial}{\partial \mu}(r q_\theta \sin\theta) = 0.$$

1.21. Equation of continuity in cylindrical co-ordinates

(Garhwal 2000)

Let ρ denote fluid density at a point $P(r, \theta, z)$ enclosed by a small parallelopiped with edges of lengths $\delta r, r\,\delta\theta, \delta z$. Let u, v, w be velocity components along AA', AP, AB, respectively. Mass of the fluid that passes in unit time across the face $APCB$ is

density . area . normal velocity

$= \rho \cdot r\,\delta\theta\,\delta z\,u$

$= f(r, \theta, z)$, say.

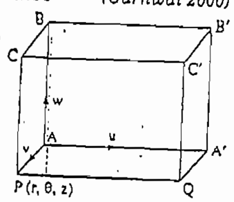


Fig. 7

Mass of the fluid that passes in unit time from the face $A'QC'B'$ is

$$f(r + \delta r, \theta, z) = f + \delta r \cdot \frac{\partial f}{\partial r}$$

Now excess of flow in over flow out from the faces $APCB$ and $A'QC'B'$ in unit time

$$= f - \left(f + \delta r \cdot \frac{\partial f}{\partial r}\right) = -\delta r \cdot \frac{\partial f}{\partial r} = -\delta r \frac{\partial}{\partial r}(\rho u\, r\,\delta\theta \cdot \delta z).$$

Similarly, the excess of flow in over flow out from the faces $AA'B'B$, $PQC'C$ and $PAA'Q$, $CC'B'B$ are, respectively,

$$-r\,\delta\theta \cdot \frac{\partial}{r\,\delta\theta}(\rho v \cdot \delta r \cdot \delta z) \quad \text{and} \quad -\delta z \frac{\partial}{\partial z}(\rho w\,\delta r \cdot r\,\delta\theta).$$

Hence total excess of flow in over flow out

$$= -\left[\delta r \cdot \frac{\partial}{\partial r}(\rho u\, r\,\delta\theta \cdot \delta z) + \delta\theta \cdot \frac{\partial}{\partial\theta}(\rho v\,\delta r \cdot \delta z) + \delta z \frac{\partial}{\partial z}(\rho w\,\delta r \cdot r\,\delta\theta)\right]$$

$$= -\left[\frac{\partial}{\partial r}(\rho u r) + \frac{\partial}{\partial\theta}(\rho v) + \frac{\partial}{\partial z}(\rho w) \cdot r\right]\delta r \cdot \delta\theta \cdot \delta z.$$

Rate of increment in mass of the fluid within the parallelopiped

$$= \frac{\partial}{\partial t}(\rho\,\delta r \cdot r\,\delta\theta \cdot \delta z)$$

$$= \frac{\partial \rho}{\partial t} \cdot r\,\delta r\,\delta\theta\,\delta z.$$

By equation of continuity,

$$\frac{\partial \rho}{\partial t} r\,\delta r\,\delta\theta\,\delta z = -\left[\frac{\partial}{\partial r}(\rho u\, r) + \frac{\partial}{\partial\theta}(\rho v) + \frac{\partial}{\partial z}(\rho w) \cdot r\right]\delta r\,\delta\theta\,\delta z$$

or

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho u\, r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0.$$

This is the required equation of continuity.

1.22. Certain Symmetrical forms of equations of continuity

1. Spherical Symmetry

The motion is symmetrical about the centre of the sphere and velocity q has only one component along the radius r. Also $q = q(r, t)$. We consider two consecutive spheres of radii r and $r + \delta r$. Mass of the fluid which passes in unit time across the inner sphere is

density . area . normal velocity

$= \rho \cdot 4\pi r^2 \cdot q = f(r, t)$, say.

Mass of the fluid that passes across the outer sphere in unit time

$$= f(r + \delta r, t) = f + \delta r \cdot \frac{\partial f}{\partial r}.$$

The excess of flow in over flow out from these two faces

$$= f - \left(f + \delta r \cdot \frac{\partial f}{\partial r}\right) = -\delta r \cdot \frac{\partial f}{\partial r}$$

$$= -\delta r \cdot \frac{\partial}{\partial r}(\rho\, 4\pi r^2 q) = -\frac{\partial}{\partial r}(\rho r^2 q)\, 4\pi\,\delta r.$$

Rate of increment in the mass of the fluid within the spheres

$$= \frac{\partial}{\partial t}(4\pi r^2 \delta r \cdot \rho)$$

$$= \frac{\partial \rho}{\partial t} \cdot 4\pi r^2\,\delta r.$$

By the def. of equation of continuity

$$\frac{\partial \rho}{\partial t} 4\pi r^2\,\delta r = -4\pi\,\delta r \cdot \frac{\partial}{\partial r}(\rho q\, r^2)$$

or

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q r^2) = 0 \quad \ldots (1)$$

This is the required equation of continuity.

Deductions : (i). If the fluid is incompressible, then the last becomes

$$0 + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 q) = 0 \quad \text{or} \quad \frac{\partial}{\partial r}(r^2 q) = 0$$

Integrating, $\quad r^2 q = \text{const.} = f(t) \quad$ or $\quad r^2 q = f(t)$.

(ii) Problem : *The particles of fluid move symmetrically in space with regard to fixed sphere, show that equation of continuity is*

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0.$$

(Kanpur 2004)

This follows from equation (1) and there replace q by u.

2. Cylindrical symmetry : In this case velocity q at any point is perpendicular to a fixed axis and is a function of r and t only, where r is perpendicular distance of the point from the axis. Consider two consecutive cylinders of radii r and $r + \delta r$ bounded by the planes at unit distance apart. Flow across the inner surface

$= \rho \cdot 2\pi r \cdot q = f(r, t)$ say.

Flow across outer surface

$$= f(r+\delta r, t) = f + \delta r \cdot \frac{\partial f}{\partial r}.$$

Excess of flow in over flow out

$$= f - \left(f + \delta r \frac{\partial f}{\partial r}\right) = -\delta r \cdot \frac{\partial f}{\partial r} = -2\pi\,\delta r \cdot \frac{\partial}{\partial r}(\rho r q)$$

Rate of increment in the mass of the fluid contained in the cylinders

$$= \frac{\partial}{\partial t}(\rho \cdot 2\pi r \cdot \delta r) = \frac{\partial \rho}{\partial t} \cdot 2\pi r\,\delta r$$

By the def. of equation of continuity,

$$\frac{\partial \rho}{\partial t} \cdot 2\pi r\,\delta r = -2\pi\,\delta r \cdot \frac{\partial}{\partial r}(\rho r q)$$

or

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q) = 0$$

This is the required equation of continuity.

Deduction : When ρ is constant, then the last gives

$$0 + \frac{1}{r} \cdot \rho \frac{\partial}{\partial r}(qr) = 0 \quad \text{or} \quad \frac{\partial}{\partial r}(rq) = 0$$

Integrating $rq = \text{const.} = f(t)$ or $rq = f(t)$.

Solved problems related to stream lines and possible liquid motion :

Problem 1. *Find the stream lines and paths of the particles for the two dimensional velocity field :*

$$u = \frac{x}{1+t}, \quad v = y, \quad w = 0.$$

Solution : We have

$$u = \frac{x}{1+t}, \quad v = y, \quad w = 0.$$

Step I. To determine stream lines.

Stream lines are the solution of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Putting the value $\frac{(1+t)}{x}dx = \frac{dy}{y} = \frac{dz}{0}$.

This $\Rightarrow \left(\frac{1+t}{x}\right)dx = \frac{dy}{y}, \quad \frac{dy}{y} = \frac{dz}{0}$

$\Rightarrow (1+t)\log x = \log y + \log a, \quad dz = 0$

$\Rightarrow x^{1+t} = ay, \quad z = b$ Ans.

These two equations represent stream lines.

Step II. To determine path lines.

Path lines are the solutions of

$$\frac{dx}{dt} = \frac{x}{1+t}, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = 0$$

This $\Rightarrow \frac{dx}{x} = \frac{dt}{1+t}, \quad \frac{dy}{y} = dt, \; dz = 0.$

Integrating, $\log x = \log(1+t) + \log a,$

$\log y = t - \log b, \; z = c.$

or $x = a(1+t), \quad (y/b) = e^t, \; z = c$

or $y = b e^{[(x/a)-1]}, \quad z = c.$ Ans.

These two equations represent path lines.

Problem 2. *Determine the streamlines and the path of the particles*

$$u = x/(1+t), \; v = y/(1+t), \; w = z/(1+t).$$

(I.A.S. 1994)

Solution : The equation of the streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or $$\frac{dx}{x/(1+t)} = \frac{dy}{y/(1+t)} = \frac{dz}{z/(1+t)}$$

or $$dx/x = dy/y = dz/z$$
(i) (ii) (iii)

By integrating (i) and (ii), we have

$\log x = \log y + \log A$, A is integration constant.

$\Rightarrow x = Ay$ ... (1)

By integrating (i) and (iii), we have

$\log x = \log z + \log B$, B is an integration constant.

$\Rightarrow x = Bz.$ ... (2)

Hence the streamlines are given by the intersection of (1) and (2). The differential equation of path lines is given by

$$q = \frac{dr}{dt}$$

This $\Rightarrow \frac{dx}{dt} = \frac{x}{1+t}, \quad \frac{dy}{dt} = \frac{y}{1+t}, \quad \frac{dz}{dt} = \frac{z}{1+t}$

$\Rightarrow \frac{dx}{x} = \frac{dt}{1+t}, \quad \frac{dy}{y} = \frac{dt}{1+t}, \quad \frac{dz}{z} = \frac{dt}{1+t}$

Integrating, we get

$\log x = \log(1+t) + \log a$

$\log y = \log(1+t) + \log b$

$\log z = \log(1+t) + \log c$

$\Rightarrow x = a(1+t), \; y = b(1+t), \; z = c(1+t)$

These give required path lines.

Problem 3. *The velocity q in three-dimensional flow field for an incompressible fluid is given by*

$$q = 2x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$$

Determine the equations of the stream lines passing through the point (1, 1, 1).

Solution : The equations of stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \Rightarrow \underset{(i)}{\frac{dx}{2x}}=\underset{(ii)}{\frac{dy}{-y}}=\underset{(iii)}{\frac{dz}{-z}}$$

From (i) and (ii), we have

$$\frac{dx}{2x}=\frac{dy}{-y} \Rightarrow \frac{dx}{x}+\frac{2dy}{y}=0$$

By integrating, we obtain

$$\log x+2\log y=\log A$$

or $\quad xy^2=A$, where A is an integration constant.

From (i) and (iii), we have

$$\frac{dx}{2x}=\frac{dz}{-z} \Rightarrow \frac{dx}{x}+\frac{2dz}{z}=0$$

By integrating, we have

$$xz^2=B, \text{ where } B \text{ is an integration constant.}$$

At the point (1, 1, 1) $A=1=B$

Hence the required stream lines are

$$xy^2=1 \quad \text{and} \quad xz^2=1. \qquad \textbf{Ans.}$$

Problem 4. *Find the equation of the stream lines for the flow*

$$\mathbf{q}=-\mathbf{i}\,(3y^2)-\mathbf{j}\,(6x)$$

at the point (1, 1).

Solution : The equations of streamline are given by

$$\frac{dx}{u}=\frac{dy}{v}$$

Here $\quad \mathbf{q}=-\mathbf{i}\,(3y^2)-\mathbf{j}\,(6x) \Rightarrow u=-3y^2, v=-6x.$

or $$\frac{dx}{-3y^2}=\frac{dy}{-6x} \Rightarrow \frac{2dx}{y^2}=\frac{dy}{x} \quad \text{or} \quad 2x\,dx=y^2\,dy$$

By integrating, we have

$$x^2=\frac{1}{3}y^3+c, \text{ where } c \text{ is an integration constant.}$$

At the point (1, 1), $\quad c=\frac{2}{3} \Rightarrow 3x^2=y^3+2.$ **Ans.**

which determines the equation of the stream lines for the flow field.

Problem 5. *The velocity field at a point in fluid is given as*

$$\mathbf{q}=(x/t, y, 0).$$

Obtain path lines and streak lines. (Meerut 2002)

Solution : Here $\mathbf{q}=(x/t, y, 0)$.

The differential equations of path lines are given by

$$\mathbf{q}=\frac{d\mathbf{r}}{dt}=\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}=\frac{x}{t}\mathbf{i}+y\mathbf{j}.$$

$$\Rightarrow \quad \frac{dx}{dt}=\frac{x}{t},\ \frac{dy}{dt}=y,\ \frac{dz}{dt}=0. \qquad \dots(1, 2, 3)$$

By integrating (1), we have

$$\frac{dx}{dt}=\frac{x}{t} \Rightarrow \log x=\log t+\log A \Rightarrow x=At. \qquad \dots(4)$$

Let (x_0, y_0, z_0) be the coordinates of the chosen fluid particle at time $t=t_0$, then

$$x_0=At_0 \Rightarrow A=\frac{x_0}{t_0}.$$

From (4), we have $\quad x=\frac{x_0}{t_0}t.$

By integrating (2), we have

$$\frac{dy}{y}=dt$$

or $$\log y=t+\log B \Rightarrow y=Be^t$$

At $\quad y=y_0, t=t_0 \Rightarrow B=y_0e^{-t_0}$

From (5), we have $$y=y_0e^{t-t_0} \qquad \dots(5)$$

By integrating (3), we have

$$\frac{dz}{dt}=0 \Rightarrow z=c \text{ i.e., } z \text{ is independent of } t \Rightarrow z=z_0.$$

Hence the path lines are given by

$$x=(x_0/t_0)\,t,\ y=y_0e^{t-t_0},\ z=z_0. \qquad \dots(6) \quad \textbf{Ans.}$$

Let the fluid particle (x_0, y_0, z_0) pass through a fixed point (x_1, y_1, z_1) at an instant of time $t=T$, where $t_0 \le T \le t$. Then the relation (6) reduces to

$$x_1=(x_0/t_0)\,T,\ y_1=y_0e^{T-t_0},\ z_1=z_0$$

or $$x_0=(x_1/T)\,t_0, y_0=y_1e^{t_0-T}, z_0=z_1 \qquad \dots(7)$$

where T is the parameter. Substituting the relation (7) into (6), we have

$$x=(x_1/T)\,t,\ y=y_1e^{t-T}, z=z_1,$$

which gives the equation of streak lines passing through the point (x_1, y_1, z_1).

Problem 6. *The velocity components in a two-dimensional flow field for an incompressible fluid are given by*

$$u=e^x\cosh y \quad \text{and} \quad v=-e^x\sinh y.$$

Determine the equation of the stream lines for this flow.

Solution : The equations of the stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v} \Rightarrow \frac{dx}{e^x\cosh y}=\frac{dy}{-e^x\sinh y} \quad \text{or} \quad dx+\coth y\,dy=0$$

By integrating, we have

$$x + \log \sinh y = \log c \quad \Rightarrow \quad \sinh y = ce^{-x}$$

where $\log c$ is an integration constant. **Ans.**

Problem 7. *Obtain the stream lines of a flow*

$$u = x; \quad v = -y.$$

Or, *If the velocity* **q** *is given by*

$$\mathbf{q} = x\mathbf{i} - y\mathbf{j},$$

determine the equation of the stream lines.

Solution: $\mathbf{q} = \mathbf{i}u + \mathbf{j}v + w\mathbf{k}$

Here we have $u = x, \quad v = -y, \quad w = 0.$

Stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{0}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dy}{-y}, \quad \frac{dx}{x} = \frac{dz}{0} \qquad (i.e., \ dz = 0)$$

Integrating these equations,

$$\log x + \log y = \log c, \quad z = c_1$$

or

$$xy = c, \quad z = c_1$$

Stream lines are given by $xy = c, \quad z = c_1$.

Problem 8. *Consider the velocity field given by*

$$\mathbf{q} = (1 + At)\,\mathbf{i} + x\mathbf{j}.$$

Find the equation of stream line at $t = t_0$ passing through the point (x_0, y_0). Also obtain the equation of path line of a fluid element which comes to (x_0, y_0) at $t = t_0$. Show that, if $A = 0$ (i.e., steady flow), the stream lines and path lines coincide.

Solution: $\mathbf{q} = (1 + At)\,\mathbf{i} + x\mathbf{j}$

and $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

This $\Rightarrow$ $u = 1 + At, \quad v = x, \quad w = 0.$

I. To determine stream lines.

These lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Stream lines at time $t = t_0$ are given by

$$\frac{dx}{1 + At_0} = \frac{dy}{x}$$

in two dimensional motion.

or

$$x\,dx = (1 + At_0)\,dy$$

Integrating

$$\frac{x^2}{2} = (1 + At_0)\,y + \frac{c}{2}$$

or

$$x^2 = 2\,(1 + At_0)\,y + c \qquad \dots (1)$$

$$x_0^2 = 2\,(1 + At_0)\,y_0 + c \qquad \dots (2)$$

(1) – (2) gives

$$x^2 - x_0^2 = 2\,(1 + At_0)\,(y - y_0) \qquad \textbf{Ans.}$$

II. To find path lines which pass through (x_0, y_0) at time $t = t_0$.

Equations of path lines are $\dot{x} = u, \quad \dot{y} = v.$

or

$$\frac{dx}{dt} = 1 + At, \quad \frac{dy}{dt} = x$$

$$\Rightarrow \quad dx = (1 + At)\,dt \qquad \dots (3)$$

$$dy = x\,dt \qquad \dots (4)$$

Integrating (3), we get

$$x = t + \frac{A}{2}t^2 + c_1 \qquad \dots (5)$$

Put $x = x_0, \quad t = t_0.$

$$x_0 = t_0 + \frac{A}{2}t_0^2 + c_1. \qquad \dots (6)$$

(5) – (6) gives

$$x - x_0 = (t - t_0) + \frac{A}{2}(t^2 - t_0^2) \qquad \dots (7)$$

Using (7) in (4),

$$dy = \left[x_0 + (t - t_0) + \frac{A}{2}(t^2 - t_0^2)\right]dt$$

Integrating,

$$y = x_0 t + \frac{t^2}{2} - t_0 t + \frac{A}{2}\left(\frac{t^3}{3} - t_0^2\,t\right) + c_2 \qquad \dots (8)$$

Putting, $y = y_0, \quad t = t_0,$ we get

$$y_0 = x_0 t_0 + \frac{t_0^2}{2} - t_0^2 + \frac{A}{2}\left(\frac{t_0^3}{3} - t_0^3\right) + c_2 \qquad \dots (9)$$

(8) – (9) gives

$$y - y_0 = x_0\,(t - t_0) + \frac{1}{2}(t^2 - t_0^2) - t_0\,(t - t_0) + \frac{A}{2}\left[\left(\frac{t^3 - t_0^3}{3}\right) - t_0^2\,(t - t_0)\right]$$

or

$$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t + t_0) - t_0 + \frac{A}{2}\left\{\left(\frac{t_0^2 + t^2 + tt_0}{3}\right) - t_0^2\right\}\right]$$

or

$$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t - t_0) + \frac{A}{6}(t^2 + tt_0 - 2t_0^2)\right] \qquad \dots (10)$$

Required path lines are given by (7) and (10).

III. Let $A = 0$.

To show that path lines and stream lines are coincident.

By step I, stream lines

$$x^2 - x_0^2 = 2(1 + At_0)(y - y_0)$$

or

$$x^2 - x_0^2 = 2(y - y_0) \qquad \dots (11)$$

By step II, path lines are given by

$$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t - t_0) + \frac{A}{6}(t^2 + tt_0 - 2t_0^2)\right], \text{ by (10)}$$

and

$$x - x_0 = (t - t_0) + \frac{A}{2}(t^2 - t_0^2), \qquad \text{by (7)}$$

This $\Rightarrow$

$$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t - t_0)\right]$$

and

$$x - x_0 = t - t_0$$

Eliminating $t - t_0$ from the last two equations,

$$y - y_0 = (x - x_0)\left[x_0 + \frac{1}{2}(x - x_0)\right]$$

or

$$2(y - y_0) = x^2 - x_0^2,$$

which is the same as equation (11). Hence stream lines and path lines are coincident.

Problem 9. *Prove that liquid motion is possible when velocity at (x, y, z) is given by*

$$u = \frac{3x^2 - r^2}{r^5}, \quad v = \frac{3xy}{r^5}, \quad w = \frac{3xz}{r^5}, \quad where \quad r^2 = x^2 + y^2 + z^2$$

and the stream lines are the intersection of the surfaces, $(x^2 + y^2 + z^2)^3 = c(y^2 + z^2)^2$, *by the planes passing through Ox.* *(Kanpur 1993, Agra 92)*

Solution : Step I. To prove that the liquid motion is possible. For this we have to show that the equation of continuity namely

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \dots (1)$$

is satisfied.

$$r^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial z}{\partial r} = \frac{z}{r}$$

$$\frac{\partial u}{\partial x} = \frac{(6x - 2x)r^5 - 5r^3x(3x^2 - r^2)}{r^{10}}, \quad \frac{\partial v}{\partial y} = \frac{3x}{r^{10}}(r^5 - 5r^3y^2),$$

$$\frac{\partial w}{\partial z} = \frac{3x}{r^{10}}(r^5 - 5r^3z^2).$$

This $\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{3x}{r^7}(3r^2 - 5x^2), \quad \frac{\partial v}{\partial y} = \frac{3x}{r^7}(r^2 - 5y^2),$$

$$\frac{\partial w}{\partial z} = \frac{3x}{r^7}(r^2 - 5z^2).$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$ Hence the result.

Step II. To determine stream lines.

Stream lines are the solutions of $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$. Putting the values,

$$\frac{dx}{3x^2 - r^2} = \frac{dy}{3xy} = \frac{dz}{3xz} = \frac{x\,dx + y\,dy + z\,dz}{x(3r^2 - r^2)} = \frac{y\,dy + z\,dz}{3x(y^2 + z^2)}.$$

This $\Rightarrow$

$$\frac{dy}{3xy} = \frac{dz}{3xz} \qquad \dots (2)$$

and

$$\frac{x\,dx + y\,dy + z\,dz}{2(x^2 + y^2 + z^2)} = \frac{y\,dy + z\,dz}{3(y^2 + z^2)} \qquad \dots (3)$$

(2) $\Rightarrow \frac{dy}{y} - \frac{dz}{z} = 0$, integrating this $\log\frac{y}{z} = \log a$

or $y = az$ ... (4), this is a plane through Ox.

Integrating (3), we get

$$\frac{1}{2}\log(x^2 + y^2 + z^2) = \frac{1}{3}\log(y^2 + z^2) + \frac{1}{6}\log b$$

or

$$(x^2 + y^2 + z^2)^3 = b(y^2 + z^2)^2. \qquad \dots (5)$$

Problem 10. *If the velocity of an incompressible fluid at the point (x, y, z) is given by*

IFoS 2009

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2 - r^2}{r^5}\right)$$

prove that the liquid motion is possible and the velocity potential is $\cos\theta/r^2$. *Also determine the stream lines.*

Solution : Given $u = \frac{3xz}{r^5}, \quad v = \frac{3yz}{r^5}, \quad w = \frac{3z^2 - r^2}{r^5}.$

Since $r^2 = x^2 + y^2 + z^2$ hence $\frac{\partial r}{\partial x} = \frac{x}{r}$ etc.

Step I. To prove that the liquid motion is possible. For this we have to prove that the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

is satisfied.

$$\frac{\partial u}{\partial x} = \frac{3z}{r^{10}}(r^5 - 5r^3x^2), \quad \frac{\partial v}{\partial y} = \frac{3z}{r^{10}}(r^5 - 5r^3y^2),$$

$$\frac{\partial w}{\partial z} = \frac{1}{r^{10}}[(6z - 2z)r^5 - 5r^3(3z^2 - r^2)z]$$

This $\Rightarrow$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{3z}{r^{10}}[2r^5 - 5r^3(r^2 - z^2)] + \frac{1}{r^{10}}(9zr^5 - 15r^3z^3) = 0.$$

Hence the result.

Step II. The show that $\phi = \cos\frac{\theta}{r^2}$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = -u\,dx - v\,dy - w\,dz$$

$$= -\frac{1}{r^5}[3xz\,dx + 3yz\,dy + (3z^2 - r^2)\,dz]$$

$$= -\frac{1}{r^5}[3z\,(x\,dx + y\,dy + z\,dz) - r^2\,dz]$$

$$= \frac{1}{r^5}\left[3z\,d\left(\frac{r^2}{2}\right) - r^2\,dz\right]$$

$$= -\frac{3z}{r^4}dr + \frac{dz}{r^3} = d\left(\frac{z}{r^3}\right).$$

Integrating, $\phi = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2}$, neglecting constant of integration.

Aliter. $\quad \frac{\partial \phi}{\partial x} = -u = \frac{-3xz}{r^5}$

Integrating w.r.t. x,

$$\phi = -\frac{3z}{2}\int (2x)\,(x^2 + y^2 + z^2)^{-5/2}\,dx$$

$$= \left(\frac{-3z}{2}\right)\left(\frac{-2}{3}\right)(x^2 + y^2 + z^2)^{-3/2}$$

or $$\phi = \frac{z}{(x^2 + y^2 + z^2)^{3/2}} = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2},$$

on neglecting constant of integration.

Step III. Stream lines are the solutions of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

Putting the values of respective terms,

$$\underset{(1)}{\frac{dx}{3xz}} = \underset{(2)}{\frac{dy}{3yz}} = \underset{(3)}{\frac{dz}{3z^2 - r^2}} = \underset{(4)}{\frac{x\,dx + y\,dy + z\,dz}{3z\,(x^2 + y^2 + z^2) - r^2 z}}$$

Taking the ratios (1) and (2), $\frac{dx}{x} = \frac{dy}{y}$.

Integration yields the result

$$\log x = \log y + \log a \quad \text{or} \quad x = ay. \qquad \ldots (5)$$

By (1) and (4),

$$\frac{dx}{3x} = \frac{x\,dx + y\,dy + z\,dz}{2r^2}$$

or $$\frac{4\,dx}{x} = 3\left(\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}\right).$$

Integrating, $\quad 4\log x = 3\log(x^2 + y^2 + z^2) + \log b$

or $$x^4 = b\,(x^2 + y^2 + z^2)^3. \qquad \ldots (6)$$

The (5) and (6) equations represent stream lines. **Ans.**

IAS 2008 Same model (12 M)

Problem 11. *Show that if velocity potential of an irrorational fluid motion is equal to $A\,(x^2 + y^2 + z^2)^{-3/2}\,z\tan^{-1}(y/x)$, the lines of flow lie on the series of the surfaces.*

$$x^2 + y^2 + z^2 = K^{2/3}\,(x^2 + y^2)^{2/3}.$$ *(Agra 2001; 2004; Kanpur 2002)*

Solution. Spherical co-ordinates are

$$x = r\sin\theta\cos\omega,\ y = r\sin\theta\sin\omega,\ z = r\cos\theta.$$

$$\phi = A\,(x^2 + y^2 + z^2)^{-3/2}\,z\tan^{-1}\frac{y}{x}$$

$$= A\,r^{-3}\,.\,r\cos\theta\tan^{-1}(\tan\omega)$$

or $\phi = A\,r^{-2}\,\omega\,.\cos\theta$. Lines of flow are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

or equivalently, $$\frac{dr}{-\frac{\partial \phi}{\partial r}} = \frac{r\,d\theta}{-\frac{1}{r}\frac{\partial \phi}{\partial \theta}} = \frac{r\sin\theta\,d\omega}{-\frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \omega}},$$

or $$\frac{dr}{\frac{2A\,\omega\cos\theta}{r^3}} = \frac{r\,d\theta}{\frac{1}{r}\frac{A}{r^2}\,\omega\sin\theta} = \frac{r\sin\theta\,d\omega}{-\frac{1}{r\sin\theta}\cdot\frac{A\cos\theta}{r^2}}$$

or $$\underset{(1)}{\frac{dr}{2\omega\cos\theta}} = \underset{(2)}{\frac{r\,d\theta}{\omega\sin\theta}} = \underset{(3)}{\frac{r\sin^2\theta\,d\omega}{-\cos\theta}}$$

By (1) and (2), $\quad \frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}\,d\theta.$

Integrating, $\quad \log r = 2\log\sin\theta + \log K$

or $$r = K\sin^2\theta = K\left(\frac{x^2 + y^2}{r^2}\right)$$

or $$r^3 = K\,(x^2 + y^2)$$

or $$(x^2 + y^2 + z^2)^{3/2} = K\,(x^2 + y^2)$$

or $$x^2 + y^2 + z^2 = K^{2/3}\,(x^2 + y^2)^{2/3}.$$

Stream lines lie on this surface. **Proved.**

2003 (12 M)

Problem 12. *Given $u = -c^2y/r^2$, $v = c^2x/r^2$, $w = 0$, where r denotes distance from z axis. Find the surfaces which are orthogonal to stream lines, the liquid being homogeneous.*

Solution : Step I : To show that liquid motion is possible, we have to show that the equation of continuity $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\right)$ is satisfied.

Here $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{2c^2y}{r^3}\cdot\frac{x}{r} - \frac{2c^2x}{r^3}\cdot\frac{y}{r} + 0 = 0$$

as $r^2 = x^2 + y^2$. Hence result I.

Step II : The surfaces orthogonal to stream lines are the solutions of

$$u dx + v dy + w dz = 0$$

i.e.,
$$-\frac{c^2 y}{r^2}dx + \frac{c^2 x}{r^2}dy + 0\, dz = 0$$

or
$$-\frac{dx}{x} + \frac{dy}{y} = 0, \text{ integrating this } \log\frac{y}{x} = \log a$$

or
$$\frac{y}{x} = a \quad \text{or} \quad y = ax.$$

This surface is orthogonal to stream lines. Ans.

Problem 13. *Show that*

$$u = -\frac{2xyz}{(x^2+y^2)^2}, \quad v = \frac{(x^2-y^2)z}{(x^2+y^2)^2}, \quad w = \frac{y}{x^2+y^2}$$

are the velocity components of a possible liquid motion. Is this motion irrotational ? *(Garhwal 2004)*

Solution : Step I : To show that the motion is possible, we have to show that the equation of continuity $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\right)$ is satisfied.

Here
$$\frac{\partial u}{\partial x} = -\frac{2yz}{(x^2+y^2)^4}\left[(x^2+y^2)^2 - 2(x^2+y^2)\, 2x^2\right]$$
$$= -\frac{2yz}{(x^2+y^2)^3}(y^2 - 3x^2),$$
$$\frac{\partial v}{\partial y} = \frac{z}{(x^2+y^2)^4}\left[-2y(x^2+y^2)^2 - (x^2-y^2)\, 2(x^2+y^2)\, 2y\right]$$
$$= -\frac{2yz}{(x^2+y^2)^3}(3x^2 - y^2)$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{2yz}{(x^2+y^2)^3}\left[(3x^2-y^2) + (y^2-3x^2) + 0\right] = 0.$$

Hence the result I.

Step II. To test the nature of the motion. The motion will be irrotational if

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\frac{2xz(x^2-3y^2)}{(x^2+y^2)^2} - \frac{2xz(3y^2-x^2)}{(x^2+y^2)^2} = 0$$

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0.$$

Hence the motion is irrotational.

Problem 14. *Given $u = -\omega y$, $v = \omega x$, $w = 0$; show that the surfaces intersecting the stream lines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist.*

Solution : Step I. To show that liquid motion is possible, we have to show that the equation of continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ is satisfied.

Here
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 + 0 + 0 = 0. \text{ Hence the result I.}$$

Step II. To show that the surfaces orthogonal to stream lines are planes through z-axis.

The required surfaces are solutions of

$$u\, dx + v\, dy + w\, dz = 0, \quad i.e.,$$
$$-\omega y\, dx + \omega x\, dy + 0.\, dz = 0$$

or
$$\frac{dx}{x} - \frac{dy}{y} = 0,$$

integrating $\log\frac{x}{y} = \log a$ or $\frac{x}{y} = a$ or $x = ay$,

which is a plane through z-axis.

Step III. To show that velocity potential ϕ does not exist.

By def.,
$$d\phi = -(u dx + v dy + w dz)$$
$$= -[-\omega y\, dx + \omega x\, dy + 0\, dz]$$

or
$$d\phi = \omega y\, dx - \omega x\, dy = M dx + N dy, \text{ say.}$$

Here
$$\frac{\partial M}{\partial y} = \omega, \; \frac{\partial N}{\partial x} = -\omega. \quad \text{Hence } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Therefore the equation is not exact so that $d\phi = \omega y dx - \omega x dy$ can not be integrated so that ϕ does not exist.

Problem 15. *In the steady motion of homogeneous liquid if the surfaces $f_1 = a_1$, $f_2 = a_2$ define the stream lines, prove that the most general values of the velocity components u, v, w are*

$$F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(y, z)}, \quad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(z, x)}, \quad F = (f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x, y)}.$$

Solution : Since the motion is steady, hence stream lines are independent of t. Therefore f_1 and f_2 are functions of x, y, z only.

$$f_1 = a_1, \; f_2 = a_2 \Rightarrow df_1 = 0, df_2 = 0 \Rightarrow$$

$$\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz = 0$$

$$\frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial z}dz = 0$$

$$\frac{dx}{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}} = \frac{-dy}{\frac{\partial f_2}{\partial z}\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial x}\frac{\partial f_1}{\partial z}} = \frac{dz}{\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}}$$

or
$$\frac{dx}{J_1} = \frac{dy}{J_2} = \frac{dz}{J_3} \quad \ldots(1)$$

where $\quad J_1 = \dfrac{\partial (f_1, f_2)}{\partial (y, z)}, \; J_2 = \dfrac{\partial (f_1, f_2)}{\partial (z, x)}, \; J_3 = \dfrac{\partial (f_1, f_2)}{\partial (x, y)}.$

But the stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \qquad \ldots (2)$$

On comparing (1) and (2), $\dfrac{u}{J_1} = \dfrac{v}{J_2} = \dfrac{w}{J_3} = F$, say.

$$\therefore \quad u = J_1 F, \; v = J_2 F, \; w = J_3 F. \qquad \ldots (3)$$

To determine the nature of F.

In order to make the liquid motion possible, the velocity components must satisfy the equation of continuity, namely.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

This $\Rightarrow \quad F\left(\dfrac{\partial J_1}{\partial x} + \dfrac{\partial J_2}{\partial y} + \dfrac{\partial J_3}{\partial z}\right) + \left(J_1 \dfrac{\partial F}{\partial x} + J_2 \dfrac{\partial F}{\partial y} + J_3 \dfrac{\partial F}{\partial z}\right) = 0$

By the property of Jacobian, $\dfrac{\partial J_1}{\partial x} + \dfrac{\partial J_2}{\partial y} + \dfrac{\partial J_3}{\partial z} = 0.$

Hence $\quad \dfrac{\partial (f_1, f_2)}{\partial (y, z)} \dfrac{\partial F}{\partial x} + \dfrac{\partial (f_1, f_2)}{\partial (z, x)} \dfrac{\partial F}{\partial y} + \dfrac{\partial (f_1, f_2)}{\partial (x, y)} \dfrac{\partial F}{\partial z} = 0$

or $\quad \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{vmatrix} = 0 \quad$ or $\quad \dfrac{\partial (F, f_1, f_2)}{\partial (x, y, z)} = 0.$

This proves that F, f_1, f_2 are not independent.

Therefore $F = F(f_1, f_2)$. Now (3) proves the required result.

Solved problems related to boundary surface.

Problem 16. *Show that the variable ellipsoid*

$$\frac{x^2}{a^2 k^2 t^4} + kt^2 \left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] = 1$$

is a possible form for the boundary surface of a liquid at any time t. *(Kanpur 1990)*

Solution : Let

$$F(x, y, z, t) = \frac{x^2}{a^2 k^2 t^4} + kt^2 \left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] - 1 = 0 \qquad \ldots (1)$$

To show that $F = 0$ is a possible form of boundary surface, it is enough to prove that

$$u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0. \qquad \ldots (2)$$

Putting the values of respective terms,

$$\frac{u\, 2x}{a^2 k^2 t^4} + vkt^2 \frac{2y}{b^2} + wkt^2 \frac{2z}{c^2} - \frac{4x^2}{a^2 k^2 t^5} + 2kt\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] = 0$$

or $\quad \dfrac{2x}{a^2 k^2 t^4}\left(u - \dfrac{2x}{t}\right) + \dfrac{2k}{b^2} t^2 y \left(v + \dfrac{y}{t}\right) + \dfrac{2k}{c^2} t^2 z \left(w + \dfrac{z}{t}\right) = 0$

Hence (2) is satisfied if we take

$$u - \frac{2x}{t} = 0, \quad v + \frac{y}{t} = 0, \quad w + \frac{z}{t} = 0,$$

i.e., if $\quad u = \dfrac{2x}{t}, \quad v = -\dfrac{y}{t}, \quad w = -\dfrac{z}{t}$

It will be a justificable step if the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

is satisfied.

Here $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = \dfrac{2}{t} - \dfrac{1}{t} - \dfrac{1}{t} = 0.$

Hence (1) is a possible form of boundary surface.

Similar Problem : *Show that the ellipsoid*

$$\frac{x^2}{a^2 k^2 t^{2n}} + kt^n \left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 1$$

is a possible form of boundary surface.

Problem 17. Show that $\quad \dfrac{x^2}{a^2} \tan^2 t + \dfrac{y^2}{b^2} \cot^2 t - 1 = 0 \qquad \ldots (1)$

is a possible form of boundary surface and find an expression for normal velocity.

(Kanpur 2004; Meerut 2003; Gharwal 2002)

Solution : To show that $F = 0$ is a possible form of boundary surface, we have to show that

$$u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0 \qquad \ldots (2)$$

Putting the values of various terms, we get

$$u \frac{2x}{a^2} \tan^2 t + v \cdot \frac{2y}{b^2} \cot^2 t + w.0 + \left(\frac{2x^2}{a^2} \tan t \sec^2 t - \frac{2y^2}{b^2} \cot t \, \operatorname{cosec}^2 t\right) = 0$$

or $\quad \dfrac{2x}{a^2} \tan^2 t \left(u + \dfrac{x \sec^2 t}{\tan t}\right) + \dfrac{2y}{b^2} \cot^2 t \left(v - \dfrac{y \operatorname{cosec}^2 t}{\cot t}\right) = 0.$

Thus (2) will be satisfied if we take

$$u + \frac{x \sec^2 t}{\tan t} = 0, \quad v - y \frac{\operatorname{cosec}^2 t}{\cot t} = 0,$$

i.e., $\quad u = \dfrac{-x}{\sin t \cos t}, \quad v = \dfrac{y}{\sin t \cos t}.$

This will be a justifiable step if the equation of continuity, namely

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \text{ is satisfied.}$$

Now

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-\frac{1}{\sin t\cos t}+\frac{1}{\sin t\cos t}+0=0.$$

Hence (1) is a possible form of boundary surface.

Second Part. Normal velocity $=\dfrac{-\partial F/\partial t}{|\nabla F|}$

$$=\frac{-\left(\dfrac{2x}{a^2}\tan t\sec^2 t-\dfrac{2y}{b^2}\cot t\operatorname{cosec}^2 t\right)}{\left[\left(\dfrac{2x}{a^2}\tan^2 t\right)^2+\left(\dfrac{2y}{b^2}\cot^2 t\right)^2\right]^{1/2}}$$

$$=-\frac{(b^2x\tan t\sec^2 t-a^2 t\cot t\operatorname{cosec}^2 t)}{(b^4x^2\tan^4 t+a^4y^2\cot^4 t)^{1/2}} \qquad \text{Ans.}$$

Problem 18. *Determine the restriction on f_1, f_2, f_3 if*

$$\frac{x^2}{a^2}f_1(t)+\frac{y^2}{b^2}f_2(t)+\frac{z^2}{c^2}f_3(t)=1$$

is a possible form of boundary surface of a liquid. (Garhwal 2003)

Solution : Let $F=\dfrac{x^2}{a^2}f_1(t)+\dfrac{y^2}{b^2}f_2(t)+\dfrac{z^2}{c^2}f_3(t)-1=0$ ... (1)

To show that $F=0$ is a possible form of boundary surface, we have to prove that ($F=0$) satisfies the condition

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial z}+w\frac{\partial F}{\partial z}=0 \qquad \text{... (2)}$$

Putting the values of respective terms,

$$\frac{x^2}{a^2}f_1'+\frac{y^2}{a^2}f_2'+\frac{z^2}{c^2}f_3'+u\frac{2x}{a^2}f_1+v\frac{2y}{b^2}f_2+w\frac{2z}{c^2}f_3=0$$

or

$$\frac{2x}{a^2}f_1\left(u+\frac{xf_1'}{2f_1}\right)+\frac{2y}{b^2}f_2\left(v+\frac{yf_2'}{2f_2}\right)+\frac{2z}{c^2}f_3\left(w+\frac{zf_3'}{2f_3}\right)=0.$$

If we take $u+x\dfrac{f_1'}{2f_1}=0,\quad v+y\dfrac{f_2'}{2f_2}=0,\quad w+\dfrac{z}{2f_2}f_3'=0$, then (2) is satisfied. This will be a justifiable step if the values of u, v, w satisfy the equation of continuity.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Putting the values,

$$-\frac{1}{2}\left[\frac{f_1'}{f_1}+\frac{f_2'}{f_2}+\frac{f_3'}{f_3}\right]=0$$

Integrating, $\log f_1f_2f_3=\log c$ or $f_1f_2f_3=c.$ Ans.

Problem 19. *Show that all necessary and sufficient conditions can be satisfied by a velocity potential of the form $\phi=\alpha x^2+\beta y^2+\gamma z^2$, and the bounding surface of the form*

$$F=ax^4+by^4+cz^4-X(t)=0,$$

where $X(t)$ is a given function of time and $\alpha, \beta, \gamma, a, b, c$ are suitable functions of the time.

Solution : Let $\phi=\alpha x^2+\beta y^2+\gamma z^2$ ... (1)

and $F(x, y, z, t)=ax^4+by^4+cz^4-X(t)=0.$ ... (2)

Step I. To prove that ϕ satisfies all the necessary conditions (*i.e.*, equation of continuity)

or

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

Putting the values of respective terms,

$$2\alpha+2\beta+2\gamma=0 \quad \text{or} \quad \alpha+\beta+\gamma=0.$$

The velocity potential ϕ has to satisfy this condition.

Step II. To prove $F=0$ satisfies the condition of boundary surface. We know that

$$u=-\frac{\partial\phi}{\partial x},\quad v=-\frac{\partial\phi}{\partial y},\quad w=-\frac{\partial\phi}{\partial z}$$

This $\Rightarrow$ $u=-2\alpha x,\quad v=-2\beta y,\quad w=-2\gamma z.$

$F=0$ will be a boundary surface if it satisfies the condition,

$$u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}+\frac{\partial F}{\partial t}=0.$$

Putting the values of respective terms,

$$-2\alpha x\cdot 4ax^3-2\beta y\cdot 4by^3-2\gamma z\cdot 4cz^3+x^4a'+y^4b'+z^4c'-X'(t)=0$$

or

$$x^4(a'-8a\alpha)+y^4(b'-8\beta b)+z^4(c'-8\gamma c)-X'(t)=0 \qquad \text{... (3)}$$

Since (2) and (3) both have to hold for all points (x, y, z) on the surface hence they should be identical. Comparing, we get

$$\underset{(4)}{\frac{a'-8a\alpha}{a}}=\underset{(5)}{\frac{b'-8\beta b}{b}}=\underset{(6)}{\frac{c'-8\gamma c}{c}}=\underset{(7)}{\frac{X'(t)}{X(t)}}$$

By (4) and (7), $\dfrac{da}{dt}-8a\alpha=\dfrac{a}{X}\cdot\dfrac{dX}{dt}$

or

$$\frac{da}{a}=8\alpha dt+\frac{dX}{X}$$

Integrating, $\log a=\log X+\int 8\alpha dt.$

Similarly, $\log b=\log X+\int 8\beta\, dt,$ by (5) and (7)

and $\log c=\log X+\int 8\gamma\, dt,$ by (6) and (7).

s.t. $\alpha+\beta+\gamma=0$

The surface $F=0$ will have to satisfy these conditions for the possible form of boundary surface.

Problem 20. *Prove that a surface of the form*

$$ax^4+by^4+cz^4-X(t)=0$$

is a possible form of boundary surface of a homogeneous liquid at time t, the velocity potential of the liquid motion being

$$\phi=(\beta-\gamma)x^2+(\gamma-\alpha)y^2+(\alpha-\beta)z^2$$

where X, α, β, γ are given functions of time and a, b, c are suitable functions of time. *(Kanpur 2003)*

Solution : Proceed as above.

Here equation of continuity $\Rightarrow$ $(\beta-\gamma)+\gamma-\alpha+\alpha-\beta=0$

Condition of boundary surface $\Rightarrow$

$$\log a=8\int(\beta-\gamma)\,dt+\log X$$

$$\log b=8\int(\gamma-\alpha)\,dt+\log X$$

$$\log c=8\int(\alpha-\beta)\,dt+\log X$$

Problem 21. *Show that*

$$\frac{x^2}{a^2}f(t)+\frac{y^2}{b^2}\phi(t)=1,$$

where $f(t)\,\phi(t)$ = const. is a possible form of the boundary surface of a liquid. *(Kanpur 1993)*

Solution : Let $\quad F=\dfrac{x^2}{a^2}f(t)+\dfrac{y^2}{b^2}\phi(t)-1=0.$ ... (1)

To prove $F=0$ is a possible form of boundary surface. For this we have to prove that

$$u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}+\frac{\partial F}{\partial t}=0. \quad \text{... (2)}$$

Putting the values,

$$u\frac{2x}{a^2}f+v\frac{2y}{b^2}\phi+w.0+\frac{x^2}{a^2}f'+\frac{y^2}{b^2}\phi'=0$$

or $$\frac{2x}{a^2}f\left(u+\frac{x}{2}\frac{f'}{f}\right)+\frac{2y\phi}{b^2}\left(v+\frac{y\phi'}{2\phi}\right)=0.$$

If we take $u+\dfrac{x}{2}\dfrac{f'}{f}=0$, $v+\dfrac{y}{2}\cdot\dfrac{\phi'}{\phi}=0$, then the condition (2) will be satisfied.

Here we get

$$u=-\frac{x}{2}\cdot\frac{f'}{f},\quad v=-\frac{y}{2}\frac{\phi'}{\phi}.$$

This will be a justifiable step if the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

is satisfied. Putting the values,

$$-\frac{1}{2}\cdot\frac{f'}{f}-\frac{1}{2}\cdot\frac{\phi'}{\phi}+0=0$$

or $$\frac{df}{f}+\frac{d\phi}{\phi}=0$$

Integrating, $\log f\phi=\log$ const. or $f\phi$ = const. which is given. Hence (1) is a possible form of boundary surface.

Solved Problems related to equation of continuity :

Problem 22. *A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders; show that the equation of continuity is*

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0,$$

where v_θ, v_z are velocities perpendicular and parallel to z. *(Meerut 1993; Kanpur 2000)*

Solution : Consider a point P whose cylindrical co-ordinates are (r,θ,z). With P as centre, construct a parallelopiped with edges of lengths dr, $r\,d\theta$, dz. Since lines of motion lie on the surface of the cylinders hence the fluid lies on the surface of the cylinders. It means that there is no velocity in the direction of dr. Equation of continuity gives

$$\frac{\partial}{\partial t}(\rho\,dr\,.\,r\,d\theta\,.\,dz)$$

$$=-\left[dr\frac{\partial}{\partial r}(\rho\,.\,0\,.\,r\,d\theta\,.\,dz)+r\,d\theta\frac{\partial}{r\,\partial\theta}(\rho v_\theta\,.\,dr\,.\,dz)+dz\frac{\partial}{\partial z}(\rho v_z\,dr\,.\,r\,.\,d\theta)\right]$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r}\left[\frac{\partial}{\partial\theta}(\rho v_\theta)+r\frac{\partial}{\partial z}(\rho v_z)\right]=0$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Problem 23. *If every particle moves on the surface of a sphere, prove that the equation of continuity is*

$$\frac{\partial\rho}{\partial t}\cos\theta+\frac{\partial}{\partial\theta}(\rho\omega\cos\theta)+\frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)=0,$$

ρ being the density, θ, φ the latitude and longitude respectively of an element and ω, ω′ the angular velocities of any element in latitude and longitude respectively. *(Garhwal 2001)*

Solution : Step I. To determine the equation of continuity in spherical co-ordinates. Consider an arbitrary point whose polar co-ordinates are (r,θ,ϕ). With P as centre, construct a parallelopiped with edges of lengths dr, $r\,d\theta$, $r\sin\theta\,d\phi$.

Let q_1, q_2, q_3 be velocity components at P along dr, r, $d\theta$, $r \sin\theta\, d\phi$, respectively. The equation of continuity gives

$$\frac{\partial}{\partial t}(\rho\, dr \,.\, r\, d\theta \,.\, r \sin\theta\, d\phi)$$

$$= -\left[dr \frac{\partial}{\partial r}(\rho q_1 \,.\, r\, d\theta \,.\, r \sin\theta\, d\phi) + r\, d\theta \frac{\partial}{r\,\partial\theta}(\rho q_2 \,.\, dr \,.\, r \sin\theta\, d\phi) + r \sin\theta\, d\phi \frac{\partial}{r \sin\theta\, \partial\phi}(\rho q_3 \,.\, dr \,.\, r\, d\theta) \right]$$

Simplifying, we get

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}(\rho q_1 r^2) + r\frac{\partial}{\partial\theta}(\rho q_2 \,.\, \sin\theta) + r\frac{\partial}{\partial\phi}(\rho q_3)\right] = 0$$

or
$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho q_2 \,.\, \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_3) = 0 \quad \ldots (1)$$

This is the equation of continuity in spherical co-ordinates.

Step II. To determine the equation of continuity in the required case.

It is given that fluid particles move on the surface of sphere, hence $q_1 = 0$.

To get the equation of continuity in present case, we have to replace θ by $90 - \theta$ in equation (1) and $d\theta$ by $d(90 - \theta) = -d\theta$.

For OP line makes an angle $90 - \theta$ with z-axis.

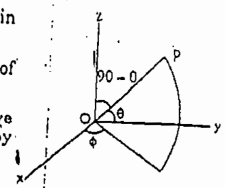


Fig. 8

$$\dot{\theta} = \omega, \quad \dot{\phi} = \omega'$$

"$q_2 = r\dot{\theta}$" gives $q_2 = r\frac{d}{dt}(90 - \theta) = -r\dot{\theta} = -r\omega$

"$q_3 = r \sin\theta\, \dot{\phi}$" gives

$$q_3 = r\sin(90 - \theta)\,\phi' = (r\cos\theta)\,\omega'$$

Putting these values in (1),

$$\frac{\partial\rho}{\partial t} + 0 + \frac{1}{r\sin(90-\theta)}\left(-\frac{\partial}{\partial\theta}\left\{\rho(-r\omega)\cos\theta\right\}\right) + \frac{1}{r\sin(90-\theta)}\frac{\partial}{\partial\phi}(\rho r\cos\theta\,\omega') = 0$$

or
$$\frac{\partial\rho}{\partial t} + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}(\rho\omega\cos\theta) + \frac{1}{\cos\theta}\frac{\partial}{\partial\phi}(\rho\cos\theta\,\omega') = 0$$

or
$$\frac{\partial\rho}{\partial t}\cos\theta + \frac{\partial}{\partial\theta}(\rho\omega\cos\theta) + \frac{\partial}{\partial\phi}(\rho\cos\theta\,\omega') = 0$$

This is the required equation of continuity.

Problem 24. *If the lines of motion are curves on the surface of cones having their vertices at the origin and the axis of z for common axis, prove that the equation of continuity is*

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial r}(\rho q_r) + \frac{2\rho q_r}{r} + \frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial\omega}(\rho q_\omega) = 0.$$

(Meerut 2002, Garhwal 2000)

Solution : Step I. To derive the equation of continuity in spherical co-ordinates. (Here write Step I of Problem 23).

Step II. To determine the equation of continuity in the required case. It is given that lines of flow lie on the surfaces of cones and hence velocity perpendicular to the surface is zero so that $q_2 = 0$. Now (1) becomes

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_3) = 0.$$

Replacing q_1 by q_r, q_3 by q_ω and ϕ by ω,

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta\,.\,\partial\omega}\frac{\partial}{}(\rho q_\omega) = 0$$

or
$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial r}(\rho q_r) + \frac{2}{r}\rho q_r + \frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial\omega}(\rho q_\omega) = 0.$$

Problem 25. *If the lines of motion are curves on the surfaces of spheres all touching the xy-plane at the origin O, the equation of continuity is*

$$r\sin\theta\frac{\partial\rho}{\partial t} + \frac{\partial\rho v}{\partial\phi} + \sin\theta\frac{\partial(\rho u)}{\partial\theta} + \rho u(1 + 2\cos\theta) = 0$$

where r is the radius CP of one of the spheres, θ the angle PCO, u the velocity in the plane PCO, v the perpendicular velocity, and φ the inclination of the plane PCO to a fixed plane through z-axis.

Solution : We consider any two consecutive spheres with centres C and C'.

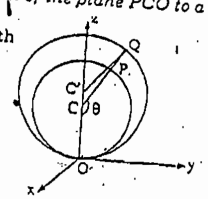


Fig. 9

Let $CP = r$, $C'Q = r + \delta r$, $\angle PCO = \theta$.

Then $CC' = \delta r$, $CQ = CP + PQ = r + PQ$

Since
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$\therefore$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Applying this formula in $\Delta C'CQ$,

$$C'Q^2 = C'C^2 + CQ^2 - 2C'C.CQ\cos(\pi - \theta)$$

or
$$(r + \delta r)^2 = (\delta r)^2 + (r + PQ)^2 + 2\delta r\,(r + PQ)\cos\theta$$

Neglecting PQ^2,

$$r\,\delta r - r\,\delta r\cos\theta = PQ\,(r + \delta r\cos\theta)$$

or
$$PQ = r\,\delta r\,(1 - \cos\theta)\,(r + \delta r\cos\theta)^{-1}$$

$$= \delta r\,(1 - \cos\theta)\left(1 + \frac{\delta r}{r}\cos\theta\right)^{-1}$$

$$= \delta r\,(1 - \cos\theta)\left(1 - \frac{\delta r}{r}\cos\theta\right)$$

$$= \delta r\,(1 - \cos\theta)$$

neglecting δr^2 and its higher powers.

$$PQ = (1 - \cos\theta)\,\delta r.$$

Since the lines of flow lie on the surfaces of the spheres, hence velocity along PQ is zero. Now we consider a parallelopiped with edges of lengths $(1 - \cos\theta)\,\delta r$, $r\,\delta\theta$, $r \sin\delta\phi$, the velocities along these elements are o, u, v respectively. The equation of continuity gives

$$\frac{\partial}{\partial t}\{\rho\,(1-\cos\theta)\,dr\,.\,rd\theta\,.\,r\sin\theta\,d\phi\}$$

$$= -\Big[(1-\cos\theta)\,dr\,.\,\frac{\partial}{(1-\cos\theta)\,\partial r}\,(\rho\,.\,o\,.\,r\,d\theta\,.\,r\sin\theta\,d\phi)$$

$$+\,rd\theta\,\frac{\partial}{r\,\partial\theta}\,\{\rho u\,(1-\cos\theta)\,dr\,.\,r\sin\theta\,.\,d\phi\}$$

$$+\,r\sin\theta\,d\phi\,\frac{\partial}{r\sin\theta\,\partial\phi}\,\{\rho v\,.\,(1-\cos\theta)\,dr\,.\,r\,d\theta\}\Big]$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r^2\sin\theta\,(1-\cos\theta)}\left[r\,\frac{\partial}{\partial\theta}\,\{\rho u\,(1-\cos\theta)\sin\theta\} + r\,(1-\cos\theta)\,\frac{\partial}{\partial\phi}\,(\rho v)\right] = 0$$

or $$r\sin\theta\,\frac{\partial\rho}{\partial t} + \frac{1}{(1-\cos\theta)}\,\frac{\partial}{\partial\theta}\,\{\rho u\,(1-\cos\theta)\sin\theta\} + \frac{\partial}{\partial\phi}\,(\rho v) = 0$$

or $$r\sin\theta\,\frac{\partial\rho}{\partial t} + \sin\theta\,\frac{\partial}{\partial\theta}\,(\rho u) + \frac{\partial}{\partial\phi}\,(\rho v) + \rho u\,(1 + 2\cos\theta) = 0.$$

For $(1-\cos\theta)\cos\theta + \sin^2\theta = (1-\cos\theta)\,(\cos\theta + 1 + \cos\theta)$.

Problem 26. *The particles of a fluid move symmetrically in space with regard to a fixed centre; prove that the equation of continuity is*

$$\frac{\partial\rho}{\partial t} + u\,\frac{\partial\rho}{\partial r} + \frac{\rho}{r^2}\,\frac{\partial}{\partial r}\,(r^2 u) = 0.$$

where u is the velocity at a distance r. *(Meerut 1992)*

Solution : Here first prove :

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\,\frac{\partial}{\partial r}\,(\rho\,q r^2) = 0 \qquad \ldots (1)$$

(This is equation (1) of Article 1.22, Case I, Page 29).

Put $q = u$ in (1), then

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\,\frac{\partial}{\partial r}\,(\rho\,u r^2) = 0$$

$$\Rightarrow \quad \frac{\partial\rho}{\partial t} + \frac{1}{r^2}\left[(r^2 u)\,\frac{\partial\rho}{\partial r} + \rho\,\frac{\partial}{\partial r}\,(ur^2)\right] = 0$$

$$\Rightarrow \quad \frac{\partial\rho}{\partial t} + u\,\frac{\partial\rho}{\partial r} + \frac{\rho}{r^2}\,\frac{\partial}{\partial r}\,(ur^2) = 0.$$

Problem 27. *If ω is the area of cross section of a stream filament, prove that the equation of continuity is*

$$\frac{\partial}{\partial t}\,(\rho\omega) + \frac{\partial}{\partial s}\,(\rho\omega q) = 0$$

Solution : Consider a volume bounded by the cross-sections through points P and Q where Q where Q is at a distance ds from P. Mass of the fluid within the volume $= \rho\omega\,ds$. By def. of continuity, rate of generation of mass

$=$ excess of flow in over flow out through this volume.

i.e., $$\frac{\partial}{\partial t}\,(\rho\omega\,ds) = -\,ds\,\frac{\partial}{\partial s}\,(\rho q\,\omega)$$

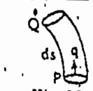


Fig. 10

or $$\frac{\partial\,(\rho\omega)}{\partial t} + \frac{\partial}{\partial s}\,(\rho\omega\,q) = 0$$

Problem 28. *A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis; show that the equation of continuity is*

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\theta}\,(\rho\omega) = 0$$

where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

Solution : Consider a point P whose polar co-ordinates are (r, θ). Let there be an elementary area $r\,\delta\theta\,\delta r$, when this area is revolved about O, then it describes a circle so that velocity OP vanishes. By equation of continuity,

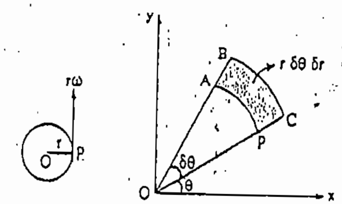


Fig. 11

$$\frac{\partial}{\partial t}\,(\rho\,r\,\delta\theta\,\delta r) = -\left[\delta r\,\frac{\partial}{\partial r}\,(\rho\,.\,0\,.\,r\,\delta\theta) + r\,\delta\theta\,\frac{\partial}{r\,\partial\theta}\,(\rho\,q\,\delta r)\right]$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r}\left[0 + \frac{\partial}{\partial\theta}\,(\rho\omega r)\right] = 0. \qquad \text{For } q = r\omega$$

or $$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\theta}\,(\rho\omega) = 0.$$

Problem 29. *Show that in the motion of a fluid in two dimensions if the current co-ordinates (x, y) are expressible in terms of initial co-ordinates (a, b) and the time, then the motion is irrotational if*

$$\frac{\partial\,(\dot{x}, x)}{\partial\,(a, b)} + \frac{\partial\,(\dot{y}, y)}{\partial\,(a, b)} = 0.$$

(Meerut 2003)

Solution : Let u, v be velocity components parallel to the axis of x and y, respectively. Then

$$\dot{x} = u,\ \dot{y} = v,\ \frac{\partial u}{\partial a} = \frac{\partial u}{\partial x}\,\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\,\frac{\partial y}{\partial a}$$

Observe that

$$\frac{\partial\,(\dot{x}, x)}{\partial\,(a, b)} + \frac{\partial\,(\dot{y}, y)}{\partial\,(a, b)} = \frac{\partial\,(u, x)}{\partial\,(a, b)} + \frac{\partial\,(v, y)}{\partial\,(a, b)} = \begin{vmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} \\ \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} \end{vmatrix} + \begin{vmatrix} \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} \end{vmatrix}$$

$$= \frac{\partial u}{\partial a}\frac{\partial x}{\partial b} - \frac{\partial x}{\partial a}\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial y}{\partial a}\frac{\partial v}{\partial b}$$

$$= \frac{\partial x}{\partial b}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a}\right) - \frac{\partial x}{\partial a}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b}\right) + \frac{\partial y}{\partial b}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial a}\right) - \frac{\partial y}{\partial a}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial b}\right)$$

$$= \frac{\partial u}{\partial y}\left(\frac{\partial x}{\partial b}\frac{\partial y}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\right) + \frac{\partial v}{\partial x}\left(\frac{\partial y}{\partial b}\frac{\partial x}{\partial a} - \frac{\partial y}{\partial a}\frac{\partial x}{\partial b}\right)$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial (x, y)}{\partial (a, b)}$$

$$= 0 \quad \text{iff} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

or iff motion is irrotational.

Problem 30. *If q is the resultant velocity at any point of a fluid which is moving irrotationally in two dimensions, prove that*

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = q\,\nabla^2 q$$

(Kanpur 2002)

Solution : Since motion is irrotational, therefore ϕ exists. Equation of continuity is

$$\nabla^2\phi = 0 \quad \text{or} \quad \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \dots (1)$$

$q = -\nabla\phi$ gives

$$q^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 \qquad \dots (2)$$

Differentiating (2) partially w.r.t. x and y respectively,

$$q\frac{\partial q}{\partial x} = \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\,\partial y} \qquad \dots (3)$$

$$q\frac{\partial q}{\partial y} = \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial y\,\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2} \qquad \dots (4)$$

Again differentiating (3) w.r.t. x and (4) w.r.t. y, we get

$$\left(\frac{\partial q}{\partial x}\right)^2 + q\frac{\partial^2 q}{\partial x^2} = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial x^3} + \left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial x^2\,\partial y} \qquad \dots (5)$$

$$\left(\frac{\partial q}{\partial y}\right)^2 + q\frac{\partial^2 q}{\partial y^2} = \left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial y^2\,\partial x} + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2 + \frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial y^3} \qquad \dots (6)$$

Adding (5) and (6),

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 + q\,\nabla^2 q = \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\} + \left\{\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right\} + 2\frac{\partial^2\phi}{\partial x\,\partial y}$$

Using (1) and noting that $\frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\phi}{\partial y^2}$, we get

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 + q\,\nabla^2 q = \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}(0) + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}(0) + 2\left[\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial x^2}\right)^2\right]$$

$$= 2\left[\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial x^2}\right)^2\right] \qquad \dots (7)$$

Squaring and adding (3) and (4),

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right] = \left(\frac{\partial\phi}{\partial x}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\,\partial x}\right)^2\right] + \left(\frac{\partial\phi}{\partial y}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right] + 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\left[\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial x\,\partial y} + \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial x\,\partial y}\right]$$

But $\frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\phi}{\partial y^2}$. Hence the last gives

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right] = \left(\frac{\partial\phi}{\partial x}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\,\partial x}\right)^2\right] + \left(\frac{\partial\phi}{\partial y}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial x^2}\right)^2\right] + 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\left[\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial x\,\partial y}\right]$$

$$= \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x\,\partial y}\right)^2\right] + 0$$

Using (2),

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right] = q^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\,\partial x}\right)^2\right]$$

or

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\,\partial x}\right)^2$$

Using this in (7),

$$\left(\frac{\partial q}{\partial x}\right) + \left(\frac{\partial q}{\partial y}\right)^2 + q\,\nabla^2 q = 2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right]$$

or

$$q\,\nabla^2 q = \left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2$$

Proved.

Problem 31. *The velocity components for a two-dimensional fluid system can be given in the Eulerian system by $u = 2x + 2y + 3t$, $v = x + y + \frac{1}{2}t$. Find the displacement of a fluid particle in the Lagrangian system.*

(Kanpur 2000, 2005)

Solution :

$$u = 2x + 2y + 3t \qquad \dots (1)$$

$$v = x + y + \frac{1}{2}t \qquad \dots (2)$$

$$u = Dx = \dot{x} = \frac{dx}{dt}, \quad v = \dot{y} = Dy = \frac{dy}{dt}, \quad \frac{d}{dt} = D.$$

By (1) and (2),

$$(D - 2)\,x - 2y = 3t \qquad \dots (3)$$

$$(D - 1)\,y - x = \frac{1}{2}t \qquad \dots (4)$$

Operationg (4) by $D-2$,

$$(D-2)(D-1)y-(D-2)x=\frac{1}{2}(D-2)t$$

or
$$(D^2-3D+2)y-(D-2)x=\frac{1}{2}(1-2t) \qquad \ldots (5)$$

(3) + (5) gives

$$(D^2-3D+2)y-2y=\frac{1}{2}+2t$$

or
$$(D^2-3D)y=\frac{1}{2}+2t$$

Auxiliary equation is given by

$$m^2-3m=0, \quad \text{this} \quad \Rightarrow m=0,3$$

$$\text{C.F.}=c_1e^{0t}+c_2e^{3t}=c_1+c_2e^{3t}$$

$$\text{P.I.}=\frac{1}{D^2-3D}\left(\frac{1}{2}+2t\right)=-\frac{1}{3D}\left(1-\frac{D}{3}\right)^{-1}\left(\frac{1}{2}+2t\right)$$

$$=-\frac{1}{3D}\left(1+\frac{D}{3}\ldots\ldots\right)\left(\frac{1}{2}+2t\right)$$

$$=-\frac{1}{3D}\left[\left(\frac{1}{2}+2t\right)+\frac{1}{3}(2)\right]=-\frac{1}{3D}\left[\frac{7}{6}+2t\right]$$

$$=-\frac{1}{3}\left(\frac{7}{6}t+t^2\right)$$

y = C.F. + P.I. gives

$$y=c_1+c_2e^{3t}-\frac{1}{3}\left(\frac{7}{6}t+t^2\right) \qquad \ldots (6)$$

$$Dy=3c_2e^{3t}-\frac{1}{3}\left(\frac{7}{6}+2t\right) \qquad \ldots (6')$$

By (4), $\quad Dy-y-x=\frac{1}{2}t$

or
$$x=Dy-y-\frac{1}{2}t$$

Using (6) and (6′),

$$x=3c_2e^{3t}-\frac{1}{3}\left(\frac{7}{6}+2t\right)-\left(c_1+c_2e^{3t}-\frac{7t}{18}-\frac{1}{3}t^2\right)-\frac{1}{2}t$$

or
$$x=-c_1+2c_2e^{3t}-\frac{7}{18}-\frac{7}{9}t+\frac{1}{3}t^2 \qquad \ldots (7)$$

By (6),
$$y=c_1+c_2e^{3t}-\frac{7}{18}t-\frac{t^2}{3}. \qquad \ldots (8)$$

Initial conditions are $x=x_0, y=y_0$ at $t=0$.

Putting in (7) and (8), we get

$$x_0=-c_1+2c_2-\frac{7}{18}, \quad y_0=c_1+c_2$$

Solving these two, we get

$$c_1=\left(\frac{2y_0-x_0}{3}\right)-\frac{7}{54}$$

$$c_2=\left(\frac{x_0+y_0}{3}\right)+\frac{7}{54}$$

Putting these values in (7) and (8), we get the required expressions :

$$x=\frac{1}{3}(x_0-2y_0)+\left[\frac{2}{3}(x_0+y_0)+\frac{7}{27}\right]e^{3t}-\frac{7}{27}-\frac{7}{9}t+\frac{t^2}{3}$$

$$y=\frac{1}{3}(2y_0+x_0)+\left[\left(\frac{x_0+y_0}{3}+\frac{7}{54}\right]e^{3t}-\frac{7}{18}t-\frac{t^2}{3}-\frac{7}{54} \qquad \text{Ans.}$$

Problem 32. *The velocities at a point in a fluid in the Eulerian system are given by $u=x+y+z+t$, $v=2(x+y+z)+t$, $w=3(x+y+z)+t$. Find the displacement of a fluid particle in the Lagrangian system. Also determine the velocity of the fluid particle at $(x_0\,y_0\,z_0)$.* *(Garhwal 2000)*

Solution : The velocity components may be expressed in terms of the displacement as

$$u=\frac{dx}{dt}=x+y+z+t, \qquad \ldots (1)$$

$$v=\frac{dy}{dt}=2(x+y+z)+t, \qquad \ldots (2)$$

$$w=\frac{dz}{dt}=3(x+y+z)+t. \qquad \ldots (3)$$

The differential equations can be written in form of operator as

$$(D-1)x-y=z+t \qquad \ldots (4)$$

$$-2x+(D-2)y=2z+t \qquad \ldots (5)$$

$$-3x-3y+(D-3)z=t \qquad \ldots (6)$$

Multiplying (4) by $(D-2)$ and adding to (5), we have

$$[(D-1)(D-2)-2]x=(D-2)z+2z+(D-2)t+t$$

$$(D^2-3D)x=Dz+1-t. \qquad \ldots (7)$$

Multiplying (4) by 2 and (5) by $(D-1)$ and adding, we have

$$[(D-1)(D-2)-2]y=(D-1)(2z+t)+2z+2t$$

or
$$(D^2-3D)y=2Dz+1+t. \qquad \ldots (8)$$

Multiplying (6) by (D^2-3D), we have

$$(D^2-3D)(D-3)z=3(D^2-3D)x+3(D^2-3D)y+(D^2-3D)t$$

From (7) and (8), we have

$$(D^2-3D)(D-3)z=3(Dz+1-t)+3(2Dz+1+t)+(D^2-3D)t$$

$$D^2(D-6)z=3. \qquad \ldots (9)$$

The solution of the differential equation (9) is given by

$$z=A+Bt+Ce^{6t}-\frac{1}{4}t^2, \qquad \ldots (10)$$

From the equations (5) and (6), we have

$$(D-2)\,y - 2z = 2x + t,$$
$$-3y + (D-3)\,z = 3x + t.$$

Solving the equations, we have

$$(D^2 - 5D)\,y = 2Dx + 1 - t \qquad \ldots (11)$$
$$(D^2 - 5D)\,z = 3Dx + 1 + t. \qquad \ldots (12)$$

From (1), we have

$$(D-1)\,x = y + z + t$$

or $$(D-1)(D^2 - 5D)\,x = (D^2 - 5D)\,y + (D^2 - 5D)\,z + (D^2 - 5D)\,t$$

or $$(D-1)(D^2 - 5D)\,x = 2Dx + 1 - t + 3Dx + 1 + t - 5$$

or $$(D^3 - 6D^2)\,x = -3 \qquad \ldots (13)$$

The solution of the differential equation becomes

$$x = A_1 + B_1 t + C_1 e^{6t} + \frac{1}{4}t^2. \qquad \ldots (14)$$

Proceeding in the same manner, we have

$$y = A_2 + B_2 t + C_2 e^{6t} \qquad \ldots (15)$$

Thus the equations (10), (14) and (15) determine the displacement of a fluid particle.

Let $x = x_0,\ y = y_0,\ z = z_0$ when $t = t_0 = 0$

The relations (14), (15) and (10) give

$$x_0 = A_1 + C_1,\quad y_0 = A_2 + C_2,\quad z_0 = A + C$$

Thus
$$x = x_0 - C_1 + B_1 t + C_1 e^{6t} + \frac{1}{4}t^2, \qquad \ldots (16)$$
$$y = y_0 - C_2 + B_2 t + C_2 e^{6t}, \qquad \ldots (17)$$
$$z = z_0 - C + Bt + Ce^{6t} - \frac{1}{4}t^2. \qquad \ldots (18)$$

Substituting these values in (1), (2) and (3), we obtain the following identities

$$B_1 + 6C_1 e^{6t} + \frac{1}{2}t = x_0 + y_0 + z_0 - (C_1 + C_2 + C) + (B_1 + B_2 + B)\,t + (C_1 + C_2 + C)\,e^{6t} + t \qquad \ldots (19)$$

$$B_2 + 6C_2 e^{6t} = 2\,(x_0 + y_0 + z_0) - 2\,(C_1 + C_2 + C) + 2\,(B_1 + B_2 + B)\,t + 2\,(C_1 + C_2 + C)\,e^{6t} + t, \qquad \ldots (20)$$

$$B + 6Ce^{6t} - \frac{1}{2}t = 3\,(x_0 + y_0 + z_0) - 3\,(C_1 + C_2 + C) + 3\,(B_1 + B_2 + B)\,t + 3\,(C_1 + C_2 + C)\,e^{6t} + t. \qquad \ldots (21)$$

Equating the coefficients of t, e^{6t} and the constant term, we have

$$\left.\begin{aligned} x_0 + y_0 + z_0 - (C_1 + C_2 + C) &= B_1, \\ C_1 + C_2 + C &= 6C_1, \\ B_1 + B_2 + B + 1 &= \frac{1}{2}, \end{aligned}\right\} \qquad \ldots (22)$$

$$\left.\begin{aligned} 2\,(x_0 + y_0 + z_0) - (C_1 + C_2 + C) &= B_2, \\ 2\,(C_1 + C_2 + C) &= 6C_2, \\ 2\,(B_1 + B_2 + B) + 1 &= 0\,. \end{aligned}\right\} \qquad \ldots (23)$$

$$\left.\begin{aligned} 3\,(x_0 + y_0 + z_0) - 3\,(C_1 + C_2 + C) &= B, \\ 3\,(C_1 + C_2 + C) &= 6C, \\ 3\,(B_1 + B_2 + B) + 1 &= -\frac{1}{2}. \end{aligned}\right\} \qquad \ldots (24)$$

From these three sets, we obtain

$$C_1 = \frac{1}{6}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right),\quad C_2 = \frac{1}{3}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right),$$
$$C = \frac{1}{2}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right).$$

Also $$B_1 = -\frac{1}{12},\quad B_2 = -\frac{1}{6},\quad B = -\frac{1}{4}.$$

Substituting these values in the relations (16), (17), and (18) and simplifying, we get

$$x = \frac{5}{6}x_0 - \frac{1}{6}y_0 - \frac{1}{6}z_0 + \frac{1}{6}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{12}t + \frac{1}{4}t^2 - \frac{1}{72},$$
$$y = -\frac{1}{3}x_0 + \frac{2}{3}y_0 - \frac{1}{3}z_0 + \frac{1}{3}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{6}t - \frac{1}{36},$$
$$z = -\frac{1}{2}x_0 - \frac{1}{2}y_0 + \frac{1}{2}z_0 + \frac{1}{2}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{4}t - \frac{1}{4}t^2 - \frac{1}{24},$$

determines the displacement of a fluid particle in Lagrangian description.

Let u_1, v_1, w_1 be the components of the velocity in the Lagrangian description.

We have

$$u_1 = \frac{\partial x}{\partial t} = \left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{12} + \frac{1}{2}t,$$
$$v_1 = \frac{\partial y}{\partial t} = 2\left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{6},$$
$$w_1 = \frac{\partial z}{\partial t} = 3\left(x_0 + y_0 + z_0 + \frac{1}{12}\right) - \frac{1}{4} - \frac{1}{2}t.$$

Thus the velocity of the fluid particle is given by

$$q_1 = u_1\mathbf{i} + v_1\mathbf{j} + w_1\mathbf{k}$$

Ans.

Problem 33. *The velocity components of flow in cylindrical co-ordinates are $(r^2 z\cos\theta,\ rz\sin\theta,\ z^2 t)$. Determine the components of acceleration of a fluid particle.*

Solution : Let u, v, w be velocity components in cylindrical co-ordinates (r, θ, z). We know that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial \theta}.$$

Given $\quad u = r^2 z \cos\theta, \quad v = rz \sin\theta, \quad w = z^2 t$

Let a_1, a_2, a_3 be components of acceleration.

$$\mathbf{a} = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3$$

$$\mathbf{a} = \frac{d}{dt}\mathbf{q} = \left(\frac{\partial}{\partial t} + \mathbf{q}.\nabla\right)\mathbf{q}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{q}.\nabla) = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + r^2 z \cos\theta \frac{\partial}{\partial r} + z \sin\theta \frac{\partial}{\partial \theta} + z^2 t \frac{\partial}{\partial z} \quad \ldots (1)$$

$$a_1 = \frac{du}{dt} - \frac{v^2}{r}; \quad a_2 = \frac{dv}{dt} + \frac{vw}{r}; \quad a_3 = \frac{dw}{dt}$$

$$a_1 = \left(\frac{\partial}{\partial t} + r^2 z \cos\theta \frac{\partial}{\partial r} + z \sin\theta \frac{\partial}{\partial \theta} + z^2 t \frac{\partial}{\partial z}\right)(r^2 z \cos\theta) - \frac{v^2}{r}$$

$$= 0 + (r^2 z \cos\theta)(2rz\cos\theta) + (z \sin\theta)(-r^2 z \sin\theta) + (z^2 t)(r^2 \cos\theta) - rz^2 \sin^2\theta$$

$$= rz^2\,[2r^2\cos^2\theta - r\sin^2\theta - \sin^2\theta + rt\cos\theta]$$

$$a_2 = \frac{dv}{dt} + \frac{uv}{r}$$

$$= \left(\frac{\partial}{\partial t} + r^2 z \cos\theta \frac{\partial}{\partial r} + z \sin\theta \frac{\partial}{\partial \theta} + z^2 t \frac{\partial}{\partial z}\right) rz \sin\theta + z^2 r^2 \sin\theta \cos\theta$$

$$= 0 + (r^2 z \cos\theta)(z \sin\theta) + (z \sin\theta)(rz \cos\theta) + (z^2 t)\, r \sin\theta + z^2 r^2 \sin\theta\cos\theta$$

$$= z^2 \sin\theta\,[2r^2\cos\theta + r\cos\theta + rt]$$

$$a_3 = \frac{dw}{dt} = \left(\frac{\partial}{\partial t} + r^2 z \cos\theta \frac{\partial}{\partial r} + z \sin\theta \frac{\partial}{\partial \theta} + z^2 t \frac{\partial}{\partial z}\right)(z^2 t)$$

$$= z^2 + 0 + 0 + z^2 t\,(2zt) = z^2\,[1 + 2zt^2]$$

Finally,

Ans. $\begin{cases} a_1 = rz^2\,[2r^2\cos^2\theta - r\sin^2\theta - \sin^2\theta + rt\cos\theta] \\ a_2 = z^2\sin\theta\,[2r^2\cos\theta + r\cos\theta + rt] \\ a_3 = z^2\,[1 + 2zt^2] \end{cases}$

Problem 34. *Give examples of irrotational and rotational flows.*

Solution. I. Cconsider fluid motion given by $u = kx,\ v = 0,\ w = 0,\ (k \neq 0)$

Then $\quad \mathbf{q} = \mathbf{i}kx$

$$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx & 0 & 0 \end{vmatrix}$$

$$\text{curl } \mathbf{q} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(0) = 0$$

$\therefore$ Motion is irrotational.

II. Consider fluid motion given by

$$u = ay,\ v = 0,\ w = 0,\ (a \neq 0)$$

$$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & 0 & 0 \end{vmatrix}$$

$$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(0 - a).$$

$\therefore \quad \text{curl } \mathbf{q} = -a\mathbf{k} \neq 0.$

Hence motion is not irrotational.

Consequently motion is rotational.

Problem 35. *If velocity distribution is*

$$\mathbf{q} = \mathbf{i}(Ax^2yt) + \mathbf{j}(By^2zt) + \mathbf{k}(Czt^2)$$

where A, B, C are constants, then find acceleration and vorticity components.

(Garhwal 2001, Kanpur 2001)

Solution : Let $\quad \mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}.$

Then $\quad u = Ax^2yt, \quad v = By^2zt, \quad w = Czt^2$

I. Let $\mathbf{a} = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3$ denote acceleration. Then

$$\mathbf{a} = \frac{d\mathbf{q}}{dt}, \quad a_1 = \frac{du}{dt} \text{ etc.}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{q}.\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial y} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + Ax^2yt\frac{\partial}{\partial x} + By^2zt\frac{\partial}{\partial y} + Czt^2\frac{\partial}{\partial z} \quad \ldots (1)$$

$a_1 = \dfrac{du}{dt}$ gives

$$a_1 = \left(\frac{\partial}{\partial t} + Ax^2yt\frac{\partial}{\partial x} + By^2zt\frac{\partial}{\partial t} + Czt^2\frac{\partial}{\partial z}\right)(Ax^2yt).$$

$$= Ax^2y + (Ax^2yt)(2Axyt) + (By^2zt)(Axt) + (Czt^2)(0)$$

$$= Ax^2y\,[1 + 2Axyt^2 + Byzt^2] \quad \ldots (2)$$

$a_2 = \dfrac{dv}{dt}$ with $\dfrac{d}{dt}$ given by (1)

$$a_2 = \left(\frac{\partial}{\partial t} + Ax^2yt\frac{\partial}{\partial x} + By^2zt\frac{\partial}{\partial y} + Czt^2\frac{\partial}{\partial z}\right)(By^2zt)$$

$$= By^2z + (Ax^2yt)(0) + (By^2zt)(2Byzt) + (Czt^2)(By^2t)$$

$$= By^2z\,[1 + 2Byzt^2 + Ct^3] \quad \ldots (3)$$

$a_3 = \dfrac{dw}{dt}$ with $\dfrac{d}{dt}$ given by (1),

$$= \left(\frac{\partial}{\partial t} + Ax^2yt\frac{\partial}{\partial x} + By^2zt\frac{\partial}{\partial y} + Czt^2\frac{\partial}{\partial z}\right)Czt^2$$

$$a_3 = 2Czt + (Ax^2y)(0) + (By^2zt)(0) + (Czt^2)(Ct^2)$$

$$= Czt\,(2 + Ct^2) \qquad \ldots (4)$$

Ans. Acceleration components are given by (2), (3) and (4).

II. Let $\mathbf{W} = \text{curl } \mathbf{q}$. Then $\mathbf{W}$ is vorticity vector.

$$\mathbf{W} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax^2yt & By^2zt & Czt^2 \end{vmatrix}$$

$$= \mathbf{i}\,(0 - By^2t) - \mathbf{j}\,(0 - 0) + \mathbf{k}\,(0 - Ax^2t)$$

Vorticity components are

$$-By^2t, \quad 0, \quad -Ax^2t. \qquad \text{Ans.}$$

Problem 36. *Test whether the motion specified by*

$$\mathbf{q} = \frac{k^2(x\mathbf{j} - y\mathbf{i})}{x^2 + y^2} \quad (k = \text{const.})$$

is a possible motion for an incompressible fluid. If so, determine the equations of stream lines. Also tell whether the motion is of the potential kind and if it determines the velocity potential. (Meerut 2002)

Solution : Here $u = \dfrac{-k^2y}{x^2+y^2}, \quad v = \dfrac{k^2x}{x^2+y^2}, \quad w = 0.$

I. Equation of continuity for incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

But
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{2k^2xy}{(x^2+y^2)^2} - \frac{2k^2xy}{(x^2+y^2)^2} + 0$$

$$= 0$$

Hence equation of continuity is satisfied.

II. Stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{0}$$

or
$$\frac{dx\,(x^2+y^2)}{-k^2y} = \frac{(x^2+y^2)\,dy}{k^2x} = \frac{dz}{0}$$

$\Rightarrow$
$$x\,dx + y\,dy = 0, \quad dz = 0$$

$\Rightarrow$
$$x^2 + y^2 = a^2, \quad z = b$$

Hence stream lines are circles whose centres lie on z-axis.

III. To test the existence of velocity potential.

$$-d\phi = u\,dx + v\,dy + w\,dz$$

$$= -k^2y\frac{dx}{x^2+y^2} + k^2x\frac{dy}{x^2+y^2}$$

$$d\phi = k^2\left[\frac{y\,dx}{x^2+y^2} - \frac{x}{x^2+y^2}\,dx\right]$$

$$= k^2\,(M\,dx + N\,dy), \text{ say}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2+y^2} + y\left[\frac{-2y}{(x^2+y^2)^2}\right] = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = -\left[\frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2}\right] = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial M}{\partial y}$$

Hence $M\,dx + N\,dy$ is exact. Therefore its solution of given by

$$\phi = \int \frac{k^2y\,dx}{x^2+y^2} + \int 0\,dy + C = \frac{k^2y}{y}\tan^{-1}\left(\frac{x}{y}\right) + C$$

Hence ϕ exists and is given by

$$\phi = k^2\tan^{-1}\left(\frac{x}{y}\right) + C \qquad \text{Ans.}$$

Problem 37. *The velocity vector in the flow field is given by*

$$\mathbf{q} = \mathbf{i}\,(Az - By) + \mathbf{j}\,(Bx - Cz) + \mathbf{k}\,(Cy - Ax)$$

where A, B, C are non-zero constants.

Determine the equations of the vortex lines. (Meerut 1993)

Solution : Let $\mathbf{W} = \mathbf{i}\xi + \mathbf{j}\eta + \mathbf{k}\zeta$ be the vorticity vector. Then $\mathbf{W} = \text{curl } \mathbf{q}$

or
$$\mathbf{W} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Az - By & Bx - Cz & Cy - Ax \end{vmatrix}$$

$$= \mathbf{i}\,(C + C) - \mathbf{j}\,(-A - A) + \mathbf{k}\,(B + B)$$

This $\Rightarrow \quad \xi = 2C, \quad \eta = 2A, \quad \zeta = 2B$

Vortex lines are given by

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

Putting the values,

$$\frac{dx}{2C} = \frac{dy}{2A} = \frac{dz}{2B}$$

or
$$\frac{dx}{C} = \frac{dy}{A} = \frac{dz}{B}$$

$\Rightarrow$
$$A\,dx - C\,dy = 0, \quad B\,dy - A\,dz = 0$$

Integrating,

$$Ax - Cy = c_1, \quad By - Az = c_2$$

Vortex lines are given by these equations.

Problem 38. *Show that $\phi = (x - t)(y - t)$ represents the velocity potential of an incompressible two dimensional fluid. Show that the stream lines at time t are the curves*

$$(x-t)^2-(y-t)^2=\text{constant}$$

and that the paths of fluid particles have the equations

$$\log(x-y)=\frac{1}{2}[(x+y)-a(x-y)^{-1}+b,$$

where a and b are constants.

Solution : Given $\quad \phi=(x-t)(y-t) \quad \ldots(1)$

I. To show that the liquid motion is possible,

$$\frac{\partial\phi}{\partial x}=y-t,\quad \frac{\partial\phi}{\partial y}=x-t$$

$$\Rightarrow\quad \frac{\partial^2\phi}{\partial x^2}=0,\quad \frac{\partial^2\phi}{\partial y^2}=0 \quad\Rightarrow\quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

$\Rightarrow \quad \nabla^2\phi=0,$ which is the equation of continuity.

Hence (1) represents velocity potential of an incompressible two dimensional fluid.

II. To determine stream lines.

$$u=-\frac{\partial\phi}{\partial x}=-(y-t)$$

$$v=-\frac{\partial\phi}{\partial y}=-(x-t)$$

Stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v}$$

$$\frac{dx}{-(y-t)}=\frac{dy}{-(x-t)}$$

or $\quad (x-t)\,dx=(y-t)\,dy$

Integrating, $\quad \frac{x^2}{2}-tx=\frac{y^2}{2}-ty+\text{const.}$

or $\quad x^2-2tx=y^2-2ty+\text{const.}$

or $\quad (x-t)^2=(y-t)^2+\text{const.}$

or $\quad (x-t)^2-(y-t)^2=\text{const.}$

which represents stream lines.

III. To determine path lines.

$$\frac{dx}{dt}=u=-\frac{\partial\phi}{\partial x}=-(y-t)$$

$$\frac{dy}{dt}=v=-\frac{\partial\phi}{\partial y}=-(x-t)$$

$$\Rightarrow\quad dx=(t-y)\,dt \quad \ldots(2)$$

$$dy=(t-x)\,dt \quad \ldots(3)$$

Upon subtraction, $\quad dx-dy=(x-y)\,dt$

or $\quad \frac{dx-dy}{x-y}=dt$

Integrating, $\quad \log(x-y)=t+\log c$

or $\quad x-y=c\,e^t \quad \ldots(4)$

(2) + (3) $\quad\Rightarrow\quad dx+dy=[2t-(x+y)]\,dt \quad \ldots(5)$

Put $x+y=u$, $dx+dy=du$, then (5) gives

$$\frac{du}{dt}+u=2t \quad \ldots(6)$$

It is of the type $\frac{dy}{dx}+Py=Q$ whose solution is

$$ye^{\int P\,dx}=c+\int Qe^{\int P\,dx}\,dx$$

Hence solution of (6) is

$$u\,e^t=k+\int 2t\,e^t\,dt$$

$$u\,e^t=k+2(t-1)\,e^t$$

or $\quad u=k\,e^{-t}+2(t-1)$

or $\quad (x+y)=\frac{kc}{x-y}+2\log\left(\frac{x-y}{c}\right)-2,$ by (4)

or $\quad \log(x-y)=\frac{1}{2}[(x+y)-kc\,(x-y)^{-1}]+1+\log c$

Taking $\quad 1+\log c=b,\quad \frac{kc}{2}=a,\quad$ we get

$$\log(x-y)=\frac{1}{2}[(x+y)-a(x-y)^{-1}]+b$$

This represents path lines.

Ex. 23. *Determine whether the motion specified by*

$$\mathbf{q} = \frac{A(x\mathbf{j} - y\mathbf{i})}{x^2 + y^2}, \quad (A = const.)$$

is a possible motion for an incompressible fluid. If so, determine the equations of the streamlines. Also, show that the motion is of potential kind. Find the velocity potential.

Solution. We know that

$$\nabla . \mathbf{q} = 0. \qquad \ldots(1)$$

or $$A\left\{-\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right\} = 0,$$

or $$A\left\{\frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}\right\} = 0,$$

which is evident. Thus the equation of continuity for an incompressible fluid is satisfied and hence it is a possible motion for an incompressible fluid.

The equation of the streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

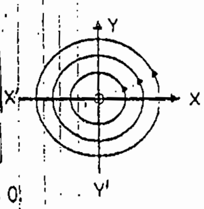
or $$\frac{dx}{-Ay/(x^2+y^2)} = \frac{dy}{Ax/(x^2+y^2)} = \frac{dz}{0}$$

or $$x\,dx + y\,dy = 0, \; dz = 0$$

By integrating, we have

$$x^2 + y^2 = \text{constant}, \; z = \text{constant}. \qquad \ldots(2, 3)$$

Thus the streamlines are circles whose centres are on Z-axis, their planes being perpendicular to the axis.



Again $$\nabla \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\frac{Ay}{(x^2+y^2)} & \frac{Ax}{(x^2+y^2)} & 0 \end{vmatrix}$$

or $$\nabla \times \mathbf{q} = \mathbf{k}\left[\frac{\partial}{\partial x}\left\{\frac{Ax}{x^2+y^2}\right\} + \frac{\partial}{\partial y}\left\{\frac{Ay}{x^2+y^2}\right\}\right]$$

or $$\nabla \times \mathbf{q} = \mathbf{k}A\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right] = 0$$

Thus the flow is of potential kind, so we can determine $\phi(x, y, z)$ such that

$$\mathbf{q} = -\nabla\phi$$

or $$\frac{\partial \phi}{\partial x} = -u = \frac{Ay}{x^2+y^2}, \; \frac{\partial \phi}{\partial y} = -v = -\frac{Ax}{x^2+y^2},$$
$$\frac{\partial \phi}{\partial z} = -w = 0, \qquad \ldots(4, 5, 6)$$

which shows that ϕ is independent of z, hence

$$\phi = \phi(x, y).$$

Integrating the relation (4), we have

$$\phi(x, y) = A \tan^{-1}(x/y) + f(y)$$

or $$\frac{\partial \phi}{\partial y} = f'(y) - Ax/(x^2+y^2).$$

Using the relation (5), we get

$$f'(y) = 0 \Rightarrow f(y) = \text{constant}.$$

Therefore $$\phi(x, y) = A \tan^{-1}(x/y).$$ **Ans.**

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Ex. 24. *Show that the velocity potential*

$$\phi = \frac{1}{2}a(x^2 + y^2 - 2z^2)$$

satisfies the Laplace equation. Also determine the streamlines.

Solution. Let ϕ be the velocity potential for the velocity field $\mathbf{q}$ then

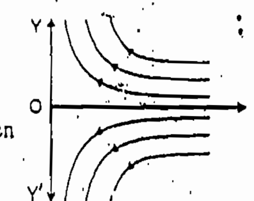
$$\mathbf{q} = -\nabla\phi = -\tfrac{1}{2}a\nabla(x^2 + y^2 - 2z^2)$$
$$\mathbf{q} = -\tfrac{1}{2}a(2x\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}).$$

Taking divergence of both the sides, we have

$$\nabla^2\phi = -\nabla \cdot \mathbf{q}$$

or $$\nabla^2\phi = -\tfrac{1}{2}a\nabla \cdot (2x\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}) = 0$$

or $$\nabla^2\phi = -\tfrac{1}{2}a(2 + 2 - 4) = 0$$



Hence Laplace equation is satisfied.

The equation of streamlines are given by

$$dx/u = dy/v = dz/w$$

or $$\underset{\text{(i)}}{dx/(-ax)} = \underset{\text{(ii)}}{dy/(-ay)} = \underset{\text{(iii)}}{dz/(2az)}$$

From (ii) and (iii), we have

$$-\log y = \tfrac{1}{2}\log z - \log C,$$

where C is an integration constant.

or $$y^2 z = C,$$

which represents a cubic hyperbola. **Proved.**

Ex. 25. *Show that*

$$u = -\frac{2xyz}{(x^2+y^2)^2}, \; v = \frac{(x^2-y^2)z}{(x^2+y^2)^2}, \; w = \frac{y}{x^2+y^2},$$

are the velocity components of a possible liquid motion. Is this motion irrotational ?

Solution. The condition for the possible liquid motion is given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

$$\Rightarrow \quad 2yz\cdot\frac{3x^2-y^2}{(x^2+y^2)^3}+2yz\cdot\frac{y^2-3x^2}{(x^2+y^2)^3}+0=0,$$

which is an identity. Hence (u, v, w) are the velocity components of a possible liquid motion.

Again the condition for irrotational motion is

$$\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}=0,\ \frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}=0 \text{ and } \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0$$

So
$$\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}=\frac{x^2-y^2}{(x^2+y^2)^2}-\frac{x^2-y^2}{(x^2+y^2)^2}=0,$$

$$\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}=-\frac{2xy}{(x^2+y^2)^2}+\frac{2xy}{(x^2+y^2)^2}=0,$$

and
$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\frac{2xz\,(3y^2-x^2)}{(x^2+y^2)^2}-\frac{2xz\,(3y^2-x^2)}{(x^2+y^2)^2}=0.$$

Thus $\nabla\times q=0$, $\Rightarrow$ that the motion is irrotational. Proved.

Ex. 26. *Find the necessary and sufficient condition that vortex lines may be at right angles to the streamlines.*

Solution. The equations of the streamlines and the vortex lines are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w},$$

and
$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta} \qquad \ldots(1,2)$$

The equation (1) and (2) are at right angles. It follows that

$$u\,\xi+v\,\eta+w\,\zeta=0$$

$$\Rightarrow \quad u\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+v\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+w\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0.$$

In order that $u\,dx+v\,dy+w\,dz$ may be a perfect differential, we have

$$u\,dx+v\,dy+w\,dz=\lambda\,d\phi=\lambda\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right)$$

$$\Rightarrow \quad u=\lambda\frac{\partial\phi}{\partial x},\ v=\lambda\frac{\partial\phi}{\partial y},\ w=\lambda\frac{\partial\phi}{\partial z},$$

which determines the necessary and sufficient condition. Ans.

Ex. 27. *In an incompressible fluid the vorticity at every point is constant in magnitude and direction; prove that the components of velocity u, v, w are the solutions of Laplace equation.*

Solution. Let $\underline{\Omega}$ be the vorticity at any point in an incompressible fluid then

$$\underline{\Omega}=\xi\,\mathbf{i}+\eta\,\mathbf{j}+\zeta\,\mathbf{k}$$

where $\xi=\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z},\ \eta=\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x},\ \zeta=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=0.$

The magnitude and direction cosines of its direction are given by

$$\Omega=\sqrt{\xi^2+\eta^2+\zeta^2} \quad\text{and}\quad \frac{\xi}{\Omega},\frac{\eta}{\Omega},\frac{\zeta}{\Omega}$$

Differentiating η partially with regard to z and ζ with regard to y and subtracting, we have

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

Hence the velocity components satisfy Laplace Equation. Ans.

Ex. 28. *Find the vorticity components of a fluid particle when velocity distribution is*

$$q=\mathbf{i}\,(k_1x^2yt)+\mathbf{j}\,(k_2y^2zt)+\mathbf{k}\,(k_3zx^2),$$

where k_1, k_2, k_3 are constants.

Solution. The vorticity components ξ,η,ζ are given by

$$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=-k_2y^2t,$$

$$\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0$$

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=-k_1x^2t. \qquad \text{Ans.}$$

Ex. 29. *Determine the equations of the vortex lines when the velocity vector of the flow field is given by*

$$q=\mathbf{i}\,(Az-By)+\mathbf{j}\,(Bx-Cz)+\mathbf{k}\,(Cy-Ax),$$

where A, B, C are constants.

Solution. The vorticity components are given by

$$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=C+C=2C,$$

$$\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=A+A=2A,$$

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=B+B=2B.$$

The equations of the vortex lines are

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta}$$

$$\Rightarrow \quad \frac{dx}{2C}=\frac{dy}{2A}=\frac{dz}{2B}$$
$$\text{(i)}\qquad\text{(ii)}\qquad\text{(iii)}$$

From (i) and (ii), we have

$$Ax - Cy = k_1, \qquad \ldots(1)$$

From (ii) and (iii), we have

$$By - Az = k_2, \text{ where } k_1 \text{ and } k_2 \text{ are integration constants.} \ldots(2)$$

Hence the vortex lines (1) and (2) are the straightlines. Ans.

Ex.30. *Investigate the nature of the liquid motion given by*

$$u = \frac{ax - by}{x^2 + y^2}, v = \frac{ay + bx}{x^2 + y^2}, w = 0.$$

Also, determine the velocity potential.

Solution. Here $u = \dfrac{ax - by}{x^2 + y^2}, v = \dfrac{ay + bx}{x^2 + y^2}, w = 0$

$$\frac{\partial u}{\partial x} = \frac{a(x^2 + y^2) - 2x(ax - by)}{(x^2 + y^2)^2} = \frac{a(y^2 - x^2) + 2bxy}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{a(x^2 + y^2) - 2y(ay + bx)}{(x^2 + y^2)^2} = \frac{a(x^2 - y^2) - 2bxy}{(x^2 + y^2)^2}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Thus the liquid motion satisfies the continuity equation hence it is a possible motion.

Let Ω be the vorticity then

$$\Omega = i\xi + j\eta + k\zeta,$$

where

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0,$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0,$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

It follows that the nature of the liquid motion is irrotational.

Let ϕ be the velocity potential, then

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = -u\,dx - v\,dy$$

$$\Rightarrow \quad d\phi = -\left\{\frac{ax - by}{x^2 + y^2}dx + \frac{ay + bx}{x^2 + y^2}dy\right\}$$

$$\Rightarrow \quad d\phi = -\left\{\frac{a(x\,dx + y\,dy)}{x^2 + y^2} + \frac{b(x\,dy - y\,dx)}{x^2 + y^2}\right\}$$

$$\Rightarrow \quad \phi = -\frac{1}{2}a\log(x^2 + y^2) + b\tan^{-1}\left(\frac{y}{x}\right). \qquad \text{Answer.}$$

Ex. 31. *If* $u\,dx + v\,dy + w\,dz = d\theta + \lambda\,d\mu$ *where* λ, θ, μ *are functions of* x, y, z *and* t, *prove that the vortex lines at any time are the lines of intersection of the surfaces* λ = *const. and* μ = *const.*

Solution. We know that

$$u\,dx + v\,dy + w\,dz = d\theta + \lambda\,d\mu$$

or

$$u\,dx + v\,dy + w\,dz = \left(\frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy + \frac{\partial \theta}{\partial z}dz + \frac{\partial \theta}{\partial t}dt\right) + \lambda\left(\frac{\partial \mu}{\partial x}dx + \frac{\partial \mu}{\partial y}dy + \frac{\partial \mu}{\partial z}dz + \frac{\partial \mu}{\partial t}dt\right)$$

Equating coefficient of dx, dy, dz and dt, we have

$$u = \frac{\partial \theta}{\partial x} + \lambda\frac{\partial \mu}{\partial x}, \; v = \frac{\partial \theta}{\partial y} + \lambda\frac{\partial \mu}{\partial y},$$

$$w = \frac{\partial \theta}{\partial z} + \lambda\frac{\partial \mu}{\partial z}, \; 0 = \frac{\partial \theta}{\partial t} + \lambda\frac{\partial \mu}{\partial t}.$$

The components of spin are,

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial z} + \lambda\frac{\partial \mu}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \theta}{\partial y} + \lambda\frac{\partial \mu}{\partial y}\right)$$

$$\Rightarrow \quad 2\xi = \lambda\frac{\partial^2 \mu}{\partial y\,\partial z} + \frac{\partial \lambda}{\partial y}\frac{\partial \mu}{\partial z} - \lambda\frac{\partial^2 \mu}{\partial y\,\partial z} - \frac{\partial \lambda}{\partial z}\frac{\partial \mu}{\partial y}$$

$$\Rightarrow \quad 2\xi = \frac{\partial \lambda}{\partial y}\frac{\partial \mu}{\partial z} - \frac{\partial \lambda}{\partial z}\frac{\partial \mu}{\partial y}$$

$$\Rightarrow \quad 2\xi = \begin{vmatrix} \dfrac{\partial \lambda}{\partial y} & \dfrac{\partial \lambda}{\partial z} \\ \dfrac{\partial \mu}{\partial y} & \dfrac{\partial \mu}{\partial z} \end{vmatrix}$$

Similarly,

$$2\eta = \begin{vmatrix} \dfrac{\partial \lambda}{\partial z} & \dfrac{\partial \lambda}{\partial x} \\ \dfrac{\partial \mu}{\partial z} & \dfrac{\partial \mu}{\partial x} \end{vmatrix} \quad \text{and} \quad 2\zeta = \begin{vmatrix} \dfrac{\partial \lambda}{\partial x} & \dfrac{\partial \lambda}{\partial y} \\ \dfrac{\partial \mu}{\partial x} & \dfrac{\partial \mu}{\partial y} \end{vmatrix}$$

Therefore

$$2\left(\xi\frac{\partial \lambda}{\partial x} + \eta\frac{\partial \lambda}{\partial y} + \zeta\frac{\partial \lambda}{\partial z}\right) = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ \lambda_x & \lambda_y & \lambda_z \\ \mu_x & \mu_y & \mu_z \end{vmatrix} = 0$$

$$\Rightarrow \quad \xi\lambda_x + \eta\lambda_y + \zeta\lambda_z = 0$$

Similarly

$$\xi\mu_x + \eta\mu_y + \xi\mu_z = 0$$

It follows that the vortex lines lie on the surfaces

$$\lambda = \text{const and } \mu = \text{const.} \qquad \text{Ans.}$$

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Ex. 32. *If the velocity of an incompressible fluid at the point* (x, y, z) *is given by* $3xz/r^5, 3yz/r^5, (3z^2 - r^2)/r^5$, *prove that the liquid motion is possible and that the velocity potential is* $\cos\theta/r^2$. *Also, determine the stream lines.*

Solution. The condition for the possible liquid motion is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

$$u = \frac{3xz}{r^5} \Rightarrow \frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{15xz}{r^6}\cdot\frac{\partial r}{\partial x} = \frac{3z}{r^5} - \frac{15x^2z}{r^7}$$

or

$$\frac{3z}{r^5} - \frac{15x^2z}{r^7} + \frac{3z}{r^5} - \frac{15y^2z}{r^7} + \frac{6z}{r^5} - \frac{15z^3}{r^5} + \frac{3z}{r^5} = 0$$

or $$\frac{15z}{r^5} - \frac{15z(x^2+y^2+z^2)}{r^7} = 0 \Rightarrow \frac{15z}{r^5} - \frac{15z}{r^5} = 0,$$

which is an identity. Hence (u, v, w) are the velocity components of a possible liquid motion.

If ϕ be the velocity potential, then

$$d\phi = (\partial\phi/\partial x)\,dx + (\partial\phi/\partial y)\,dy + (\partial\phi/\partial z)\,dz$$

or $$d\phi = -(u\,dx + v\,dy + w\,dz)$$

or $$d\phi = -\frac{1}{r^5}\{3xz\,dz + 3yz\,dy + (3z^2 - r^2)\,dz\}$$

or $$d\phi = -\frac{1}{r^5}\{3z(x\,dx + y\,dy + z\,dz) - r^2\,dz\}$$

or $$d\phi = -\frac{3z}{2}\frac{d(x^2+y^2+z^2)}{r^5} + \frac{dz}{r^3}$$

or $$d\phi = -\frac{3z}{2}\frac{d(r^2)}{r^5} + \frac{dz}{r^3} = -\frac{3z}{2}\cdot\frac{2r\,dr}{r^5} + \frac{dz}{r^3} = d\left(\frac{z}{r^3}\right)$$

By integrating, we have

$$\phi = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2},$$

constant of integration vanishes. Proved.

The equations to the streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or $$\frac{dx}{3xz/r^5} = \frac{dy}{3yz/r^5} = \frac{dz}{(3z^2 - r^2)/r^5}$$

or $$\underset{\text{(i)}}{\frac{dx}{3xz}} = \underset{\text{(ii)}}{\frac{dy}{3yz}} = \underset{\text{(iii)}}{\frac{dz}{3z^2 - (x^2+y^2+z^2)}} = \underset{\text{(iv)}}{\frac{x\,dx + y\,dy + z\,dz}{2z(x^2+y^2+z^2)}}$$

From (i) and (ii), we have

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \log x = \log y + \log c \Rightarrow x = cy, \quad \ldots(1)$$

From (i) and (iv), we have

$$\frac{dx}{3x} = \frac{x\,dx + y\,dy + z\,dz}{2(x^2+y^2+z^2)}$$

By integrating, we have

$$\frac{2}{3}\log x = \frac{1}{2}\log(x^2+y^2+z^2) + \log D,$$

where D is an arbitrary constant.

$$x^{2/3} = D(x^2+y^2+z^2)^{1/2}. \quad \ldots(2)$$

Thus the equation (1) and (2) represents the stream lines. Ans.

Ex. 33. *For an incompressible fluid $u = -\omega y$, $v = \omega x$, $w = 0$, show that the surfaces intersecting the streamlines orthogonally exist and are the*

planes through Z-axis, although the velocity potential does not exist. Discuss the nature of flow.

Solution. The motion will be possible if it satisfies the equation of continuity, that is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

which is true from the given relation. Hence the motion is a possible one.

The differential equation to the lines of flow are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \Rightarrow \frac{dx}{-\omega y} = \frac{dy}{\omega x} = \frac{dz}{0}$$

or $$x\,dx + y\,dy = 0 \text{ and } dz = 0$$

By integrating, we have

$$x^2 + y^2 = \text{const., and } z = \text{const.}$$

The surfaces which cut the stream lines orthogonally are

$$u\,dx + v\,dy + w\,dz = 0$$

or $$-\omega y\,dx + \omega x\,dy = 0$$

By integrating, we have

$$dx/x - dy/y = 0 \Rightarrow \log(x/y) = \log c,$$

where c is an arbitrary constant.

Therefore $x = cy$, which represents a plane through Z-axis and cuts the stream line orthogonally.

The velocity potential will exist if $u\,dx + v\,dy + w\,dz$ is a perfect differential. But $u\,dx + v\,dy + w\,dz$ is not a perfect differential, therefore, the surfaces intersecting streamlines orthogonally exist and are the planes through Z-axis, although the velocity potential does not exist. Further

$$\nabla \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k}.$$

Hence the flow is not of the potential kind. It shows that a rigid body rotating about Z-axis with constant angular velocity $\omega\mathbf{k}$ gives the same type of motion. Ans.

2

EQUATIONS OF MOTION

Theorem 1. Euler's equation of motion : *To derive Euler's Dynamical equations.* *(Kanpur 2000, 2002; Meerut 1991)*

Proof : Let a closed surface S enclosing a volume V of a non-viscous fluid be moving with the fluid so that S contains the same number of fluid particles at any time t. Consider a point P inside S. Let ρ be the fuid density, $\mathbf{q}$ the fluid velocity and dV the elementary volume enclosing P. Since the mass $\rho\, dV$ remains unchanged throughout the motion so that

$$\frac{d}{dt}(\rho\, dV) = 0 \qquad \ldots (1)$$

The entire momentum M of the volume V is

$$\mathbf{M} = \int_V \mathbf{q}\, \rho\, dV$$

or $\quad$ momentum = mass × velocity

$$\frac{d\mathbf{M}}{dt} = \int \left[\frac{d\mathbf{q}}{dt}\rho\, dV + \frac{d}{dt}(\rho\, dV)\,\mathbf{q} \right]$$

using (1),

$$\frac{d\mathbf{M}}{dt} = \int \frac{d\mathbf{q}}{dt}\rho\, dV. \qquad \ldots (2)$$

Let $\mathbf{n}$ be the unit outward normal vector on the surface element dS. Suppose $\mathbf{F}$ is the external force per unit mass acting on the fluid and p the pressure at any point on the element ds. Total surface force is

$$\int_V \mathbf{F}\rho\, dV + \int_S p\,(-\mathbf{n})\, dS$$

[For pressure acts along inward normal]

$$= \int_V \mathbf{F}\rho\, dV + \int_V -\nabla p\, dV, \qquad \text{by Gauss Theorem}$$

$$= \int_V (\mathbf{F}\rho - \nabla p)\, dV. \qquad \ldots (3)$$

By Newton's second law of motion,

rate of change of momentum = total applied force

i.e., $$\int \frac{d\mathbf{q}}{dt}\rho\, dV = \int (\mathbf{F}\rho - \nabla p)\, dV, \qquad \text{by (2) and (3)}$$

or $$\int \left[\frac{d\mathbf{q}}{dt}\rho - \mathbf{F}\rho + \nabla p \right] dV = 0$$

Since S is arbitrary and so V is arbitrary so that the integrand of the last integral vanishes,

i.e., $$\frac{d\mathbf{q}}{dt}\rho - \mathbf{F}\rho + \nabla p = 0$$

or $$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots (4)$$

This equation is known as Euler's equation of motion. If we write

$$\mathbf{q} = \mathbf{q}(u, v, w), \quad \mathbf{F} = \mathbf{F}(X, Y, Z)$$

then the cartesian equivalent of (4) is

$$\frac{d}{dt}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) = (\mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z) - \frac{1}{\rho}\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)p$$

This $$\Rightarrow \frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{dv}{dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{dw}{dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

with $$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

Deduction : (i) To derive symmetrical form.

Here we have $\mathbf{q} = u$, $\nabla = \dfrac{\partial}{\partial r}$,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r}, \quad \mathbf{F} = F.$$

Now (4) becomes

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)u = F - \frac{1}{\rho}\frac{\partial p}{\partial r}.$$

(ii) To derive Lamb's hydrodynamical equation. By (4),

$$\left(\frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla\right)\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots (5)$$

or $$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\,\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

$\because$ $$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2\,[\mathbf{q}\times\text{curl}\,\mathbf{q} + (\mathbf{q}\cdot\nabla)\,\mathbf{q}]$$

using this in (5),

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - \mathbf{q}\times\text{curl}\,\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

writing $\mathbf{W} = \text{curl}\,\mathbf{q}$, we obtain

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) + \mathbf{W}\times\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

This is known as Lamb's hydrodynamical equation. The chief advantage of the is that it is invariant under a change of co-ordinate system.

(iii) **Euler's equation in cylindrical co-ordinates.**

Euler's equation of motion is

$$\frac{d\mathbf{q}}{dt} = \frac{D\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \dots (1)$$

Let (q_r, q_θ, q_z) be the velocity components and (F_r, F_θ, F_z) be the components of external force in r, θ, z directions. Then we know that

$$\frac{D\mathbf{q}}{Dt} = \left(\frac{Dq_r}{Dt} - \frac{q_\theta^2}{r}, \frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r}, \frac{Dq_z}{Dt}\right)$$

$$\mathbf{F} = (F_r, F_\theta, F_z), \quad \nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z}\right)$$

Substituting in (1) and equating the coefficient of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we obtain Euler's equations of motion in cylindrical coordinates as :

$$\left.\begin{aligned} \frac{Dq_r}{Dt} - \frac{q_\theta^2}{r} &= F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r} &= F_\theta - \frac{1}{\rho}\frac{\partial p}{r\partial \theta} \\ \frac{Dq_z}{Dt} &= F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \qquad \dots (2)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r \frac{\partial}{\partial t} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z \frac{\partial}{\partial z}. \qquad \dots (3)$$

(iv) **Euler's equations of motion in spherical polar coordinates :** *(Garhwal 2001)*

Euler's equation of motion is

$$\frac{d\mathbf{q}}{dt} = \frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \dots (1)$$

Let (q_r, q_θ, q_ϕ) be the velocity components and (F_r, F_θ, F_ϕ) be the components of external force in r, θ, ϕ directions. Then we know that

$$\frac{D\mathbf{q}}{Dt} = \left(\frac{Dq_r}{Dt} - \frac{q_\theta^2 - q_\phi^2}{r}, \frac{Dq_\theta}{Dt} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r}, \frac{Dq_\phi}{Dt} + \frac{q_\theta q_\phi \cot\theta}{r}\right)$$

$$\mathbf{F} = (F_r, F_\theta, F_\phi), \quad \nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial r}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\right).$$

Substituting in (1) and equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ we obtain Euler's equations of motion in spherical polar coordinates as :

$$\left.\begin{aligned} \frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r} &= F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{Dq_\theta}{Dt} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r} &= F_\theta - \frac{1}{\rho}\frac{\partial p}{r\partial \theta} \\ \frac{Dq_\phi}{Dt} + \frac{1}{r} q_\theta q_\phi \cot\theta &= F_\phi - \frac{1}{\rho}\cdot\frac{1}{r\sin\theta}\cdot\frac{\partial p}{\partial \phi} \end{aligned}\right\} \qquad \dots (2)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r \frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}.$$

2.1. Definition

The velocity $\mathbf{q}$ is called Beltrami vector if $\mathbf{q}$ is parallel to $\mathbf{W}$, *i.e.*, if $\mathbf{q} \times \mathbf{W} = 0$.

Def. A fluid is said to be barotropic if $p = f(\rho)$.

2.2. Def. Conservative field of force :

In a conservative field of force, the work done by a force $\mathbf{F}$ in taking a unit mass from a point A to a point B is independent of the path, *i.e.*,

$$\int_{ACB} \mathbf{F} \cdot d\mathbf{r} = \int_{ADB} \mathbf{F} \cdot d\mathbf{r} = -\underset{\text{say}}{\Omega},$$

Here Ω is a scalar function and is known as potential function. It can be proved that

$$\mathbf{F} = -\nabla\Omega.$$

Theorem 2. Pressure equation (Bernoulli's equation for unsteady motion). When velocity potential exists and forces are conservative and derivable from a potential Ω, the equations of motion can always be integrated and the solution is

$$\int \frac{dp}{\rho} - \frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + \Omega = F(t).$$

(Kanpur 2001, 2004; Garhwal 2004)

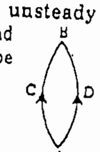

Fig. 2.1

Proof : Existence of velocity potential $\Rightarrow$ the motion is irrotational and $\mathbf{q} = -\nabla\phi$.

Forces are conservative $\Rightarrow \mathbf{F} = -\nabla\Omega$.

Let $\quad P = \int_0^p \frac{dp}{\rho}$, then $\frac{dP}{dp} = \frac{1}{\rho}$ so that $\nabla p = \Sigma \frac{i\,\partial P}{\partial x}$.

or $\quad \nabla P = \Sigma i \frac{dP}{dp}\frac{\partial p}{\partial x} = \Sigma \frac{i}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\nabla p \quad$ or $\quad \nabla P = \frac{1}{\rho}\nabla p$

By Euler's equation,

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \quad \text{or} \quad \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\nabla\Omega - \nabla P$$

or $$\frac{\partial}{\partial t}(-\nabla\phi) + \nabla(\Omega + P) + (\mathbf{q}\cdot\nabla)\mathbf{q} = 0$$

or $$\nabla\left(-\frac{\partial \phi}{\partial t} + \Omega + P\right) + \frac{1}{2}\nabla q^2 - \mathbf{q} \times \text{curl}\,\mathbf{q} = 0$$

For $$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2\,[\mathbf{q} \times \text{curl}\,\mathbf{q} + (\mathbf{q}\cdot\nabla)\,\mathbf{q}]$$

or $$\nabla\left(\Omega + p + \frac{1}{2}q^2 - \frac{\partial \phi}{\partial t}\right) = 0$$

[For curl $\mathbf{q} = \nabla \times \mathbf{q} = \nabla \times (-\nabla\phi) = -$ curl grad $\phi = 0$].

Multiplying scalarly by $d\mathbf{r}$ and noting the $d\mathbf{r} \cdot \nabla = d$, we get

$$d\left(\Omega + P + \frac{1}{2}q^2 - \frac{\partial \phi}{\partial t}\right) = 0$$

Integrating, $\Omega + P + \frac{1}{2}q^2 - \frac{\partial \phi}{\partial t} = F(t)$

where $F(t)$ is a constant of integration.

or $$\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{\partial \phi}{\partial t} = F(t) \qquad \ldots (1)$$

The equation is known as *Bernoulli's equation for unsteady irrotational motion.* This is also known as pressure equation. *(Kanpur 2005)*

If fluid is incompressible then (1) $\Rightarrow$

$$\Omega + \frac{p}{\rho} + \frac{1}{2}q^2 - \frac{\partial \phi}{\partial t} = F(t). \text{ For } \int \frac{dp}{\rho} = \frac{1}{\rho}\int dp = \frac{p}{\rho}.$$

Deduction : Suppose the motion is steady.

Then $\frac{\partial \phi}{\partial t} = 0$. Now (1) becomes

$$\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 = F(t) = C = \text{absolute constant}$$

or $$\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 = C.$$

This is known as *Bernoulli's equation for steady motion.* *(Kanpur 2000)*

If ρ = constant, then

$$\Omega + \frac{p}{\rho} + \frac{1}{2}q^2 = \text{const.}$$

Ex. *Derive Bernoulli's equation for unsteady motion of an incompressible fluid and hence derive expression for steady motion.* *(Garhwal 2004)*

Solution : Here write the above proof and its deduction complete.

Problem 1. *Show that the velocity field*

2006 model (2M)

$$u(x, y) = \frac{B(x^2 - y^2)}{(x^2 + y^2)^2}, \quad v(x, y) = \frac{2Bxy}{(x^2 + y^2)^2}, \quad w = 0$$

satisfies the equation of motion for an inviscid incompressible flow. Determine the pressure associated with this velocity field. B is constant.

Solution : Euler's equation of moion in absence of external forces is

$$\frac{dq}{dt} = -\frac{1}{\rho}\nabla p$$

or, $$\left(\frac{\partial}{\partial t} + q \cdot \nabla\right) q = -\frac{1}{\rho}\nabla p$$

But motion is two dimensional as $w = 0$ and $q = ui + vj$

$$\therefore \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) q = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

Putting the values,

$$\left[\frac{\partial}{\partial t} + \frac{B(x^2 - y^2)}{(x^2 + y^2)^2}\frac{\partial}{\partial x} + \frac{2Bxy}{(x^2 + y^2)^2}\frac{\partial}{\partial y}\right](ui + vj) = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

As u, v are independent of t, by assumption.

$\frac{\partial u}{\partial t} = 0 = \frac{\partial v}{\partial t}$. Hence the last gives

$$\frac{B}{(x^2 + y^2)^2}\left[(x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right](ui + vj) = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

This $\Rightarrow$ $$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{B}{(x^2 + y^2)^2}\left[(x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right]\frac{B(x^2 - y^2)}{(x^2 + y^2)^2} \qquad \ldots (1)$$

and $$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{B}{(x^2 + y^2)^2}\left[(x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right]\frac{2Bxy}{(x^2 + y^2)^2} \qquad \ldots (2)$$

But $$\frac{\partial}{\partial x}\left\{\frac{x^2 - y^2}{(x^2 + y^2)^2}\right\} = \frac{2x(3y^2 - x^2)}{(x^2 + y^2)^3} \qquad \ldots (3)$$

$$\frac{\partial}{\partial y}\left\{\frac{x^2 - y^2}{(x^2 + y^2)^2}\right\} = -\frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3} \qquad \ldots (4)$$

$$\frac{\partial}{\partial x}\left\{\frac{2xy}{(x^2 + y^2)^2}\right\} = \frac{2y(y^2 - x^2)}{(x^2 + y^2)^3} \qquad \ldots (5)$$

$$\frac{\partial}{\partial y}\left\{\frac{2xy}{(x^2 + y^2)^2}\right\} = \frac{2x(x^2 - y^2)}{(x^2 + y^2)^3} \qquad \ldots (6)$$

Writing (1) with the help of (3) and (4),

$$\frac{\partial p}{\partial x} = \frac{-2\rho B^2}{(x^2 + y^2)^5}[(x^2 - y^2)x(3y^2 - x^2) - 2xy^2(3x^2 - y^2)]$$

or $$\frac{\partial p}{\partial x} = \frac{2\rho B^2 x}{(x^2 + y^2)^3} \qquad \ldots (7)$$

Writing (2) with the help of (5) and (6),

$$\frac{\partial p}{\partial y} = \frac{-2\rho B^2}{(x^2 + y^2)^5}[(x^2 - y^2)y(y^2 - x^2) + 2x^2y(x^2 - y^2)]$$

or $$\frac{\partial p}{\partial y} = \frac{2B^2 y(x^2 + y^2)}{(x^2 + y^2)^4} \qquad \ldots (8)$$

Differentiating (7) and (8) partially w.r.t. y and x we find that

$$\frac{\partial^2 p}{\partial y\,\partial x} = \frac{\partial^2 p}{\partial x\,\partial y} \quad \text{(Prove it)}$$

This proves that velocity field satisfies the equation of motion.

$$dp = -\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Using (7) and (8),

$$dp = 2\rho B^2\left[\frac{x\,dx}{(x^2+y^2)^3} - \frac{y(x^2-y^2)}{(x^2+y^2)^4}dy\right]$$

$$= 2\rho B^2\,[M\,dx + N\,dy], \text{ say} \qquad \ldots (9)$$

$$\frac{\partial M}{\partial y} = -\frac{6xy}{(x^2+y^2)^4} = \frac{\partial N}{\partial x}$$

$\therefore$ $M\,dx + N\,dy$ is exact.

$$\int (M + dx + N\,(dy)) = \int \frac{x\,dx}{(x^2+y^2)^3} + \int 0\,dy$$

$$= \frac{1}{2}\int 2x\,(x^2+y^2)^{-3}\,dx + c = -\frac{1}{4(x^2+y^2)^2} + c$$

In view of this (9) becomes,

$$p = -\frac{2\rho B^2}{4(x^2+y^2)^2} + c_1$$

or,

$$p = -\frac{\rho B^2}{2(x^2+y^2)^2} + c_1 \qquad \text{Ans.}$$

This is the required expression for pressure.

Problem 2. *The particle velocity for a fluid motion referred to rectangular axis is given by the components*

$$u = A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a},\quad v = 0,\quad w = A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a},$$

where A is a constant. Show that this is a possible motion of an incompressible fluid under no body forces in an infinite fixed rigid tube, $-a \le x \le a$, $0 \le z \le 2a$. Also, find the pressure associated with this velocity field.

Solution : The equations of motion for a two-dimensional steady, inviscid, incompressible flow under no body force, in cartesian coordinates, are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \qquad \ldots (1, 2, 3)$$

Here $\quad u = A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a},\; v = 0,\; w = A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}. \qquad \ldots (4)$

From equation (2), it follows that the pressure p is independent of y i.e., $p = p\,(x, z)$.

Using (4) into (1) and (3), we have

$$\left(A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\right)\cdot\left(-\frac{\pi A}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\right) + \left(A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right)$$

$$\times\left(-\frac{\pi A}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

or $\quad \frac{\pi A^2}{2a}\left[\cos\frac{\pi x}{2a}\sin\frac{\pi x}{2a}\cos^2\frac{\pi z}{2a} + \cos\frac{\pi x}{2a}\sin\frac{\pi x}{2a}\sin^2\frac{\pi z}{2a}\right] = \frac{1}{\rho}\frac{\partial p}{\partial x}$

or $\quad \frac{\pi A^2}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi x}{2a} = \frac{1}{\rho}\frac{\partial p}{\partial x}. \qquad \ldots (5)$

and $\quad \left(A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\right)\left(\frac{\pi A}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right) + \left(A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right)$

$$\times\left(\frac{\pi A}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

or $\quad \frac{\pi A^2}{2a}\left[\cos\frac{\pi z}{2a}\sin\frac{\pi z}{2a}\cos^2\frac{\pi x}{2a} + \cos\frac{\pi z}{2a}\sin\frac{\pi z}{2a}\sin^2\frac{\pi x}{2a}\right] = -\frac{1}{\rho}\frac{\partial p}{\partial z},$

or $\quad \frac{\pi A^2}{2a}\cos\frac{\pi z}{2a}\sin\frac{\pi z}{2a} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \qquad \ldots (6)$

The equations (5) and (6) show that the velocity components satisfy the equations of motion.

Again, $\quad dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz$

or $\quad dp = \frac{\pi\rho A^2}{2a}\left[\cos\frac{\pi x}{2a}\sin\frac{\pi x}{2a}\,dx - \cos\frac{\pi z}{2a}\sin\frac{\pi z}{2a}\,dz\right].$

By integrating, we have

$$p = \frac{1}{2}\rho A^2\left[\cos^2\frac{\pi z}{2a} - \cos^2\frac{\pi x}{2a}\right] + C,$$

where C is an integration constant. This gives the reqruied pressure distribution. Ans.

Problem 3. *Determine the pressure, if the velocity field $q_r = 0$, $q_\theta = Ar + B$, $q_z = 0$, satisfies the equation of motion $\rho\frac{q_\theta^2}{r} = \frac{dp}{dr}$, where A and B are arbitrary constants.*

Solution : $\quad \frac{dp}{dr} = \rho\frac{1}{r}\left(Ar + \frac{B}{r}\right)^2$

$$\frac{dp}{dr} = \rho\left(A^2 r + \frac{B^2}{r^3} + 2AB\frac{1}{r}\right)$$

By integrating, we have

$$p = \rho\left(\frac{1}{2}A^2 r^2 - \frac{B^2}{2r^2} + 2AB\log r\right) + C,$$

where C is an integration constant.

Ans.

Problem 4. *For an inviscid incompressible, steady flow with negligible body forces, velocity components in spherical polar coordinates are given by*

$$u_r = V\left(1-\frac{R^3}{r^3}\right)\cos\theta,\quad u_\theta = -V\left(1+\frac{R^3}{2r^3}\right)\sin\theta,$$

$u_\phi = 0$. *Show that it is a possible solution of momentum equations (i.e., equations of motion) R and V are constants.*

Solution : Write $u_r = u,\ u_\theta = v,\ u_\phi = w$. Then

$$u = V\left(1-\frac{R^3}{r^3}\right)\cos\theta,\quad v = -V\left(1+\frac{R^3}{2r^3}\right)\sin\theta,\quad w = 0.$$

To show that velocity components satisfy equation of momentum, we have to show that the velocity components satisfy Euler's equation of motion.

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla p$$

or

$$\left[\frac{\partial}{\partial t} + q\cdot\nabla\right]q = F - \frac{1}{\rho}\nabla p$$

By assumption, q is independent of t, $\frac{\partial q}{\partial t} = 0$ and body force F is negligible.

$$\therefore \qquad (q\cdot\nabla)\,q = -\frac{1}{\rho}\nabla p \qquad \dots(1)$$

Write
$$D = u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Putting the values of u, v, w,

$$D = V\left[\left(1-\frac{R^3}{r^3}\right)\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\frac{\partial}{\partial\theta}\right] \qquad \dots(2)$$

Spherical polar equivalent of (1) is

$$Du - \frac{(v^2+w^2)}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$Dv + \frac{uv}{r} - \frac{w\cot\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}$$

$$Dw + \frac{uw}{r} + \frac{vw}{r}\cot\theta = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}$$

Since $w = 0$, hnce the above equations become

$$Du - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \dots(3)$$

$$Dv + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \dots(4)$$

$$0 = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} \qquad \dots(5)$$

$$(5) \Rightarrow \frac{\partial p}{\partial\phi} = 0 \Rightarrow p = f(r,\theta).$$

By (3), $\frac{\partial p}{\partial r} = \frac{\rho v^2}{r} - \rho Du$

Putting the values

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{1}{r}\left\{V\left(1+\frac{R^3}{2r^3}\right)\sin\theta\right\}^3 + VD\left(1-\frac{R^3}{r^3}\right)\cos\theta$$

With D given by (2), simplifying, we get

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{3V^2R^3}{2r^4}\left(1+\frac{R^3}{2r^3}\right)\sin^2\theta - \frac{3V^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cos^2\theta \qquad \dots(6)$$

Similarly (4) gives

$$\frac{1}{\rho}\frac{\partial p}{\partial\theta} = \frac{3V^2R^3}{2r^3}\left(1-\frac{R^3}{r^3}\right)\sin\theta\cos\theta + \frac{3V^2R^3}{2r^3}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cos\theta \qquad \dots(7)$$

(Calculate it)

Differentiating (6) partially w.r.t. θ and simplifying, we get

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r} = \left(\frac{9V^2R^3}{r^4} - \frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \qquad \dots(8)$$

Differentiating (7) partially w.r.t. r and simplifying, we get

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta} = \left(\frac{9V^2R^3}{r^4} - \frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \qquad \dots(9)$$

Since (8) and (9) are identical hence equation of motion is satisfied.

Problem 5. *The velocity components*

$$u_r(r,\theta) = -V\left(1-\frac{a^2}{r^2}\right)\cos\theta,$$

$$u_\theta(r,\theta) = V\left(1+\frac{a^2}{r^2}\right)\sin\theta$$

satisfy equations of motion for a two dimensional inviscid incompressible flow, Find the pressure associated with velocity field. V and a are constants.

Solution : Euler's equation of motion in absence of external forces is

$$\frac{dq}{dt} = -\frac{1}{\rho}\nabla p. \qquad \dots(1)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + q\cdot\nabla.$$

But $u_r = u,\ u_\theta = v$ are independent of t.

$$\therefore \qquad \frac{\partial q}{\partial t} = 0.$$

Now (1) becomes

$$(q\cdot\nabla)\,q = -\frac{1}{\rho}\nabla p \qquad \dots(2)$$

Write $u_r = u,\ u_\theta = v,\ u_z = w$. Then

Problem 4. *For an inviscid incompressible, steady flow with negligible body forces, velocity components in spherical polar coordinates are given by*

$$u_r = V\left(1 - \frac{R^3}{r^3}\right)\cos\theta,\quad u_\theta = -V\left(1 + \frac{R^3}{2r^3}\right)\sin\theta,$$

$u_\phi = 0$. Show that it is a possible solution of momentum equations (i.e., equations of motion) R and V are constants.

Solution : Write $u_r = u$, $u_\theta = v$, $u_\phi = w$. Then

$$u = V\left(1 - \frac{R^3}{r^3}\right)\cos\theta,\quad v = -V\left(1 + \frac{R^3}{2r^3}\right)\sin\theta,\quad w = 0.$$

To show that velocity components satisfy equation of momentum, we have to show that the velocity components satisfy Euler's equation of motion.

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla p$$

or

$$\left[\frac{\partial}{\partial t} + q\cdot\nabla\right]q = F - \frac{1}{\rho}\nabla p$$

By assumption, q is independent of t, $\frac{\partial q}{\partial t} = 0$ and body force F is negligible.

$$\therefore\quad (q\cdot\nabla)\,q = -\frac{1}{\rho}\nabla p \qquad \ldots(1)$$

Write

$$D = u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Putting the values of u, v, w,

$$D = V\left[\left(1 - \frac{R^3}{r^3}\right)\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\left(1 + \frac{R^3}{2r^3}\right)\sin\theta\frac{\partial}{\partial\theta}\right] \qquad \ldots(2)$$

Spherical polar equivalent of (1) is

$$Du - \frac{(v^2 + w^2)}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$Dv + \frac{uv}{r} - \frac{w\cot\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}$$

$$Dw + \frac{uw}{r} + \frac{vw}{r}\cot\theta = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}$$

Since $w = 0$, hnce the above equations become

$$Du - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(3)$$

$$Dv + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \ldots(4)$$

$$0 = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} \qquad \ldots(5)$$

$(5) \Rightarrow \frac{\partial p}{\partial\phi} = 0 \Rightarrow p = f(r, \theta)$.

By (3), $\frac{\partial p}{\partial r} = \frac{\rho v^2}{r} - \rho Du$

Putting the values

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{1}{r}\left\{V\left(1 + \frac{R^3}{2r^3}\right)\sin\theta\right\}^3 + VD\left(1 - \frac{R^3}{r^3}\right)\cos\theta$$

With D given by (2), simplifying, we get

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{3V^2R^3}{2r^4}\left(1 + \frac{R^3}{2r^3}\right)\sin^2\theta - \frac{3V^2R^3}{r^4}\left(1 - \frac{R^3}{r^3}\right)\cos^2\theta \qquad \ldots(6)$$

Similarly (4) gives

$$\frac{1}{\rho}\frac{\partial p}{\partial\theta} = \frac{3V^2R^3}{2r^3}\left(1 - \frac{R^3}{r^3}\right)\sin\theta\cos\theta + \frac{3V^2R^3}{2r^3}\left(1 + \frac{R^3}{2r^3}\right)\sin\theta\cos\theta \qquad \ldots(7)$$

(Calculate it)

Differentiating (6) partially w.r.t. θ and simplifying, we get

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r} = \left(\frac{9V^2R^3}{r^4} - \frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \qquad \ldots(8)$$

Differentiating (7) partially w.r.t. r and simplifying, we get

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta} = \left(\frac{9V^2R^3}{r^4} - \frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \qquad \ldots(9)$$

Since (8) and (9) are identical hence equation of motion is satisfied.

Problem 5. *The velocity components*

$$u_r(r, \theta) = -V\left(1 - \frac{a^2}{r^2}\right)\cos\theta,$$

$$u_\theta(r, \theta) = V\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

satisfy equations of motion for a two dimensional inviscid incompressible flow. Find the pressure associated with velocity field. V and a are constants.

Solution : Euler's equation of motion in absence of external forces is

$$\frac{dq}{dt} = -\frac{1}{\rho}\nabla p \qquad \ldots(1)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + q\cdot\nabla$$

But $u_r = u$, $u_\theta = v$ are independent of t.

$$\frac{\partial q}{\partial t} = 0.$$

Now (1) becomes

$$(q\cdot\nabla)\,q = -\frac{1}{\rho}\nabla p \qquad \ldots(2)$$

Write $u_r = u$, $u_\theta = v$, $u_z = w$. Then

$$u = -V\left(1-\frac{a^2}{r^2}\right)\cos\theta,\ v = V\left(1+\frac{a^2}{r^2}\right)\sin\theta,\ w=0.$$

Write $\quad D = u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial \theta}+w\frac{\partial}{\partial z}$. But $w=0$

$$\therefore \quad D = u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial \theta}.$$

Putting the values of u, v, we get

$$D = V\left[-\left(1-\frac{a^2}{r^2}\right)\cos\theta\frac{\partial}{\partial r}+\left(\frac{1}{r}+\frac{a^2}{r^3}\right)\sin\theta\frac{\partial}{\partial\theta}\right] \qquad \ldots (3)$$

Cylindrical equivalent of (2) is

$$Du-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots (4)$$

$$Dv+\frac{uv}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad \ldots (5)$$

$$Dw=-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \ldots (6)$$

But $w=0 \Rightarrow Dw=0 \Rightarrow \frac{\partial p}{\partial z}=0 \Rightarrow p=p(r,\theta)$

Putting the values in (4) and (5),

$$-VD\left(1-\frac{a^2}{r^2}\right)\cos\theta-\frac{V^2}{r}\left(1+\frac{a^2}{r^2}\right)^2\sin^2\theta=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots (7)$$

$$VD\left(1+\frac{a^2}{r^2}\right)\sin\theta-\frac{V^2}{r}\left(1-\frac{a^2}{r^4}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} \qquad \ldots (8)$$

Simplifying (7) with the help of (3),

$$-\frac{1}{\rho}\frac{\partial p}{\partial r}=\frac{2V^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta-\frac{2V^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta \qquad \ldots (9)$$

Simplifying (8) with the help of (3),

$$-\frac{1}{\rho}\frac{\partial p}{\partial \theta}=\frac{2a^2V^2}{r^2}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta+\frac{2a^2V^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta \qquad \ldots (10)$$

Differentiating (9) w.r.t. θ and simplifying,

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r}=\frac{8V^2a^2}{r^3}\sin\theta\cos\theta \qquad \ldots (11)$$

Differentiating (10) partially w.r.t. r,

$$\frac{\partial^2 p}{\partial r\,\partial\theta}=\frac{8}{r^3}V^2a^2\sin\theta\cos\theta \qquad \ldots (12)$$

Evidently R.H.S. of (11) and (12) are equal. This proves that the given velocity components satisfy equations of motion.

II. To find pressure p.

$$dp=\frac{\partial p}{\partial r}dr+\frac{\partial p}{\partial\theta}d\theta$$

Putting the values from (9) and (10),

$$-\frac{\rho\,dp}{2V^2a^2}=\left[\frac{1}{r^2}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta-\frac{1}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta\right]dr$$
$$+\left\{\frac{1}{r^2}\left(1-\frac{a^2}{r^2}\right)+\frac{1}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta\right\}d\theta$$
$$=M\,dr+N\,d\theta \qquad \ldots (13)$$

It can be seen that $\frac{\partial M}{\partial\theta}=\frac{\partial N}{\partial r}$. (Prove it)

Hence $M\,dr+N\,d\theta$ is exact. Solution of (13) is given by

$$\frac{-\rho}{2V^2a^2}p=\int\left[\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)\cos^2\theta-\left(\frac{1}{r^2}+\frac{a^2}{r^5}\right)\sin^2\theta\right]dr$$

$$=\cos^2\theta\left(-\frac{1}{2r^2}+\frac{a^2}{4r^4}\right)-\sin^2\theta\left(-\frac{1}{2r^2}-\frac{a^2}{4r^4}\right).$$

$$p=-\frac{2V^2a^2}{\rho}\left[-\frac{1}{2r^2}\cos\theta+\frac{a^2}{4r^4}\right]+c,$$

Bernoulli's Theorem 3. Bernoulli's equation for steady motion : If (i) the forces are conservative (ii) motion is steady (iii) density ρ is a function of pressure p only, then the equation of motion is

$$\int\frac{dp}{\rho}+\frac{1}{2}q^2+\Omega=C, \text{ C being absolute constant.}$$

(Kanpur 2002, 2003; Meerut 1992)

Proof : Step I. Forces are conservative $\Rightarrow \mathbf{F}=-\nabla\Omega$. Motion is steady $\Rightarrow \frac{\partial q}{\partial t}=0$, density is a function of pressure p only $\Rightarrow$ there exists a relation of the type $\quad P=\int_c^p\frac{dp}{\rho}$ so that $\quad \nabla P=\frac{1}{\rho}\nabla p.$

By Euler's equation, $\quad \frac{dq}{dt}=-\nabla\Omega-\nabla P$

or $\quad \frac{\partial q}{\partial t}+(q\,.\,\nabla)\,q=-\nabla(\Omega+P) \quad$ or $\quad \nabla(\Omega+P)+(q\,.\,\nabla)\,q=0.$

But $\quad \nabla(q\,.\,q)=2\,[q\times\text{curl}\,q+(q\,.\,\nabla)\,q]$

$$\therefore \quad \nabla(\Omega+P)+\frac{1}{2}\nabla q^2-q\times\text{curl}\,q=0$$

or $$\nabla\left(\Omega+P+\frac{1}{2}q^2\right)=q\times\text{curl}\,q. \qquad \ldots (1)$$

Step II. Multiplying (1) scalarly by q and noting that

$$q\,.\,(q\times\text{curl}\,q)=(q\times q)\,.\,\text{curl}\,q=0.$$

For $q\times q=0$, we obtain $q\,.\,\nabla\left(\Omega+P+\frac{1}{2}q^2\right)=0.$

The solution of this is $\Omega + P + \frac{1}{4}q^2 = \text{const.} = C$

or, $$\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 = C.$$

Proved.

Theorem 4. If the motion of an ideal fluid, for which density is a function of pressure p only, is steady and the external forces are conservative, then there exists a family of surfaces which contain the stream lines and vortex lines.

Proof : Step I. $\nabla\left(\Omega + P + \frac{1}{2}q^2\right) = \mathbf{q} \times \text{curl } \mathbf{q}.$ ... (1)

Here write step I of Theorem 3.

Step II. Write $\mathbf{W} = \text{curl } \mathbf{q}$ = vorticity vector.

Then $\nabla\left(\Omega + P + \frac{1}{2}q^2\right) = \mathbf{q} \times \mathbf{W}.$ ... (2)

Write $\nabla\left(\Omega + P + \frac{1}{2}q^2\right) = \mathbf{N}.$

Then $\mathbf{N} = \mathbf{q} \times \mathbf{W}.$

Thus $\Rightarrow$ $\mathbf{N} \cdot \mathbf{q} = 0 = \mathbf{N} \cdot \mathbf{W}.$ [For $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, if any two of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are equal]

$\Rightarrow$ $\mathbf{N}$ is perpendicular to both $\mathbf{q}$ and $\mathbf{W}$.

Also $\mathbf{N}$ perpendicular to the family of surfaces

$$\Omega + P + \frac{1}{2}q^2 = \text{const.} = C.$$

[For ∇f is perpendicular everywhere to f = const.]

This leads to the conclusion that $\mathbf{q}$ and $\mathbf{W}$ both are tangential to the surface

$$\Omega + P + \frac{1}{2}q^2 = C.$$

It means, that the surfaces $\Omega + P + \frac{1}{2}q^2 = C$ contains stream lines and vortex lines.

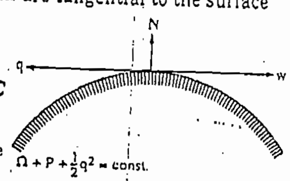

Fig. 2.2

Remark : The above theorem can also be restated as follows :

To prove that for steady motion of an invisical isotropic fluid

$$(\rho = f(p)), \int \frac{dp}{\rho} + \frac{1}{2}q^2 + \Omega = \text{const.}$$

over a surface containing the stream lines and vortex lines. Comment on the nature of this constant.

Theorem 5. Lagrange's equation of motion. To obtain Lagrange's equation of motion. *(Kanpur 2003, Garhwal 2000)*

Proof : Let initially a fluid particle be at (a, b, c) at time $t = t_0$, when its volume is dV_0 and density is ρ_0. After a lapse of time t, let he same fluid particle be at (x, y, z) when its volume is dV and density is ρ. The equation of continuity is

$$\rho J = \rho_0 \quad \text{... (1)}$$

where $J = \frac{\partial(x, y, z)}{\partial(a, b, c)}$. The components of acceleration are

$$\ddot{x} = \frac{\partial^2 x}{\partial t^2}, \quad \ddot{y} = \frac{\partial^2 y}{\partial t^2}, \quad \ddot{z} = \frac{\partial^2 z}{\partial t^2}.$$

Let the external forces be conservative so that $\mathbf{F} = -\nabla\Omega$.

But Euler's equation of motion,

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p = -\nabla\Omega - \frac{1}{\rho}\nabla p.$$

Its cartesian equivalent is

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial \Omega}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial^2 z}{\partial t^2} = -\frac{\partial \Omega}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Multiplying these equations by

$$\frac{\partial x}{\partial a}, \frac{\partial y}{\partial a}, \frac{\partial z}{\partial a}$$

respectively and then adding columnwise.

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial \Omega}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a}. \quad \text{... (2)}$$

Replacing a by b and c respectively, we get two more equations

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial \Omega}{\partial b} - \frac{1}{\rho}\frac{\partial p}{\partial b} \quad \text{... (3)}$$

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 r}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial \Omega}{\partial c} - \frac{1}{\rho}\frac{\partial p}{\partial c} \quad \text{... (4)}$$

The equations (1), (2), (3) and (4) together represent Lagrange's hydrodynamical equations.

Theorem 6. Helmholtz vorticity equation. If the external forces are conservative and density is a function of pressure p only, then

$$\frac{d}{dt}\left(\frac{\mathbf{W}}{\rho}\right) = \left(\frac{\mathbf{W}}{\rho} \cdot \nabla\right)\mathbf{q}.$$ *(Garhwal 2003, Kanpur 2000)*

Proof : $\mathbf{F}$ is conservative $\Rightarrow$ $\mathbf{F} = -\nabla\Omega$

ρ is a function of p only $\Rightarrow$ three exists a relation of the type

$$P = \int_\rho^p \frac{dp}{\rho}$$

$$\Rightarrow \quad \nabla P = \Sigma i \frac{\partial P}{\partial x} = \Sigma i \frac{dP}{dp}\frac{\partial p}{\partial x} = \Sigma \frac{1}{\rho} i \frac{\partial p}{\partial x}$$

$$\Rightarrow \quad \nabla P = \frac{1}{\rho}\nabla p.$$

By Euler's equations of motion.

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p.$$

or $$\frac{\partial \mathbf{q}}{dt} + (\mathbf{q}\,.\,\nabla)\,\mathbf{q} = -\nabla\Omega - \nabla P.$$

But $$\nabla(\mathbf{q}.\mathbf{q}) = 2[\mathbf{q}\times \text{curl}\,\mathbf{q} + (\mathbf{q}.\nabla)\,\mathbf{q}]$$

$\therefore$ $$\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2}\nabla q^2 - \mathbf{q}\times\text{curl}\,\mathbf{q} = -\nabla(\Omega + P)$$

or $$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\Omega + P + \frac{1}{2}q^2\right) = \mathbf{q}\times\mathbf{W}.$$

Taking curl of both sides and noting that curl grad $\equiv 0$, we obtain

$$\text{curl}\,\frac{\partial \mathbf{q}}{\partial t} = \frac{\partial}{\partial t}\,\text{curl}\,\mathbf{q} = \frac{\partial \mathbf{W}}{\partial t} = \text{curl}\,(\mathbf{q}\times\mathbf{W})$$

or $$\frac{\partial \mathbf{W}}{\partial t} = \mathbf{q}\,(\nabla.\mathbf{W}) - \mathbf{W}\,(\nabla.\mathbf{q}) + (\mathbf{W}.\nabla)\,\mathbf{q} - (\mathbf{q}.\nabla)\,\mathbf{W}.$$

But $\nabla.\mathbf{W} = \text{div curl}\,\mathbf{q} \equiv 0$ and equation of continuity is

$$\frac{d\rho}{dt} + \rho\,(\nabla.\mathbf{q}) = 0$$

Hence $$\frac{\partial \mathbf{W}}{\partial t} = 0 + \frac{\mathbf{W}}{\rho}\frac{d\rho}{dt} + (\mathbf{W}.\nabla)\,\mathbf{q} - (\mathbf{q}.\nabla)\,\mathbf{W}$$

or $$\left[\frac{\partial}{\partial t} + \mathbf{q}.\nabla\right]\mathbf{W} = \frac{\mathbf{W}}{\rho}\frac{d\rho}{dt} + (\mathbf{W}.\nabla)\,\mathbf{q}$$

or $$\frac{d\mathbf{W}}{dt} = \frac{\mathbf{W}}{\rho}\frac{d\rho}{dt} + (\mathbf{W}.\nabla)\,\mathbf{q}.$$

or $$\frac{1}{\rho}\frac{d\mathbf{W}}{dt} - \frac{\mathbf{W}}{\rho^2}\frac{d\rho}{dt} = (\mathbf{W}.\nabla)\,\mathbf{q}\,\frac{1}{\rho}$$

or $$\frac{d}{dt}\left(\frac{\mathbf{W}}{\rho}\right) = \left(\frac{\mathbf{W}}{\rho}\,.\,\nabla\right)\mathbf{q}. \qquad \ldots(1)$$

This is called Helmholtz vorticity equation. If we write
$\mathbf{W} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$, $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$,

then the cartesian form of (1) is

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left(\frac{\xi}{\rho}\right) = \frac{1}{\rho}\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)u$$

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left(\frac{\eta}{\rho}\right) = \frac{1}{\rho}\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)v$$

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left(\frac{\zeta}{\rho}\right) = \frac{1}{\rho}\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)w.$$

Remark : For ρ = const. (1) *was originally given by Stoke and Helmholtz and later on exended to the above form by Nanson.* *(Meerut 1992)*

Theorem 7. Cauchy's integrals : Lagrange's hydrodynamical equations are

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial a}\frac{\partial z}{\partial a} = -\frac{\partial\Omega}{\partial a} - \frac{1}{\rho}\,.\,\frac{\partial p}{\partial a}$$

with two similar equations.

If we write $Q = \Omega + \int_c^p \frac{dp}{\rho}$, then the last becomes

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial Q}{\partial a} \qquad \ldots(1)$$

Similarly we have

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \qquad \ldots(2)$$

$$\frac{\partial^2 y}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \qquad \ldots(3)$$

Put $$\dot{x} = \frac{\partial x}{\partial t} = u,\ \dot{y} = \frac{\partial y}{\partial t} = v,\ \dot{z} = \frac{\partial y}{\partial t} = w.$$

Now (2) and (3) are expressible as

$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial b} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \qquad \ldots(4)$$

$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial c} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial c} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \qquad \ldots(5)$$

Eliminating Q between (4) and (5), we have

$$\frac{\partial}{\partial c}\ \text{L.H.S. of (4)} = \frac{\partial}{\partial b}\ \text{L.H.S. of (5)}.$$

i.e. $$\frac{\partial^2 u}{\partial c\,\partial t}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial t}\frac{\partial^2 x}{\partial b\,\partial c} + \frac{\partial^2 v}{\partial c\,\partial t}\frac{\partial y}{\partial b} + \frac{\partial v}{\partial t}\frac{\partial^2 y}{\partial b\,\partial c} + \frac{\partial^2 w}{\partial t\,\partial c}\frac{\partial z}{\partial b} + \frac{\partial w}{\partial t}\frac{\partial^2 z}{\partial b\,\partial c}$$

$$= \frac{\partial^2 u}{\partial b\,\partial t}\frac{\partial x}{\partial c} + \frac{\partial u}{\partial t}\frac{\partial^2 x}{\partial b\,\partial c} + \frac{\partial^2 v}{\partial b\,\partial t}\frac{\partial y}{\partial c} + \frac{\partial v}{\partial t}\frac{\partial^2 y}{\partial c\,\partial b} + \frac{\partial^2 w}{\partial b\,\partial t}\frac{\partial z}{\partial c} + \frac{\partial w}{\partial t}\frac{\partial^2 z}{\partial b\,\partial c}$$

or $$\left(\frac{\partial^2 u}{\partial b\,\partial t} - \frac{\partial^2 u}{\partial c\,\partial t}\frac{\partial x}{\partial b}\right) + \left(\frac{\partial^2 v}{\partial b\,\partial t}\frac{\partial y}{\partial c} - \frac{\partial^2 v}{\partial c\,\partial t}\frac{\partial y}{\partial b}\right) + \left(\frac{\partial^2 w}{\partial b\,\partial t}\frac{\partial z}{\partial c} - \frac{\partial^2 w}{\partial c\,\partial t}\frac{\partial z}{\partial b}\right) = 0$$

or $$\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) - \frac{\partial u}{\partial b}\frac{\partial^2 x}{\partial t\,\partial c} + \frac{\partial u}{\partial c}\frac{\partial^2 x}{\partial t\,\partial b}\right]$$

$$+\left[\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c} - \frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right) - \frac{\partial v}{\partial b}\frac{\partial^2 y}{\partial t\,\partial c} + \frac{\partial v}{\partial c}\frac{\partial^2 y}{\partial t\,\partial b}\right]$$

$$+\left[\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right) - \frac{\partial w}{\partial b}\frac{\partial^2 z}{\partial t\,\partial c} + \frac{\partial w}{\partial c}\frac{\partial^2 z}{\partial t\,\partial b}\right] = 0$$

But $$\frac{\partial u}{\partial b}\frac{\partial^2 x}{\partial t\,\partial c} = \frac{\partial u}{\partial b}\frac{\partial u}{\partial c},\quad \frac{\partial u}{\partial c}\frac{\partial^2 x}{\partial t\,\partial b} = \frac{\partial u}{\partial c}\frac{\partial u}{\partial b}.$$

Hence terms outside the brackets cancel so that

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) + \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c} - \frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right) + \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\frac{\partial z}{\partial c} - \frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right) = 0.$$

Integrating w.r.t. t,

$$\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) + \left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c} - \frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right) + \left(\frac{\partial w}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right) = c \qquad \ldots(6)$$

Initially, *i.e.*, at $t = 0$, $x = a$, $y = b$, $z = c$ so that

$$\frac{\partial x}{\partial a} = 1,\ \frac{\partial y}{\partial b} = 1,\ \frac{\partial z}{\partial c} = 1,\ \frac{\partial x}{\partial b} = 0,\ \frac{\partial y}{\partial c} = 0 \text{ etc.}$$

Subjecting (6) to this condition,

$$(0-0)_0 + \left(0 - \frac{\partial v}{\partial c}\cdot 1\right)_0 + \left(\frac{\partial w}{\partial b}\cdot 1 - 0\right)_0 = c$$

or
$$c = \left(\frac{\partial w}{\partial b}\right)_0 - \left(\frac{\partial v}{\partial c}\right)_0 = \xi_0$$

where
$$W = \text{curl } q = \xi i + \eta j + \zeta k$$

$$\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) + \left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c} + \frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right) + \left(\frac{\partial w}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right) = \xi_0$$

or
$$\left[\frac{\partial x}{\partial c}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial u}{\partial z}\frac{\partial x}{\partial b}\right) - \frac{\partial x}{\partial b}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial c}\right)\right]$$
$$+\left[\frac{\partial y}{\partial c}(\ldots) - \frac{\partial y}{\partial b}(\ldots)\right] + \left[\frac{\partial z}{\partial c}(\ldots) - \frac{\partial z}{\partial b}(\ldots)\right] = \xi_0$$

or
$$\frac{\partial(y,z)}{\partial(b,a)}\xi + \frac{\partial(z,x)}{\partial(b,c)}\eta + \frac{\partial(x,y)}{\partial(b,c)}\zeta = \xi_0 \quad \ldots (7)$$

Similarly
$$\frac{\partial(y,z)}{\partial(c,a)}\xi + \frac{\partial(z,x)}{\partial(c,a)}\eta + \frac{\partial(x,y)}{\partial(c,a)}\zeta = \eta_0 \quad \ldots (8)$$

$$\frac{\partial(y,z)}{\partial(a,b)}\xi + \frac{\partial(y,z)}{\partial(a,b)}\eta + \frac{\partial(y,z)}{\partial(a,b)}\zeta = \zeta_0 \quad \ldots (9)$$

Multiplying (7), (8), (9) by

$$\frac{\partial x}{\partial a},\ \frac{\partial x}{\partial b},\ \frac{\partial x}{\partial c}$$

respectively and adding columnwise,

$$\xi\frac{\partial(x,y,z)}{\partial(a,b,c)} + \eta\frac{\partial(z,x,x)}{\partial(a,b,c)} + \zeta\frac{\partial(x,y,z)}{\partial(a,b,c)} = \xi_0\frac{\partial x}{\partial a} + \eta_0\frac{\partial x}{\partial b} + \zeta_0\frac{\partial x}{\partial c}.$$

But $\quad \rho\dfrac{\partial(x,y,z)}{(a,b,c)} = \rho J = \rho_0$

Hence
$$\xi\frac{\rho_0}{\rho} = \xi_0\frac{\partial x}{\partial a} + \eta_0\frac{\partial x}{\partial b} + \zeta_0\frac{\partial x}{\partial c}$$

or
$$\frac{\xi}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial x}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial x}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial x}{\partial c} \quad \ldots (10)$$

Similarly
$$\frac{\eta}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial y}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial y}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial y}{\partial c} \quad \ldots (11)$$

and
$$\frac{\zeta}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial z}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial z}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial z}{\partial c}. \quad \ldots (12)$$

The euations (10), (11) and (12) are called *Cauchy integrals*. The vector form of these equations is

$$\frac{W}{\rho} = \left(\frac{\xi_0}{\rho_0}\frac{\partial}{\partial a} + \frac{\eta_0}{\rho_0} + \frac{\zeta_0}{\rho_0}\frac{\partial}{\partial c}\right)r$$

or
$$\frac{W}{\rho} = \left(\frac{W}{\rho}\cdot\nabla\right)_0 r.$$

Deduction 1. *To prove the principle of permanence of irrotational motion.*

Proof : If $W_0 = 0$, *i.e.* if $\xi_0 = \eta_0 = \zeta_0 = 0$, then (10), (11), (12)

$$\Rightarrow \quad \xi = \eta = \zeta = 0$$

i.e., $\quad W_0 = 0 \quad \Rightarrow \quad W = 0$

This proves that if the motion be irrotational initially, then it is always irrotational for all time. This establishes the principle of irrotational motion for all time *t*.

Deduction 2. *To prove Cauchy's integrals are the integrals of Helmholtz vorticity equations.*

Or *To prove Helmholtz equations with the help of Cauchy's integrals.*

Proof : (10) $\times \dfrac{\partial u}{\partial x}$ + (11) $\times \dfrac{\partial u}{\partial y}$ + (12) $\times \dfrac{\partial u}{\partial z}$ gives

$$\frac{1}{\rho}\left(\xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z}\right) = \frac{\xi_0}{\rho_0}\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a}\right) + \ldots + \ldots$$
$$= \frac{\xi_0}{\rho_0}\frac{\partial u}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial u}{\partial b} + \frac{\zeta_0}{r_0}\frac{\partial u}{\partial c} = \frac{d}{dt}\left(\frac{\xi}{\rho}\right), \text{ according to (10).}$$

$\therefore$
$$\frac{1}{\rho}\left(\xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z}\right) = \frac{d}{\partial t}\left(\frac{\xi}{\rho}\right)$$

Similarly
$$\frac{1}{\rho}\left(\xi\frac{\partial v}{\partial x} + \eta\frac{\partial v}{\partial y} + \zeta\frac{\partial v}{\partial z}\right) = \frac{d}{\partial t}\left(\frac{\eta}{\rho}\right)$$

and
$$\frac{1}{\rho}\left(\xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} + \zeta\frac{\partial w}{\partial z}\right) = \frac{d}{\partial t}\left(\frac{\zeta}{\rho}\right).$$

This is equivalent to single vector equation;

$$\frac{1}{\rho}\left(\xi\frac{\partial q}{\partial x} + \eta\frac{\partial q}{\partial y} + \zeta\frac{\partial q}{\partial z}\right) = \frac{d}{dt}\left(\frac{W}{\rho}\right)$$

or
$$\frac{1}{\rho}(W\cdot\nabla)\,q = \frac{d}{dt}\left(\frac{W}{\rho}\right)$$

or
$$\frac{d}{dt}\left(\frac{W}{\rho}\right) = \left(\frac{W}{\rho}\cdot\nabla\right)q.$$

This is known as Helmholtz vorticity equation.

Theorem 8. Equations for Impulsive Action : To obtain general equations of motion for impulsive action. *(Kanpur 2002, 2003, 2005)*

Proof : Consider an arbitrary closed surface *S* moving with a non-viscous fluid such that it encloses a volume *V*. Let q_1 and q_2 be fluid velocities at *P* within *S* just before the impulse and just after the impulse. Let ρ be fluid density at *P*. Suppose *I* is the external impulse per unit mass and $\tilde{\omega}$ the impulse pressure on a surface element *dS*. Also let *n* be unit outward normal vecotr.

Change of momentum = Total impulsive forces

i.e.,
$$\int \rho\,(q_2 - q_1)\,dV = \int I\rho\,dV + \int -n\tilde{\omega}\,dS$$

[For ω acts along inward normal]

By Gauss theorem the last gives
$$\int [\rho\,(q_2 - q_1) - I\rho + \nabla\tilde{\omega}]\,dV = 0.$$

Since the surface S is arbitrary and hence the integrand of the last integral vanishes.

$\therefore$
$$\rho\,(q_2 - q_1) - I\rho + \nabla\tilde{\omega} = 0$$

or
$$q_2 - q_1 = I - \frac{1}{\rho}\nabla\tilde{\omega} \qquad \dots (1)$$

This is the required equation for impulsive action. If
$I = I\,(X, Y, Z)$, $q_2 = q_2\,(u, v, w)$, $q_1 = q_1\,(u_0, v_0, w_0)$,
then the cartesian equivalent of (1) is
$$u - u_0 = X - \frac{1}{\rho}\frac{\partial\tilde{\omega}}{\partial x},\quad v - v_0 = Y - \frac{1}{\rho}\frac{\partial\tilde{\omega}}{\partial y},\quad w - w_0 = Z - \frac{\partial\tilde{\omega}}{\partial z}.$$

Deduction (i). Vorticity in a non-vicous incompressible fluid is never generated by impulses if the extenal forces are conservative.

Proof : External impulses are conservative $\Rightarrow$ $I = -\nabla\Omega$.

Fluid is incompressible $\Rightarrow \rho$ is constant.

By (1),
$$q_2 - q_1 = -\nabla\left(\Omega + \frac{\tilde{\omega}}{\rho}\right)$$
or
$$\nabla \times (q_2 - q_1) = 0 \text{ as } \nabla \times \nabla = \text{curl grad} = 0$$
or
$$\text{curl } q_2 = \text{curl } q_1 \quad \text{or} \quad W_2 = W_1.$$

From this the required result follows.

(ii) To prove $\nabla^2\tilde{\omega} = 0$ under suitable conditions.

Proof : Let the external impulse be absent so that $I = 0$. Also let ρ be constant. Then (1) gives
$$q_2 - q_1 = -\nabla\left(\frac{\tilde{\omega}}{\rho}\right) \qquad \dots (2)$$
or
$$\nabla\,.\,(q_2 - q_1) = -\nabla\,.\,\nabla\left(\frac{\tilde{\omega}}{\rho}\right) = -\nabla^2\left(\frac{\tilde{\omega}}{\rho}\right)$$
or
$$\nabla^2\tilde{\omega} = \rho\,[-\nabla.q_2 + \nabla.q_1] = \rho\,[-0 + 0] \quad \text{or} \quad \nabla^2\tilde{\omega} = 0$$

For $\nabla.q_1 = 0 = \nabla.q_2$ is the equation of continuity.

Remark : If the motion is irrotational, then
$$-\nabla\phi_2 + \nabla\phi_1 = -\nabla\left(\frac{\tilde{\omega}}{\rho}\right), \quad \text{by (2)}$$
or
$$\nabla\,[\rho\,(\phi_2 - \phi_1) - \tilde{\omega}] = 0.$$

Integrating, $\rho\,(\phi_2 - \phi_1) - \tilde{\omega} = 0$, neglecting constant of integration

or
$$\tilde{\omega} = \rho\,(\phi_2 - \phi_1).$$

(iii) To prove $\tilde{\omega} = \rho\phi$ under suitable conditions. Let the external impulse be absent so that $I = 0$. Also let ρ be constant and motion starts from rest. Then (1) gives
$$q_2 - 0 = 0 - \frac{1}{\rho}\nabla\tilde{\omega}.$$

Since the motion starts from rest by the application of impulsive pressure hence it must be irrotational. Then $q = -\nabla\phi$.

$\therefore$
$$-\nabla\phi = -\frac{1}{\rho}\nabla\tilde{\omega} \quad \text{or} \quad \nabla\,(\rho\phi - \tilde{\omega}) = 0.$$

Integrating it, $\rho\phi - \tilde{\omega} = 0$, neglecting constant of integration.

or
$$\tilde{\omega} = \rho\phi. \qquad \textit{(Kanpur 2001)}$$

If $\rho = 1$, then $\tilde{\omega} = \phi$.

Remark : If $I = 0$; $q_2 = 0$, then
$$(1) \Rightarrow -q_1 = -\frac{1}{\rho}\nabla\tilde{\omega}.$$

Further if velocity has one component, then this gives
$$u = \frac{1}{\rho}\frac{\partial\tilde{\omega}}{\partial x} = \frac{1}{\rho}\frac{d\tilde{\omega}}{dx}$$
or
$$d\tilde{\omega} = \rho u\,dx.$$

This equation is very important for further study.

Def. Flow : Consider any two points A and B in a fluid. The value of the integral
$$\int_A^B (u\,dx + v\,dy + w\,dz) = \int_A^B q.dr$$
taken along any path in the fluid, is called flow from A to B along that path. If the motion is irrotational, then the flow is
$$\int_A^B q.dr = \int_A^B -\nabla\phi.dr = -\int_A^B d\phi = \phi_A - \phi_B$$
where ϕ_A and ϕ_B denote velocity potentials at A and B, respectively.

Def. Circulation :

Flow along a closed path c is defined as circulation.
$$\text{circulation} = \int_c q.dr.$$

If the motion is irrotational, then circulation $= \phi_A - \phi_B = \phi_A - \phi_A = 0$.

For a closed path, points A and B coincide.

Theorem 9. Kelvin's Circulation theorem : The circulation along any closed path moving with the fluid is constant for all times if the external forces are conservative and density ρ is function of pressure p only.

(Garhwal 2004; Meerut 2002; Agra 2001, 2004)

Proof : Let c be a closed path and cir denotes circulation. Then

$$\text{cir} = \int_c q.dr.$$

$$\frac{d}{dt}[\text{cir}] = \int_c \left[\frac{dq}{dt}.dr + q.\frac{d}{dt}(dr)\right]$$

$$= \int_c \left[\frac{dq}{dt}.dr + q.dq\right] \quad \left[\text{For } q.\frac{d}{dt}(dr) = q.d\left(\frac{dr}{dt}\right)\right]$$

$$= \int_c \left[\left(F - \frac{1}{\rho}\nabla p\right).dr + d\left(\frac{1}{2}q^2\right)\right]$$

by Euler's equation.

$$= \int_c \left(-\nabla\Omega - \frac{1}{\rho}\nabla p\right).dr + d\left(\frac{1}{2}q^2\right)$$

$$= \int_c \left[\left(-\frac{dp}{\rho} - d\Omega\right) + d\left(\frac{1}{2}q^2\right)\right] \text{as } dr.\nabla = d$$

$$= -\left[\Omega - \frac{1}{2}q^2 + \int\frac{dp}{\rho}\right]_c = 0. \quad \dots (1)$$

For, on R.H.S. of (1), the quantities involved are single valued and on passing once round the circuit, the change expressed in (1) is zero. Thus $\frac{d}{dt}$ (cir) = 0.

This ⇒ circulation is constant along c for all times.

Theorem 10. Permanence of irrotational motion : *If the motion of a non viscous fluid is once irrotational, it remains irrotational even afterwards provided the external forces are conservative and density ρ is a function of pressure p only.*

(Garhwal 2001, 2002, 2004)

Proof : Let c denote a closed path moving with the fluid and cir denotes circulation.

Then cir $= \int_c q.dr = \int_S n.\text{curl } q\, dS$, by Stoke's theorem.

Suppose motion is once irrotational. Then cir along c is zero. By Kelvin's theorem cir is constant for all times along c. Consequently cir along c is zero for all times,

i.e., cir = 0 ∀ t along c

Then $\int_S n.\text{curl } q\, dS = 0$. Also S is arbitrary.

Hence $n.$curl $q = 0$ or curl $q = 0$, this ⇒ motion is irrotational for all times. Hence motion is permanently irrotational.

Theorem 11. *To obtain equation of energy.* *(Garhwal 2004)*

Proof : Consider an arbitrary closed surface S moving with a non-viscous fluid s.t. it encloses a volume V. Let n be the unit inward drawn normal vector on an

element dS. Let the force be conservative so that $F = -\nabla\Omega$. Since force potential Ω is supposed to be independent of time, so that

$$\frac{\partial\Omega}{\partial t} = 0. \text{ Further } \frac{d}{dt} = \frac{\partial}{\partial t} + (q.\nabla)$$

Hence
$$\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + (q.\nabla)\,\Omega = (q.\nabla)\,\Omega \quad \dots (1)$$

Let T, W, I denote kinetic energy, potential energy and intrinsic energy, respectively. Since Ω is force potential per unit mass hence

$$W = \int \Omega\, dm = \int \Omega\,\rho\, dV$$

$$T = \int \frac{1}{2}\rho q^2\, dV = \frac{1}{2}\int q^2 \rho\, dV$$

Since elementary mass remains invariant throughout the motion hence

$$\frac{d}{dt}(\rho\, dV) = 0.$$

Now
$$\frac{dT}{dt} = \frac{1}{2}\int \frac{dq^2}{dt}\rho\, dV + \frac{1}{2}\int q^2.0 = \int q.\frac{dq}{dt}\rho\, dV + 0$$

$$[\text{as } q^2 = q^2 = q.q]$$

$$\frac{dW}{dt} = \int \frac{d\Omega}{dt}.\rho\, dV + \int \Omega.0 = \int \frac{d\Omega}{dt}\rho\, dV + 0$$

Intrinsic energy E per unit mass of the fluid is defined as the work done by the unit mass of the fluid against external pressure p under the supposed relation between pressure and density from its actual state to some standard state in which pressure and density are p_0 and ρ_0 respectively. Then

$$I = \int E\rho\, dV, \quad E = \int_V^{V_0} p\, dV \text{ where } V\rho = 1$$

$$= \int_\rho^{\rho_0} p\,d\left(\frac{1}{\rho}\right) = -\int_\rho^{\rho_0} \frac{p}{\rho^2}d\rho$$

or
$$E = \int_{\rho_0}^{\rho} \frac{p}{\rho^2}d\rho. \quad \text{Hence } dE = \frac{p}{\rho^2}d\rho$$

$$\frac{dI}{dt} = \int\left[\frac{dE}{dt}\rho\, dV + E\frac{d}{dt}(\rho\, dV)\right] = \int \frac{dE}{dt}\rho\, dV + 0$$

$$= \int \frac{dE}{d\rho}\frac{d\rho}{dt}\rho\, dV = \int \frac{p}{\rho^2}\frac{d\rho}{dt}\rho\, dV = \int \frac{p}{\rho}\frac{d\rho}{dt}dV$$

$$= \int \frac{p}{\rho}(-\rho\,\nabla.q)\, dV \quad \text{as } \frac{d\rho}{dt} + \rho\,\nabla.q = 0$$

is the equation of continuity.

or $$\frac{dI}{dt} = -\int p\,(\nabla.\mathbf{q})\,dV$$

Finally, $$\frac{dT}{dt} = \int \mathbf{q}\,.\,\frac{d\mathbf{q}}{dt}\,\rho\,dv,\quad \frac{dW}{dt} = \int \frac{d\Omega}{dt}\,\rho\,dV, \qquad \ldots(2)$$

$$\frac{dI}{dt} = -\int p\,(\nabla.\mathbf{q})\,dV \qquad \ldots(3)$$

By Euler's equation, $\dfrac{d\mathbf{q}}{dt} = -\nabla\Omega - \dfrac{1}{\rho}\nabla p$

$$\therefore\quad \mathbf{q}\,.\,\frac{d\mathbf{q}}{dt}\,\rho\,dV = -[(\mathbf{q}.\nabla)\,\Omega]\,\rho\,dV - (\mathbf{q}.\nabla p)\,dV$$

Integrating over V and using (2),

$$\frac{dT}{dt} + \int (\mathbf{q}.\nabla\Omega)\,\rho\,dV + \int (\mathbf{q}.\nabla p)\,dV = 0$$

or $$\frac{dT}{dt} + \int \frac{d\Omega}{dt}\,\rho\,dV + \int (\mathbf{q}.\nabla p)\,dV = 0, \text{ by (1)}$$

or $$\frac{dT}{dt} + \frac{dW}{dt} + \int (\mathbf{q}.\nabla p)\,dV = 0, \qquad \ldots(4), \text{ by (2)}.$$

But $$\nabla.(p\mathbf{q}) = p\,\nabla.\mathbf{q} + \mathbf{q}.\nabla p$$

or $$\int \nabla.(p\mathbf{q})\,dV - \int p\,\nabla.\mathbf{q}\,dV = \int (\mathbf{q}.\nabla p)\,dV$$

or $$\int -\hat{\mathbf{n}}.(p\mathbf{q})\,dS + \frac{dI}{dt} = \int (\mathbf{q}.\nabla p)\,dV, \text{ by (3)},$$

as $\hat{\mathbf{n}}$ is inward normal.

Now (4) becomes $$\frac{d}{dt}(T + W + I) - \int \mathbf{n}.(p\mathbf{q})\,dS = 0.$$

or $$\frac{d}{dt}(T + W + I) = \int_S \mathbf{n}.(p\mathbf{q})\,dS$$

This is energy equation. This proves that : rate of change of total energy (K.E. + Potential + Intrinsic) of a portion of fluid is equal to the work done by external pressure on the boundary provided the external forces are conservative.

Corollary. Principle of energy for incompressible fluids. In the present case $I = 0$. Hence the rate of change of total energy (K.E. + P.E.) is equal to the work done by the pressure on the boundary.

WORKING RULES

In order to solve the equations of motion, we adopt the following techniques :

(1) Equation of motion is $\dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = F - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$

where $F = -\dfrac{\partial\Omega}{\partial x}$

(2) Equation of continuity (i) $x^2 v = F(t)$ for spherical symmetry if ρ = const.

(ii) $xv = F(t)$ for cylindrical symmetry if ρ = const.

(iii) $\dfrac{\partial\rho}{\partial t} + \dfrac{\partial \rho v}{\partial x} = 0$ (general case)

(3) Generally the fluid is assumed to be at rest at infinity, *i.e.*, $x = \infty$, $v = 0$; $p = \Pi$, say.

(4) If r be the radius of cavity (or hollow sphere), then $x = r$, $v = \dot{r}$, $p = 0$.

(5) When $r = a$, $v = 0$ so that $F(t) = 0$

(6) Boyle's law : $p_1 V_1 = p_2 V_2$ = const. Its alternate form is $p = k\rho$.

(7) Flux = cross sectional area . normal velocity . density.

(8) Equation of impulsive action is $d\bar{\omega} = \rho v\,dx = \rho v'\,dr'$

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \Gamma\left(\frac{p+1}{2}\right).\Gamma\left(\frac{q+1}{2}\right)\Big/2\Gamma\left(\frac{p+q+2}{2}\right)$$

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \text{ and } \Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$

(9) K.E. of the liquid = work done = $\displaystyle\int -p\,dV$.

(10) If a sphere of radius a is annihilated, then when $x = a$, $p = 0$ so that $v = \dot{x} = 0$.

(11) If a problem contains external and internal radii, *i.e.*, R and r, then subject the result (which is obtained from the integration of the equation of motion) to the two boundary conditions for R and r. In this way we obtain an equation free from constant of integration C and pressure p. Again we integrate this equation to obtain the required result.

SOLVED EXAMPLES

Problem 1. *A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being Π, show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is*

$$\Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right].$$

If $R = a\,(2 + \cos nt)$, show that, to prevent cavitation in the fluid, Π must not be less than $3\rho a^2 n^2$.

Solution : Equation of motion is $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial x}$ and equation of continuity is $x^2 v = F(t)$ so that $\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$.

Hence $\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$ as ρ is constant.

Integrating w.r.t. x, $\quad \frac{-F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C. \quad \ldots(1)$

Boundary conditions are

(2) when $x = \infty$, $p = \Pi$, $v = 0$, $\quad \ldots(2)$

(3) When $x = R$, $p = p$, $v = \dot{R}$, $\quad \ldots(3)$

Also $x^2 v = F(t) = R^2\dot{R}$

$$\therefore \quad F'(t) = 2R(\dot{R})^2 + R^2\ddot{R}.$$

Subjecting (1) to the conditions (2) and (3),

$$0 + 0 = -\frac{\Pi}{\rho} + C \text{ and}$$

$$-\frac{F'(t)}{R} + \frac{1}{2}(\dot{R})^2 = -\frac{p}{\rho} + C = -\frac{p}{\rho} + \frac{\Pi}{\rho}$$

or $$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{1}{2}(\dot{R})^2 + \frac{1}{R}[2R(\dot{R})^2 + R^2\ddot{R}]$$

or $$p = \Pi + \frac{1}{2}\rho\,[3(\dot{R})^2 + 2R\ddot{R}] \quad \ldots(4)$$

Now $$\frac{d^2R^2}{dt^2} + (\dot{R})^2 = \frac{d}{dt}(2R\dot{R}) + \dot{R}^2 = 2\dot{R}^2 + 2R\ddot{R} + \dot{R}^2$$

Now (4) becomes $$p = \Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \dot{R}^2\right] \quad \ldots(5)$$

Second part : Let $R = a(2 + \cos nt)$ ... (6). Let there be no cavitation in the fluid everywhere on the surface so that $p > 0$. Then we have to prove that $\Pi > 3\rho a^2 n^2$.

We have $\dot{R} = -an\sin nt$, $\ddot{R} = -an^2\cos nt$.

Observe that $2R\ddot{R} + 3\dot{R}^2 = 2a(2 + \cos nt)(-an^2\cos nt) + 3a^2n^2\sin^2 nt$

$$= a^2n^2[-4\cos nt - 2\cos^2 nt + 3\sin^2 nt]$$

$$= a^2n^2[-4\cos nt - 2 + 5\sin^2 nt]$$

using this in (4)

$$p = \Pi + \frac{1}{2}\rho a^2n^2(-4\cos nt - 2 + 5\sin^2 nt). \quad \ldots(7)$$

As $\cos nt$ varies from -1 and 1 and so R varies from a to $3a$, by (6). Thus sphere shrinks from $R = 3a$ to $R = a$ and so there is a possibility of cavitation. Also p is minimum when $nt = 0$ or $2m\pi$.

$$p_{min} = \Pi + \frac{1}{2}\rho a^2n^2(-4 - 2 + 0), \text{ by (7)} = \Pi - 3\rho a^2n^2$$

$$p > 0 \Rightarrow p_{min} > 0 \Rightarrow \Pi - 3\rho a^2n^2 > 0 \Rightarrow \Pi > 3\rho a^2n^2$$

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Problem 2. *An infinite mass of fluid acted on by a force $\mu r^{-3/2}$ per unit mass is directed to the origin. If initially the fluid is a rest and there is a cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $(2/5\mu)^{1/2}c^{5/4}$.*

Solution : Let v be the velocity, p the pressure at a distance x from the origin, then the equations of motion and continuity are respectively.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

and $$x^2 v = F(t) \text{ so that } v = \frac{F(t)}{x^2}, \ \frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$$

$$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v\right)^2 = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Integrating, $\quad -\frac{F'(t)}{x} + \frac{1}{2}v^2 = \frac{2\mu}{\sqrt{x}} - \frac{p}{\rho} + C. \quad \ldots(1)$

Boundary conditions are

2. When $x = \infty$, $v = 0$, $p = 0$

3. When $x = r$, (radius of cavity), $p = 0$, $v = \dot{r}$

4. When $r = c$, $v = 0$ so that $F(t) = 0$,

5. Let T be the required time of filling the cavity.

Subjecting (1) to the conditions (2) and (3),

$$0 + 0 = 0 - 0 + C \quad \text{and} \quad \frac{-F'(t)}{r} + \frac{1}{2}(\dot{r}^2) = \frac{2\mu}{\sqrt{r}} - 0 + C$$

or $$\frac{-F'(t)}{r} + \frac{1}{2}\dot{r}^2 = \frac{2\mu}{\sqrt{r}}.$$

Since $\quad r^2(\dot{r}) = F(t)$, $r^2\,dr = F(t)\,dt$.

Multiplying by $\quad 2F(t)\,dt$ or $2r^2\,dr$,

$$\frac{-2F'(t)F(t)\,dt}{r} + \frac{F^2(t)}{r^4}\cdot r^2\,dr = \frac{4\mu}{\sqrt{r}}\cdot r^2\,dr$$

or $$d\left[\frac{-F^2(t)}{r}\right] = 4\mu r^{3/2}\,dr$$

Integrating, $\quad \frac{-F^2(t)}{r} = 4\mu\cdot\frac{2}{5}r^{5/2} + A. \quad \ldots(6)$

Subjecting (6) to (4), $\quad 0 = \frac{8\mu}{5}c^{5/2} + A$

Now (6) $\Rightarrow \quad \frac{-(r^2\dot{r})^2}{r} = \frac{8\mu}{5}(r^{5/2} - c^{5/2})$

$$\Rightarrow \quad \frac{dr}{dt} = -\left[\frac{8\mu}{5r^3}\cdot(c^{5/2} - r^{5/2})\right]^{1/2}$$

[negative sign is taken as velocity increases when r decreases]

$$-\int_c^0 \frac{r^{3/2}}{(c^{5/2}-r^{5/2})^{1/2}}\,dr = \int_0^T \left(\frac{8\mu}{5}\right)^{1/2} dt$$

or
$$T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^c \frac{r^{3/2}\,dr}{(c^{5/2}-r^{5/2})^{1/2}} \quad \ldots (7)$$

Put $r^{5/2} = c^{5/2}\sin^2\theta$, $\frac{5}{2} r^{3/2}\,dr = c^{5/2}.2\sin\theta\cos\theta\,d\theta$.

$$T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \frac{4}{5} c^{5/2} \cdot \frac{\sin\theta\cos\theta\,d\theta}{c^{5/4}\cos\theta} = \left(\frac{5}{8\mu}\right)^{1/2} \cdot \frac{4}{5} c^{5/4} (-\cos\theta)_0^{\pi/2}$$

or
$$T = \left(\frac{2}{5\mu}\right)^{1/2} . c^{5/4}$$ Proved.

Aliter : Equation of continuity is $x^2 u = r^2 v$ ... (1) where u is velocity at distance x and v is velocity at a distance r. K.E., T of liquid when radius of cavity is r :

$$T = \int_r^\infty \frac{1}{2}(4\pi x^2\,dx\,.\,\rho)\,u^2$$

$$= 2\pi\rho \int_r^\infty x^2 \left(\frac{r^2 v}{x^2}\right)^2 dx$$

$$= 2\pi\rho v^2 r^4 \int_r^\infty \frac{dx}{x^2} = 2\pi\rho v^2 r^4 \left(-\frac{1}{x}\right)_r^\infty$$

$$= 2\pi\rho v^2 r^3.$$

If Ω is force potential due to external forces, then
$$-\frac{\partial\Omega}{\partial x} = \frac{\mu}{x^{3/2}} \quad \text{as } \mathbf{F} = -\nabla\Omega.$$

Integrating $\qquad \Omega = \frac{2\mu}{x^{1/2}}$

Work done by external forces
$$= \int_r^c \Omega\,dm = \int_r^c \frac{2\mu}{\sqrt{x}}(4\pi x^2\,dx\,.\,\rho)$$

$$= 8\pi\rho\mu \int_r^c x^{3/2}\,dx$$

$$= \frac{16}{5}\pi\rho\mu\,(c^{5/2} - r^{5/2})$$

By principle of energy, work done = K.E.

or
$$\frac{16}{5}\pi\rho\mu\,(c^{5/2} - r^{5/2}) = 2\pi\rho v^2 r^3$$

$$v = \frac{dr}{dt} = -\left(\frac{8\mu}{5}\right)^{1/2} \left[\frac{c^{5/2} - r^{5/2}}{r^3}\right]^{1/2}$$

$$\text{Time} = T = -\left(\frac{5}{8\mu}\right)^{1/2} \int_c^0 \frac{r^{3/2}\,dr}{(c^{5/2}-r^{5/2})^{1/2}}$$

$$= \left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}.$$

Problem 3. *Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; if V and v be the corresponding velocities of the stream, and if the motion be supposed to be that of divergence from the vertex of the cone, prove that*

$$\frac{v}{V} = \frac{D^2}{d^2} e^{(v^2 - V^2)/2k}$$

where k is the pressure divided by the density and supposed to be constant.

(Meerut 1993)

Solution : Let u be the velocity at a distance x from the end A, the equation of motion is

$$u\frac{\partial u}{\partial x} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

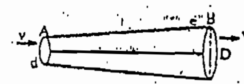

Fig. 2.3

(Since the motion is steady)

or
$$\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = -\frac{k}{\rho}\frac{\partial\rho}{\partial x} \quad \text{as } p = k\rho$$

Integrating, $\qquad \frac{1}{2}u^2 = -k\log\rho + c$

or
$$\log\rho - \log A_1 = -\frac{u^2}{2k} \quad \text{or} \quad \rho = A_1 e^{-u^2/2k} \quad \ldots (1)$$

Boundary conditions are

(i) $\rho = \rho_1$ when $u = v$

(ii) $\rho = \rho_2$ when $u = V$.

Subjecting (1) to (i) and (ii) we obtain $\rho_1 = A_1 e^{-v^2/2k}$ and $\rho_2 = A_1 e^{-V^2/2k}$

This $\qquad \Rightarrow \quad \frac{\rho_1}{\rho_2} = e^{(V^2 - v^2)/2k} \quad \ldots (2)$

By the equation of continuity

Flux at A = Flux at B

$$\pi\left(\frac{d}{2}\right)^2 v\,.\,\rho_1 = \pi\left(\frac{D}{2}\right)^2 . V\,.\,\rho_2$$

or
$$\frac{\rho_1}{\rho_2} = \frac{V}{v} \cdot \frac{D^2}{d^2}$$

Now (2) becomes $\frac{V}{v}\cdot\frac{D^2}{d^2}=e^{(V^2-v^2)/2k}$

or $$\frac{v}{V}=\frac{D^2}{d^2}e^{(v_1^2-V^2)/2k}$$

Problem 4. *A mass of homogeneous liquid is moving so that the velocity at any point is proportional to the time, and that the pressure is given by*

$$\frac{p}{\rho}=\mu xyz+\frac{1}{2}t^2(y^2z^2+z^2x^2+x^2y^2)$$

prove that this motion may have been generated from rest by finite natural forces independent of time; and show that if the direction of motion at every point coincides with the direction of acting forces, each particle of the liquid describes a curve which is the intersection of two hyperbolic cylinders.

Solution : Velocity is proportional to time, *i.e.*, $q=\lambda t$ ... (1).

Also $$\frac{p}{\rho}=\mu xyz-\frac{1}{2}t^2(y^2z^2+z^2x^2+x^2y^2) \quad \dots (2)$$

Step I. Let the motion be generated from rest by finite natural force **F** (conservative force), then there exists velocity potential ϕ s.t. $\mathbf{q}=-\nabla\phi$. To prove that **F** is independent of time.

By pressure equation, $\frac{p}{\rho}+\frac{1}{2}q^2+\Omega-\frac{\partial\phi}{\partial t}=F(t)$

$$\therefore \quad \frac{p}{\rho}=\frac{\partial\phi}{\partial t}-\Omega-\frac{1}{2}\lambda^2t^2+F(t). \quad \dots (3)$$

$$q=\lambda t, \quad \mathbf{q}=-\nabla\phi \Rightarrow \phi=t\,f(x,y,z).$$

Write $\frac{\partial f}{\partial x}=f_x$ etc.; (3) is expressible as

$$\frac{p}{\rho}=f-\Omega-\frac{1}{2}\lambda^2t^2+F(t). \quad \dots (4)$$

Comparing (2) and (4), $f-\Omega=\mu xyz$, $\lambda^2=\Sigma y^2z^2$, $F(t)=0$

Now $\lambda^2t^2=q^2=(\nabla\phi)^2=t^2(\nabla f)^2=t^2(f_x^2+f_y^2+f_z^2)$

or $$\lambda^2=f_x^2+f_y^2+f_z^2 \quad \text{or} \quad \Sigma f_x^2=\Sigma y^2z^2$$

or $$\Sigma(f_x^2-y^2z^2)=0.$$

This $\Rightarrow f_x^2-y^2z^2=0,\ f_y^2-z^2x^2=0,\ f_z^2=\Sigma y^2z^2=0$

$\Rightarrow f=xyz$

We have seen that $f-\Omega=\mu xyz$, this $\Rightarrow$

$$xyz-\Omega=\mu xz \quad \text{or} \quad \mathbf{F}=-\nabla\Omega=\nabla(\mu-1)xyz$$

or $$\mathbf{F}=(\mu-1)\nabla(xyz). \quad \dots (5)$$

This $\Rightarrow$ **F** is independent of t.

Step II. Let the direction of motion coincide with the direction of acting force so that

$$\frac{u}{F_1}=\frac{v}{F_2}=\frac{w}{F_3}. \quad \dots (6)$$

To prove that stream lines are the intersection of two hyperbolic cylinders. Equations of stream lines are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}.$$

Using (6), $$\frac{dx}{F_1}=\frac{dy}{F_2}=\frac{dz}{F_3}.$$

By (5), $$\frac{dx}{(\mu-1)yz}=\frac{dy}{(\mu-1)zx}=\frac{dz}{(\mu-1)xy}$$

or $$\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{xy} \quad \text{or} \quad x\,dx=y\,dy=z\,dz.$$

This $\Rightarrow$ $x\,dx=y\,dy,\ x\,dx=z\,dz.$

Integration yields the result $x^2-y^2=a^2,\ x^2-z^2=b^2$. This represents two distinct hyperbolic cylinders. Hence the result.

Problem 5. *Air, obeying Boyle's law, is in motion in a uniform tube of small section, prove that if ρ be the density and v the velocity at a distance x from a fixed point at time t,*

$$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}[(v^2+k)\rho], \quad \text{where } k=\frac{p}{\rho}.$$

(Kanpur 2004, Meerut 2003, Garhwal 2002, 2003)

Solution : Equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0. \quad \dots (1)$$

Equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}. \quad \dots (2)$$

By Boyle's law, pr. vol. = const.

But vol. density = mass.

Hence pr. vol. = const, $\quad$ vol. $=\frac{\text{mass}}{\rho}$

$$\therefore \quad \text{pr.}\ \frac{\text{mass}}{\rho}=\text{const.}$$

This $\Rightarrow \frac{p}{\rho}=$ const. $=k$, say $\Rightarrow p=k\rho$.

By (2), $$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{k}{\rho}\frac{\partial \rho}{\partial x} \quad \dots (3)$$

To determine $\frac{\partial^2\rho}{\partial t^2}$.

By (1), $$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial}{\partial t}\left[-\frac{\partial}{\partial x}(\rho v)\right]=-\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}(\rho v)\right]$$

$$\frac{\partial^2 \rho}{\partial t^2} = -\frac{\partial}{\partial x}\left[v\frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial t}\right]$$

$$= -\frac{\partial}{\partial x}\left[v\left(-\frac{\partial \rho v}{\partial x}\right) + \rho\left(-\frac{k}{\rho}\frac{\partial \rho}{\partial x} - v\frac{\partial v}{\partial x}\right)\right]$$

$$= \frac{\partial}{\partial x}\left[v\frac{\partial \rho v}{\partial x} + k\frac{\partial \rho}{\partial x} + \rho v\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\rho v . v) + \frac{\partial}{\partial x}(k\rho)\right] = \frac{\partial^2}{\partial x^2}(\rho v^2 + k\rho)$$

or
$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}[\rho(v^2 + k)]$$

Problem 6. *An elastic fluid, the weight of which is neglected obeying Boyle's is in motion in a uniform straight tube; show that on the hypothesis of parallel sections the velocity at any time t at a distance r from a fixed point in the tube is defined by the equation*

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t} + v^2\frac{\partial v}{\partial r}\right) = k\frac{\partial^2 v}{\partial r^2}.$$

Solution : Boyle's law is $\quad \frac{p}{\rho} = k$ as volume $\propto \frac{1}{\rho}$.

Equations of continuity and motion are

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} = 0 \qquad \dots(1)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r},$$

i.e.,
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{k}{\rho}\frac{\partial \rho}{\partial r} \qquad \dots(2)$$

To determine $\frac{\partial^2 v}{\partial t^2}$. By (2), we get

$$-\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial t}\left[v\frac{\partial v}{\partial r} + \frac{k}{\rho}\frac{\partial \rho}{\partial r}\right] = \frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t} + \frac{k}{\rho}\frac{\partial \rho}{\partial t}\right)^*$$

$$= \frac{\partial}{\partial r}\left[v\left(-v\frac{\partial v}{\partial r} - \frac{k}{\rho}\frac{\partial \rho}{\partial r}\right) + \frac{k}{\rho}\left(-\frac{\partial \rho v}{\partial r}\right)\right], \quad \text{by (1), (2).}$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial r}\left[v^2\frac{\partial v}{\partial r} + \frac{vk}{\rho}\frac{\partial \rho}{\partial r} + \frac{kv}{\rho}\frac{\partial \rho}{\partial r} + k\frac{\partial v}{\partial r}\right]$$

$$^*\ \frac{\partial}{\partial t}\left[v\frac{\partial v}{\partial r} + \frac{k}{\rho}\frac{\partial \rho}{\partial r}\right] = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial r}\left(\frac{1}{2}v^2\right) + \frac{\partial}{\partial r}(k\log\rho)\right]$$

$$= \frac{\partial^2}{\partial t\,\partial r}\left(\frac{1}{2}v^2 + k\log\rho\right)$$

$$= \frac{\partial}{\partial r}\frac{\partial}{\partial t}\left(\frac{1}{2}v^2 + k\log\rho\right)$$

$$= \frac{\partial}{\partial r}\left[v\frac{\partial v}{\partial t} + \frac{k}{\rho}\frac{\partial \rho}{\partial t}\right]$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial r}\left(v^2\frac{\partial v}{\partial r} + \frac{2kv}{\rho}\frac{\partial \rho}{\partial r}\right) + k\frac{\partial^2 v}{\partial r^2}$$

$$= \frac{\partial}{\partial r}\left[v^2\frac{\partial v}{\partial r} + 2v\left(-\frac{\partial v}{\partial t} - v\frac{\partial v}{\partial r}\right)\right] + k\frac{\partial^2 v}{\partial r^2}, \quad \text{by (2).}$$

$$= \frac{\partial}{\partial r}\left[-v^2\frac{\partial v}{\partial r} - 2v\frac{\partial v}{\partial t}\right] + k\frac{\partial^2 v}{\partial r^2}$$

or
$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial r}\left(v^2\frac{\partial v}{\partial r} + 2v\frac{\partial v}{\partial t}\right) = k\frac{\partial^2 v}{\partial r^2}.$$

Problem 7. *A mass of liquid surrounds a solid sphere of radius a, and its outer surface, which is a concentric sphere of radius b, is subject to a given constant pressure Π, no other forces being action on the liquid. Then solid sphere suddenly shrinks into a concentric sphere; it is required to determine the subsequent motion and the impulsive action on the sphere.*

Solution : Equations of motion and continuity are

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \dots(1)$$

$$x^2 v = F(t). \qquad \dots(2)$$

Hence $\quad \frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right).$

Integrating w.r.t. x, we get

$$-\frac{F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C \qquad \dots(3)$$

Since the liquid is contained between two spheres $r = a, r = b$, so we suppose that r and R are internal and external radii at any time t and the corresponding velocities are u and U, respectively. Boundary conditions are

when $\quad x = r, \quad v = u = \dot{r}, \quad p = 0. \qquad \dots(4)$

(Since pressure vanishes on the internal boundary)

When $\quad x = R, \quad v = \dot{R} = U, \quad p = \Pi, \qquad \dots(5)$

(Since outer surface is subjected to constant pressure Π).

$r = a, \quad v = \dot{r} = 0 \quad$ so that $F(t) = 0. \qquad \dots(6)$

Subjecting (3) to the conditions (4) and (5),

$$-\frac{F'(t)}{r} + \frac{1}{2}u^2 = 0 + C$$

$$-\frac{F'(t)}{R} + \frac{1}{2}U^2 = -\frac{\Pi}{\rho} + C.$$

Also $r^2 u = F(t) = R^2 U$ upon subtraction,

$$F'(t)\left\{\frac{1}{R} - \frac{1}{r}\right\} + \frac{1}{2}.F^2\left\{\frac{1}{r^4} - \frac{1}{R^4}\right\} = \frac{\Pi}{\rho}. \qquad \dots(7)$$

Since $r^2 u = F(t) = R^2 U$ i.e., $r^2 dr = F(t)\,dt = R^2 dR.$

Multiplying (7) by $2F(t)\,dt = 2r^2\,dr = 2R^2\,dR$, we get

$$2FF'\left\{\frac{1}{R}-\frac{1}{r}\right\}dt + F^2\left\{\frac{dr}{r^2}-\frac{dR}{R^2}\right\} = \frac{\Pi}{\rho}\cdot 2r^2\,dr$$

or
$$d\left[\left(\frac{1}{R}-\frac{1}{r}\right)F^2\right] = \frac{\Pi}{\rho}\cdot 2r^2\,dr.$$

Integrating, $\left(\frac{1}{R}-\frac{1}{r}\right)F^2(t) = \frac{2}{3}\frac{r^3}{\rho}\cdot\Pi + A.$

Subjecting this to (6), $0 = \frac{2}{3}\frac{a^3}{\rho}\cdot\Pi + A.$

Subtracting, we get

$$\left(\frac{1}{R}-\frac{1}{r}\right)F^2(t) = \frac{2\Pi}{3\rho}\cdot(r^3-a^3)$$

$$\left(\frac{R-r}{rR}\right)(r^2u)^2 = \frac{2\Pi}{3\rho}\cdot(a^3-r^3)$$

or
$$r^3u^2\left(\frac{R-r}{R}\right) = \frac{2\Pi}{3\rho}(a^3-r^3) \qquad \dots(8)$$

with
$$R^3 - r^3 = b^3 - a^3.$$

For total mass of liquid is constant

$\Rightarrow$ volume of liquid at any time t = volume of liquid initially.

$$\Rightarrow \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3.$$

To determine the equation of impulsive action. Equation of impulsive action is

$$d\varpi = \rho v\,dx = \frac{\rho F}{x^2}\,dx$$

$$\int_0^{\varpi} d\varpi = \int_r^R \frac{\rho F}{x^2}\,dx = -\rho F\left\{\frac{1}{R}-\frac{1}{r}\right\} = \rho r^2 u\left\{\frac{1}{r}-\frac{1}{R}\right\}$$

or
$$\varpi = \rho r^2 u\left(\frac{1}{r}-\frac{1}{R}\right)$$

The whole impulse on the surface of the sphere is

$$4\pi r^2\varpi = 4\pi r^2\rho r^2 u\left(\frac{1}{r}-\frac{1}{R}\right) = 4\pi r^3\cdot\rho u\left(\frac{R-r}{R}\right) \qquad \dots(9)$$

(8) and (9) are the required equations. **Ans.**

Problem 8. *An infinite fluid in which a spherical hollow shell of radius a is initially at rest under the action of no forces. If a constant pressure Π is applied at infinity, show that the time of filling up the cavity is $\pi^2 a\left(\frac{\rho}{\Pi}\right)^{1/2}\cdot 2^{5/6}\cdot[\Gamma(1/3)]^{-3}$.* (Meerut 2002)

Solution : The equations of motion and continuity are

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \dots(1)$$

$$x^2 v = F(t).$$

Hence
$$\frac{F'(t)}{x^2} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Inegrating w.r.t. x,
$$-\frac{F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C. \qquad \dots(2)$$

Let T be the time of filling up the cavity. Boundary conditions are :

(i) when $x = \infty$, $v = 0$, $p = \Pi$, (since constant pressure Π is applied at infinity)

(ii) when $x = r$ = radius of cavity, $v = u = \dot r$, $p = 0$,

[The pressure vanishes on the surface of cavity]

(iii) When $r = a$, so that $\dot r = u = 0$, $F(t) = 0$.

Subjecting (2) to the condition (i),
$$0 = -\frac{\Pi}{\rho} + C \quad\text{or}\quad C = \frac{\Pi}{\rho}.$$

Subjecting (2) to (ii),
$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = 0 + C$$

or
$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = \frac{\Pi}{\rho} \quad \text{s.t.}\quad r^2u = F(t) = r^2\dot r$$

or
$$\frac{-F'(t)}{r} + \frac{1}{2}\cdot\frac{F^2}{r^4} = \frac{\Pi}{\rho}.$$

Multiplying this by $2F\,dt = 2r^2\,dr$,
$$\frac{2FF'}{r}\,dt - \frac{F^2}{r^2}\,dr = -\frac{\Pi}{\rho}\cdot 2r^2\,dr$$

or
$$d\left(\frac{F^2}{r}\right) = -\frac{\Pi}{\rho}\cdot 2r^2\,dr.$$

Integration yields $\frac{F^2}{r} = -\frac{2\Pi}{3\rho}r^3 + A.$

Subjecting this to (iii),
$$0 = -\frac{2\Pi}{3\rho}a^3 + A$$

Hence
$$\frac{F^2}{r} = \frac{2\Pi}{3\rho}(a^3-r^3) \quad\text{or}\quad \frac{r^4\dot r^2}{r} = \frac{2\Pi}{3\rho}(a^3-r^3)$$

or
$$\frac{dr}{dt} = -\left[\frac{2\Pi}{3\rho}\cdot\frac{a^3-r^3}{r^3}\right]^{1/2}$$

[Negative sign is taken as velocity increases when r decreases].

$$\int_0^T dt = -\int_a^0\left[\frac{3\rho}{2\Pi}\cdot\frac{r^3}{a^3-r^3}\right]^{1/2}dr$$

or
$$T = \left(\frac{3\rho}{2\Pi}\right)^{1/2}\cdot I, \qquad \dots(3)$$

where
$$I = \int_0^a\left(\frac{r^3}{a^3-r^3}\right)^{1/2}dr.$$

Put $r^3 = a^3\sin^2\theta$, $3r^2\,dr = 2a^3\sin\theta\cos\theta\,d\theta$

$$I = \int_0^{\pi/2} \frac{a^{3/2} \sin\theta}{a^{3/2} \cos\theta} \cdot \frac{2a^3 \sin\theta \cos\theta \, d\theta}{3r^2}$$

$$= \int_0^{\pi/2} \frac{2a^3 \sin^2\theta \, d\theta}{3 (a \sin^{2/3}\theta)^2}$$

$$= \frac{2a}{3} \int_0^{\pi/2} (\sin\theta)^{2/3} (\cos\theta)^0 \, d\theta$$

$$= \frac{2a}{3} \cdot \frac{\Gamma\left(\frac{5}{6}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{6} + \frac{1}{2}\right)} = \frac{a}{3} \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(1 + \frac{1}{2}\right)}$$

$$= \frac{a}{3} \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{5}{6}\right)}{\left(\frac{1}{3}\right) \Gamma\left(\frac{1}{3}\right)} = \frac{a\sqrt{\pi}}{\Gamma\left(\frac{1}{3}\right)} \cdot \Gamma\left(\frac{5}{6}\right) \qquad \dots (4)$$

Recall that

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \quad \Gamma(n)\,\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi}\, \frac{\Gamma(2n)}{2^{2n-1}}$$

For $n = \frac{1}{3}$, $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sin(\pi/3)} = \frac{2\pi}{\sqrt{3}}$

and $$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right)}{2^{-1/3}}$$

Hence $$\Gamma\left(\frac{5}{6}\right) = \frac{\sqrt{\pi} \cdot 2^{1/3}}{\Gamma\left(\frac{1}{3}\right)} \cdot \frac{2\pi}{\sqrt{3} \cdot \Gamma\left(\frac{1}{3}\right)} = \frac{2^{4/3}}{\sqrt{3}} \cdot \pi^{2/3} \cdot \frac{1}{\left[\Gamma\left(\frac{1}{3}\right)\right]^2}$$

using this in (4), $$I = a\pi^2 \cdot \frac{2^{4/3}}{\sqrt{3}} \cdot \frac{1}{\left[\Gamma\left(\frac{1}{3}\right)\right]^2}$$

Now (3) is reduced to

$$T = \pi^2 a \left(\frac{\rho}{\Pi}\right)^{1/2} \cdot \frac{1}{\left[\Gamma\left(\frac{1}{3}\right)\right]^3} \cdot 2^{5/6}$$ Proved.

Aliter : Let v be velocity when radius of cavity is r. Similarly u is the velocity when radius is x. Equation of continuity is

$$x^2 u = r^2 v \qquad \dots (1)$$

$$\text{K.E.} = \int \frac{1}{2} (4\pi x^2 \, dx \,.\, \rho)\, u^2$$

$$= 2\pi\rho \int_r^\infty x^2 \left(\frac{r^2 v}{x^2}\right)^2 dx = 2\pi\rho r^4 v^2 \int_0^\infty \frac{dx}{x^2}$$

$$\text{K.E.} = 2\pi\rho\, r^4 v^2 \left(-\frac{1}{x}\right)_r^\infty = 2\pi\rho\, r^3 v^2$$

Work done by outer pressure

$$= \int \Pi\,(4\pi x^2 dx) = 4\pi\,\Pi \int_r^a x^2 \, dx$$

$$= \frac{4\pi}{3} \Pi\,(a^3 - r^3)$$

By principle of energy,

$$2\pi\rho r^3 v^2 = \frac{4\pi}{3} \Pi\,(a^3 - r^3)$$

or $$v = -\frac{dr}{dt} = \left(\frac{2\Pi}{3\rho}\right)^{1/2} \left(\frac{a^3 - r^3}{r^3}\right)^{1/2}$$

$$\int_0^t dt = -\int_a^0 \left(\frac{3\rho}{2\Pi}\right)^{1/2} \cdot \frac{r^{3/2}\, dr}{(a^3 - r^3)^{1/2}}$$

From this the required result follows.

Problem 9. *A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is u_0 to become $\rho_0\, \phi\,(vt - x)$. Prove that the velocity u (at time t and distance x from the origin) is given by* $v + \frac{(u_0 - v)\,\phi\,(vt)}{\phi\,(vt - x)}$.

Solution : Equation of continuity is

$$\frac{\partial\rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0. \qquad \dots (1)$$

we have to prove $$u = v + \frac{(u_0 - v)\,\phi\,(vt)}{\phi\,(vt - x)}$$

Given $$\rho = \rho_0\, \phi\,(vt - x) \qquad \dots (2)$$

and $$u = u_0 \text{ when } x = 0. \qquad \dots (3)$$

$$(2) \Rightarrow \frac{\partial \rho}{\partial t} = \rho_0\, v \phi'\,(vt - x), \quad \frac{\partial\rho}{\partial x} = -\rho_0\, \phi'\,(vt - x)$$

Putting these values in (1), we get

$$\rho_0\, v\phi'\,(vt - x) + u\,[-\rho_0 \phi'\,(vt - x)] + \rho_0\, \phi\,(vt - x)\, \frac{\partial u}{\partial x} = 0$$

or $$(v - u)\,\phi' + \phi\, \frac{\partial u}{\partial x} = 0 \quad \text{or} \quad \frac{du}{v - u} + \frac{\phi'\,(vt - x)}{\phi\,(vt - x)}\, dx = 0$$

Integrating, $$-\log(v - u) - \log \phi\,(vt - x) = -\log A$$

or $$(v - u)\,\phi\,(vt - x) = A \qquad \dots (4)$$

In view of (3), this $\Rightarrow$ $(v - u_0)\,\phi\,(vt) = A$

$$\therefore \quad (v - u)\,\phi\,(vt - x) = (v - u_0)\,\phi\,(vt)$$

or
$$v - u = \frac{(v - u_0)\,\phi(vt)}{\phi(vt - x)}$$

or
$$u = v + \frac{(u_0 - v)\,\phi(vt)}{\phi(vt - x)}$$

Problem 10. *A stream in a horizontal pipe after passing a contraction in the pipe at which its sectional area is A, is delivered at atmospheric pressure at a place, where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth* $\frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$ *below the pipe, s being the delivery per second.* (*Garhwal 2000*)

Solution : Let v and V be the the velocity of the stream at two cross-sections. The equation of continuity is given by

flux at the first cross section = flux at the second cross section

i.e., $A v \rho = B V \rho = s$, (given)

[For flux = cross section area × density . normal velo.]

Also $\rho = 1$ for stream.

Hence $v = \frac{s}{A}$, $V = \frac{s}{B}$.

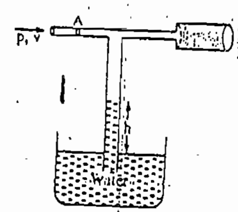

Fig. 2.4

The equation of motion is $u\,\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ as the motion is steady,

or
$$\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = -\frac{\partial p}{\partial x} \quad \text{as } \rho = 1.$$

Integrating,
$$\frac{1}{2}u^2 = -p + C \qquad \dots (1)$$

Boundary conditions are :

(i) $u = V$, $p = \Pi$.

(Since stream is deliverd at atmospheric pressure $p = \Pi$, say at a place where cross-sectional area is B).

(ii) $u = v$, $p = p$.

In view of (i) and (ii), (1) gives $\frac{1}{2}V^2 = -\Pi + C$
$$\frac{1}{2}v^2 = -p + C.$$

Upon subtraction, $\Pi - p = \frac{1}{2}(v^2 - V^2) = \frac{1}{2}\left(\frac{s^2}{A^2} - \frac{s^2}{B^2}\right)$

or
$$\Pi - p = \frac{s^2}{2}\left(\frac{1}{A^2} - \frac{1}{B^2}\right). \qquad \dots (2)$$

Let h be the height of water column in the side tube which is sucked from a reservoir, then $\Pi - p$ = difference of pressure $= \rho g h = gh$ as $\rho = 1$.

Now (2) $\Rightarrow gh = \frac{s^2}{2}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$ or $h = \frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$.

This concludes the problem.

Problem 11. *Show that the rate per unit of time at which work is done by the internal pressure between the parts of a compressible fluid obeying Boyle's law is*
$$\iiint p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx\,dy\,dz,$$
where p is the pressure and (u, v, w) the velocity at any point, and the integration extends through the volume of the fluid.

Solution : Let W denote work done, then rate of work done is $\frac{dW}{dt}$. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and $dV = dx\,dy\,dz$.

Then we have to prove that $\frac{dW}{dt} = \int p\,(\nabla . \mathbf{q})\,dV$. ... (1)

We know that $W = \int -p\,d V$

and
$$\frac{d\rho}{dt} + \rho\,(\nabla . \mathbf{q}) = 0 \text{ (equation of continuity)} \qquad \dots (2)$$

Hence $\frac{dW}{dt} = \int -\frac{dp}{dt}\,dV = -\int \frac{d\,k\rho}{dt}\,dV$ as $p = \rho k$ (Boyle's law)

or $\frac{dW}{dt} = -k\int \frac{d\rho}{dt}\,dV = -k\int -\rho\,(\nabla . \mathbf{q})\,dV$, by (2)
$$= \int k\rho\,(\nabla . \mathbf{q})\,dV = \int p\,(\nabla . \mathbf{q})\,dV$$

Hence the result (1).

Problem 12. *A spherical mass of fluid of radius b has a concentric spherical cavity of radius a, which contains gas at pressure p whose mass may be neglected; at every point of the external boundary of the liquid an impulsive pressure* $\overline{\omega}$ *per unit area is applied. Assuming that the gas obeys Boyle's law, show that when the liquid first comes to rest, the radius of the internal spherical surface will be* $a\,\exp\{-\overline{\omega}^2/2p\rho\,a^2\,(b - a)\}$ *where* ρ *is the density of the liquid.*

Solution : Equation of impulsive action is $d\overline{\omega} = \rho v\,dx$ and equation of continuity is
$$x^2 v = F(t).$$
$$d\overline{\omega} = \rho\,F(t)\,.\,\frac{dx}{x^2}$$

This $\Rightarrow \int_0^{\varpi} d\varpi = \int_a^b \rho F \dfrac{dx}{x^2}$

or $$\varpi = \rho F \left\{-\frac{1}{x}\right\}_a^b = \left(\frac{b-a}{ab}\right)\rho F(t). \quad \ldots (1)$$

Let r be the radius of internal spherical cavity and p_1 the pressure there. Since gas obeys Boyle's law hence

$$\frac{4}{3}\pi r^3 p_1 = \frac{4}{3}\pi a^3 p \quad \text{or} \quad p_1 = \frac{a^3 p}{r^3}.$$

[Internal cavity of radius a contains gas at pressure p].

Finally, the liquid is at rest.

$$\text{Gain in K.E.} = \int_a^b \frac{1}{2}(4\pi x^2\,dx\,.\,\rho)\,v^2 = 2\pi\rho \int_a^b .\, x^2 \frac{F^2}{x^4}\,dx \quad \text{as } x^2 v = F(t)$$

$$= 2\pi\rho\left(-\frac{1}{x}\right)_a^b F^2 = 2\pi\rho F^2\left\{\frac{b-a}{ab}\right\}$$

$$= 2\pi\rho\left(\frac{b-a}{ab}\right).\frac{\varpi^2 a^2 b^2}{\rho^2(b-a)^2}$$

$$= 2\pi\, ab\, \varpi^2/\rho\,(b-a).$$

Work done in compressing the gas from radius a to radius r is $\int_a^r -p\,dV$ in usual notation

$$= -\int_a^r \frac{4\pi r^2\,dr}{r^3}\,.\,a^3 p = -4\pi p a^3 \log\left(\frac{r}{a}\right).$$

But gain in K.E. = work done.

$$\therefore \quad \frac{2\pi ab\varpi^2}{\rho(b-a)} = -4\pi p a^3 \log\left(\frac{r}{a}\right)$$

or $$\log\left(\frac{r}{a}\right) = \frac{\varpi^2 b}{2a^2 p\,\rho\,(b-a)}$$

Hence $$r = a \exp\left\{\frac{\varpi^2 b}{2a^2 p\,\rho\,(b-a)}\right\}.$$

Problem 13. *Two equal closed cylinders, of height c, with their bases in the same horizontal plane, are filled, one with water, and the other with air of such a density as to support a column h of water, h < c. If a communication be open between them at their bases, the height x, to which the water rises, is given by the equation*

$$cx - x^2 + ch \log\left(\frac{c-x}{c}\right) = 0.$$

(*Meerut 1992*)

Solution : Suppose that the cylinders A and B are filled with water and gas respectively. Let k be the cross section of each cylinder. The water and gas both are at rest before and after the communication is allowed between the cylinders. Hence initial and final both K.E. are zero. Change in K.E. = 0.

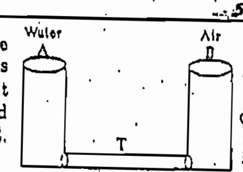

Fig. 1.5

This $\Rightarrow$ Total work done = change in K.E.

$\Rightarrow$ Total work done = 0.

Initial potential energy due to water in A

= 'Mgh' in usual notation.

$$= \int_0^c (kz\rho)\,gdz = \frac{1}{2}kg\rho c^2$$

and final potential energy due to watr of height $c - x$ in A and height x in B is

$$\int_0^{c-x} (kz\rho)\,gdz + \int_0^x (kz\rho)\,gdz = \frac{1}{2}kg\rho\,[(c-x)^2 + x^2].$$

Now work done by gravity = loss in potential energy

= Initial P.E. − Final P.E.

$$= \frac{1}{2}kg\rho\,[c^2 - (c-x)^2 - x^2] = kg\rho\,(cx - x^2)$$

$\therefore$ Work done by gravity $= kg\rho\,(cx - x^2)$. ... (2)

Also some work is done against the compression of air in B. Let p be the pressure of the gas when the height of water level in B is y. By Boyle's law,

$$P_1V_1 = P_2V_2 \quad \text{or} \quad pk\,(c-y) = h\rho g\,.\,kc.$$

This $\Rightarrow p = \dfrac{h\rho gc}{c-y}$, ρ being density of water.

[For pressure $= hdg$ = height, density, g and initial pressure of the gas in B is equal to pressure due to a column h of water (given)].

Work done against the compression of gas in B

$$= \int_0^x -p\,dV, \text{ in usual notation.}$$

$$= \int_0^x -\left(hg\rho\frac{c}{c-y}\right)kdy, \quad dV = kdy$$

$$= hg\rho ck \log\left(\frac{c-h}{c}\right). \quad \ldots (3)$$

Equating the sum of (2) and (3) to (1),

$$kg\rho\,[cx - x^2] + hg\rho ck \log\left(\frac{c-x}{c}\right) = 0$$

$$cx - x^2 + ch \log\left(\frac{c-x}{c}\right) = 0$$

Problem 13 (a). *Water oscillates in a bent uniform tube in a vertical plane. If O be the lowest point of the tube, AB the equilibrium level of water, α, β the inclinations of the tube to the horizontal at A, B and OA = a, OB = b, the period of oscillation is given by*

$$2\pi\left\{\frac{(a+b)}{g\,(\sin\alpha+\sin\beta)}\right\}^{1/2}$$

(Kanpur 2001)

Solution : Suppose O is the lowest point of the tube, AB the equilibrium level of water, h the height of AB above O, α, β the inclinations of the tube to the horizontal at A and B respectively and θ, the inclination at a distance s from O. Let $OA = a$,

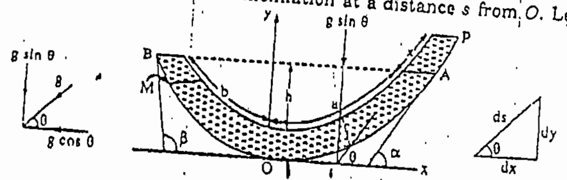

Fig. 2.6

$OB = b$, $AP = x$. Let water in the tube be displaced at small distance x from its equilibrium position so that $AP = x$. After displacement

(i) $p = \Pi$, $y = h + x\sin\alpha$, $s = OP = a + x$ at P,

(ii) $p = \Pi$, $y = h - x\sin\beta$, $s = OM = -(b - x)$ at M.

Let u denote velocity. Equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

or,

$$\frac{\partial u}{\partial s}=0 \qquad \ldots(1)$$

This ⇒ u is independent of s.

Equation of motion is

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial s}=-g\sin\theta-\frac{1}{\rho}\frac{\partial p}{\partial s}$$

But

$$\frac{\partial u}{\partial s}=0,\quad \sin\theta=\frac{\partial y}{\partial s}$$

Hence we have

$$\frac{\partial u}{\partial t}=-g\frac{\partial y}{\partial s}-\frac{\partial}{\partial s}\left(\frac{p}{\rho}\right)$$

Integrating w.r.t. s,

$$\int\frac{\partial u}{\partial t}\,ds=-gy-\frac{p}{\rho}+f(t)$$

or

$$s\,\frac{\partial u}{\partial t}=-gy-\frac{p}{\rho}+f(t)$$

($f(t)$ is constant of integration)

Applying (i) and (ii),

$$(a+x)\frac{\partial u}{\partial t}=-g\,(h+x\sin\alpha)-\frac{\Pi}{\rho}+f(t)$$

$$-(b-x)\frac{\partial u}{\partial t}=-g\,(h-x\sin\beta)-\frac{\Pi}{\rho}+f(t)$$

Upon subtraction,

$$(a+b)\frac{\partial u}{\partial t}=-gx\,(\sin\alpha+\sin\beta)$$

Since

$$u=\dot{x},\quad \dot{u}=\ddot{x}$$

$$(a+b)\,\ddot{x}=-gx\,(\sin\alpha+\sin\beta)$$

or

$$\ddot{x}=-\mu x \qquad \ldots(1),\quad \text{where } \mu=\frac{g\,(\sin\alpha+\sin\beta)}{(a+b)}$$

(1), represents S.H.M. Its time period T is given by

$$T=\frac{2\pi}{\sqrt{\mu}}=2\pi\left[\frac{(a+b)}{g\,(\sin\alpha+\sin\beta)}\right]^{1/2}$$

Problem 14. *A given quantity of liquid moves, under no forces, in a smooth conical tube having a small vertical angle, and the distances of its nearer and farther extremities for the vertex at time t are r and r′, show that*

$$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left[3-\frac{r}{r'}-\frac{r^2}{r'^2}-\frac{r^3}{r'^3}\right]=0,$$

the pressure at the two surfaces being equal. Show also that the preceding equation results from supposing the vis viva of the mass of liquid to be constant; and that the velocity of inner surface is given by the equation

$$V^2=\frac{Cr'}{r^3\,(r'-r)},\quad r'^3-r^3=c^3.$$

C and c being constants.

Solution : At any time t, let p be the pressure at a distance x from the vertex and v the velocity there. The equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=0-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

and the equation of continuity is

$$(x\tan\alpha)^2\,v=F(t),$$

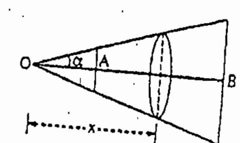

Fig. 2.7

i.e.,

$$x^2v=f(t)$$

where $f(t)=\dfrac{F(t)}{\tan^2\alpha}$. $\left(\text{Here } \dfrac{r}{h}=\dfrac{r}{x}=\tan\alpha\right)$

Hence,

$$\frac{f'(t)}{x}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating w.r.t. x,

$$-\frac{f'(t)}{x}-\frac{1}{2}v^2=-\frac{p}{\rho}+C. \qquad \ldots(1)$$

Boundary conditions are

(i) when $x = r, p = p, v = \dot{r} = u$, say

(ii) when $x = r', v = U = \dot{r}', p = p$

[Since the pressure at the two ends is equal].

Subjecting (1) to the conditions (i) and (ii),

$$-\frac{f'(t)}{r} + \frac{1}{2}u^2 = -\frac{p}{\rho} + C$$

$$-\frac{f'(t)}{r'} + \frac{1}{2}U^2 = -\frac{p}{\rho} + C$$

Upon subtraction

$$\left(\frac{1}{r'} - \frac{1}{r}\right) f'(t) + \frac{1}{2}(u^2 - U^2) = 0.$$

But $\quad r^2 u = f(t) = r'^2 U$

$$\therefore \quad \left(\frac{1}{r'} - \frac{1}{r}\right)\frac{d}{dt}(r^2 u) + \frac{1}{2}u^2\left(1 - \frac{r^4}{(r')^4}\right) = 0, \quad u = \dot{r}$$

or $$\left(\frac{r - r'}{rr'}\right)\left[2r\left(\frac{dr}{dt}\right)^2 + r^2\frac{d^2r}{dt^2}\right] + \frac{1}{2}\left(\frac{dr}{dt}\right)^2\left[\frac{r'^4 - r^4}{r'^4}\right] = 0$$

Dividing by $(r - r')/2r'$, we get

$$\frac{2}{r}\left[2r\left(\frac{dr}{dt}\right)^2 + r^2\frac{d^2r}{dt^2}\right] - \left(\frac{dr}{dt}\right)^2 \frac{(r + r')(r^2 + r'^2)}{r'^3} = 0$$

or $$2r\frac{d^2r}{dt^2} + \left(\frac{dr}{dt}\right)^2\left[3 - \frac{r}{r'} - \frac{r^3}{r'^3} - \frac{r^2}{r'^2}\right] = 0.$$

This proves the first required result.

Second Part : The vis-via = 2K.E.

$$= 2\int_r^{r'} \frac{1}{2}(\pi x^2 \tan^2 \alpha)\, dx . \rho v^2 = \pi\rho \tan^2 \alpha \int_r^{r'} \frac{x^2}{x^4} f^2(t)\, dx$$

$$= \pi\rho \tan^2 \alpha f^2(t)\left(\frac{1}{r} - \frac{1}{r'}\right).$$

By the principle of conservation of vis-via,

$$\pi\rho \tan^2 \alpha f^2(t) . \left(\frac{1}{r} - \frac{1}{r'}\right) = \text{const.} = C_1$$

or $$\left(\frac{1}{r} - \frac{1}{r'}\right)(r^2 u)^2 = C_2$$

or $$\left(\frac{r' - r}{rr'}\right) r^4 u^2 = C_2$$

or $$u^2 = \frac{r' C_2}{(r' - r) r^3}$$

Replacing u by V and C_2 by C we get

$$V^2 = Cr'/r^3 (r' - r).$$

Again, since mass is constant and so is volume.

This $\Rightarrow \frac{1}{3}(\pi r'^2 \tan \alpha . r' - \pi r^2 \tan^2 \alpha . r) = \text{const.}$

or $\quad r'^3 - r^3 = \text{const.} = c^3$, say. For volume $= \frac{\pi}{3}(\text{radius})^2 h$

This concludes the problem.

Problem 15. *A portion of homogeneous fluid is confined between two concentric spheres of radii A and a, and is attracted towards their centre by a force varying inversely as the square of the distance the inner spherical surface is suddenly annihilated, and when the radii of the inner and outer surfaces of the fluid are r and R, the fluid impinges on a solid ball concentric with their surfaces; prove that the impulsive pressure at any point of the ball for different values of R and r varies as*

$$\left[(a^2 - r^2 - A^2 + R^2)\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{1/2}$$

Solution : The equation of continuity is $x^2 v = F(t)$ so that $\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$. Equation of motion is

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = F - \frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\mu}{x^2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

[as μ/x^2 is a force towards the centre].

or $$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\mu}{x^2} - \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating w.r.t. x, $\quad \frac{-F'(t)}{x} + \frac{1}{2}v^2 = \frac{\mu}{x} - \frac{p}{\rho} + C. \quad \text{... (1)}$

Let r and R be internal and external radii at any time t. Boundary conditions are

(i) when $x = r, v = \dot{r} = u$ say $p = 0$.
(Since pressure vanishes on the internal surface).

(ii) when $x = R, v = \dot{R} = U$ say, $p = 0$.
(Since pressure vanishes on the surface of a annihilated sphere).

(iii) when $r = a, R = A$, the velocity is zero so that $F(t) = 0$.

Subjecting (1) to the conditions (i), and (ii),

$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = \frac{\mu}{r} + C$$

$$\frac{-F'(t)}{R} + \frac{1}{2}U^2 = \frac{\mu}{R} + C$$

Upon subtraction

$$\left\{\frac{1}{R} - \frac{1}{r}\right\} F'(t) + \frac{1}{2}(u^2 - U^2) = \mu\left(\frac{1}{r} - \frac{1}{R}\right)$$

or $$\left(\frac{1}{R} - \frac{1}{r}\right) F'(t) + \frac{1}{2}F^2\left\{\frac{1}{r^4} - \frac{1}{R^4}\right\} = \mu\left\{\frac{1}{r} - \frac{1}{R}\right\}.$$

[For $r^2 u = F(t) = R^2 U$]

Multiplying by $2Fdt$ or equivalently by $2R^2dR = 2r^2\, dr$, we obtain

$$2FF'\left\{\frac{1}{R}-\frac{1}{r}\right\}dt + F^2\left\{\frac{dr}{r^2}-\frac{dR}{R^2}\right\} = 2\mu\,(r\,dr - R\,dR)$$

or
$$d\left[\left\{\frac{1}{R}-\frac{1}{r}\right\}F^2\right] = 2\mu\,(r\,dr - R\,dR).$$

Integrating, $\left(\frac{1}{R}-\frac{1}{r}\right)F^2(t) = \mu\,(r^2 - R^2) + C_1$

Subjecting this to (iii), $0 = \mu\,(a^2 - A^2) + C_1$

$$\therefore \quad \left(\frac{1}{R}-\frac{1}{r}\right)F^2 = \mu\,(r^2 - R^2 - a^2 + A^2). \qquad \ldots (2)$$

The equation of impulsive action is

$$d\bar{\omega} = \rho\, v\, dx = \frac{\rho\, F(t)}{x^2}\, dx$$

This $\Rightarrow$
$$\int_0^{\bar{\omega}} d\bar{\omega} = \int_r^R \rho\,(F)\,\frac{dx}{x^2} \;\Rightarrow\; \bar{\omega} = \left\{\frac{1}{r}-\frac{1}{R}\right\}\rho\, F(t).$$

Putting the values of $F(t)$ from (2) in this equation,

$$\bar{\omega} = \left(\frac{1}{r}-\frac{1}{R}\right)\rho\left[\frac{\mu\,(r^2 - R^2 - a^2 + A^2)}{\left(\frac{1}{R}-\frac{1}{r}\right)}\right]^{1/2}$$

or
$$\bar{\omega} = \rho\left[\mu\,(a^2 - r^2 - A^2 + R^2)\left(\frac{1}{r}-\frac{1}{R}\right)\right]^{1/2}$$

or
$$\bar{\omega} \text{ various as } \left[(a^2 - r^2 - A^2 + R^2)\left(\frac{1}{r}-\frac{1}{R}\right)\right]^{1/2}$$

Problem 16. *A sphere of radius a is surrounded by infinite liquid of density ρ, the pressure at infinity being Π. The sphere is suddenly annihilated. Show that pressure at a distance r from the centre immediately falls to $\Pi\left(1-\frac{a}{r}\right)$.* (Kanpur 1992)

Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius $a/2$, the impulsive pressure sustained by the surface of this sphere is $\left[\frac{7}{6}\Pi\rho a^2\right]^{1/2}$.

Solution : The equation of motion is $x^2 v = F(t)$ so that $\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$. Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or
$$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating w.r.t. x, $\quad -\frac{F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C \qquad \ldots (1)$

Boundary conditions are

(i) when $x = \infty$, $p = \Pi$, $v = 0$.

(ii) when $x = a$, $v = \dot{x} = 0$, $p = 0$, $t = 0$

[Since the sphere of radius a is annihilated and pressure vanishes on the annihilated sphere]

Immediately after annihilation, the liquid has no time to move. So we suppose

(iii) when $t = 0$, $x = r$, $v = 0$, $p = p_0$, where $r > a$.

We want to prove $\quad p_0 = \Pi\left(1-\frac{a}{r}\right).$

Subjecting (1) to (i) and (ii), $\quad 0 = -\frac{\Pi}{\rho} + C$

and
$$-\frac{F'(0)}{a} + 0 = 0 + C.$$

This $\Rightarrow$
$$\frac{-F'(0)}{a} = C = \frac{\Pi}{\rho}. \qquad \ldots (2)$$

In view of (iii), (1) gives $\frac{-F'(0)}{r} = -\frac{p_0}{\rho} + C$

or
$$\frac{a\Pi}{\rho}\cdot\frac{1}{r} = -\frac{p_0}{\rho} + \frac{\Pi}{\rho}, \quad \text{by (2)}$$

or
$$p_0 = \Pi\left(1-\frac{a}{r}\right).$$

Second Part : Let $\bar{\omega}$ be the required impulsive pressure. Then we have to prove that $\bar{\omega} = \left[\frac{7}{6}\Pi\rho a^2\right]^{1/2}$

First we shall determine velocity on the inner surface. Let r be the radius of inner surface. Then

(iv) when $x = r$, $v = \dot{r} = u$ say, $p = 0$ when $r < a$

[Note the difference of (ii) and (iv).]

Since pressure vanishes on the inner surface. In view of the above condition, (1) gives

$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = C = \frac{\Pi}{\rho}$$

or
$$\frac{-F'(t)}{r} + \frac{1}{2}\cdot\frac{F^2}{r^4} = \frac{\Pi}{\rho} \quad \text{as } r^2 u = F(t)$$

Multiplying by $2F(t)\,dt$ or equivalently by $2r^2\,dr$, we obtain

$$\frac{-2F\,F'\,dt}{r} + \frac{F^2\,dr}{r^2} = \frac{\Pi}{\rho}\,2r^2\,dr.$$

or
$$d\left(\frac{-F^2}{r}\right) = \frac{\Pi}{\rho}\,2r^2\,dr.$$

Integrating $\quad \dfrac{-F^2}{r} = \dfrac{2}{3}\dfrac{\Pi}{\rho} \cdot r^3 + C_1$

or $\quad -r^3 u^2 = \dfrac{2}{3}\dfrac{\Pi}{\rho} \cdot r^3 + C_1.$

In view of (ii), this

$\Rightarrow \quad 0 = \dfrac{2}{3}\dfrac{\Pi}{\rho} a^3 + C_1$

$\therefore \quad -r^3 u^2 = \dfrac{2}{3}\dfrac{\Pi}{\rho} \cdot r^3 - \dfrac{2\Pi}{3\rho} a^3$

or $\quad r^3 u^2 = \dfrac{2}{3}\dfrac{\Pi}{\rho}(a^3 - r^3)$. or $u^2 = \dfrac{2}{3}\dfrac{\Pi}{\rho}\left(\dfrac{a^3}{r^3} - 1\right)$

$\therefore \quad [(u^2)_{r=a/2}] = \dfrac{2}{3}\dfrac{\Pi}{\rho} \cdot (8-1)$

or $\quad [(u)_{r=a/2}] = \left[\dfrac{14\Pi}{3\rho}\right]^{1/2}$

Equation of impulsive action is $d\bar{\omega} = \rho\, v\, dx$.

This $\Rightarrow d\bar{\omega} = \rho\, u\, dr \Rightarrow \displaystyle\int_0^{\bar{\omega}} d\bar{\omega} = \rho\,(u)_{r=a/2} \int_0^{a/2} dr$

$\Rightarrow \bar{\omega} = \rho\left[\dfrac{14\Pi}{3\rho}\right]^{1/2} \cdot \dfrac{a}{2} = \left[\dfrac{7}{6}\rho\,\Pi\, a^2\right]^{1/2}.$

Problem 17. *A sphere whose radius at time t is $b + a\cos nt$ is surrounded by liquid extending to infinity under no forces. Prove that the pressure at a distance r from the centre is less than the pressure at an infinite distance by*

$$\frac{na^2}{r}(b + a\cos nt)\left\{a(1 - 3\sin^2 nt) + b\cos nt + \frac{a}{2r^3}(b + a\cos nt)^3 \sin^2 nt\right\}$$

Solution : Let Π be the pressure at infinity and p_0 at a distance r. Then we have to prove that

$$\frac{\Pi - p_0}{\rho} = \frac{na^2}{r}(b + a\cos nt)\left[a(1 - 3\sin^2 nt) + b\cos nt + \frac{a}{2r^3}(b + a\cos nt)^3 \sin^2 nt\right] \quad \ldots (1)$$

Equation of continuity is $x^2 v = F(t)$ so that $\dfrac{\partial v}{\partial t} = \dfrac{F'(t)}{x^2}$.

Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right).$$

Integrating, $$\frac{-F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C \quad \ldots (2)$$

Boundary conditions are

(i) when $x = \infty$, $v = 0$, $p = \Pi$

(ii) when $x = r$, $v = \dot{r} = u$ say, $p = p_0$

Subjecting (2) to (i), $\quad 0 = -\dfrac{\Pi}{\rho} + C$ or $C = \dfrac{\Pi}{\rho}$.

$\therefore \quad \dfrac{-F'(t)}{x} + \dfrac{1}{2}v^2 = \dfrac{\Pi - p}{\rho}$

Subjecting this to (ii),

$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = \frac{\Pi - p_0}{\rho}$$

$$\frac{-F'(t)}{r} + \frac{1}{2}\frac{F^2}{r^4} = \frac{\Pi - p_0}{\rho} \quad \ldots (3)$$

Let R be the radius at any time t. Then

$R = b + a\cos nt$. Also let $U = \dot{R}$. We have

$r^2\dot{r} = r^2 u = F(t) = R^2\dot{R}$, $\dot{R} = -na\sin nt$.

$F(t) = R^2\dot{R} = (b + a\cos nt)^2(-na\sin nt)$

$F'(t) = 2(b + a\cos nt)\, n^2a^2\sin^2 nt - an^2\cos nt\,(b + a\cos nt)^2$.

Putting these in (3), we get

$$\frac{\Pi - p_0}{\rho} = -\frac{1}{r}[2(b + a\cos nt)\, n^2a^2\sin^2 nt - (b + a\cos nt)^2 n^2 a\cos nt]$$
$$+ \frac{1}{2} \cdot \frac{(b + a\cos nt)^4}{r^4} \cdot n^2a^2\sin^2 nt$$
$$= (b + a\cos nt)\frac{n^2 a}{r}\Big[-2a\sin^2 nt + (b + a\cos nt)\cos nt$$
$$+ (b + a\cos nt)^3 \cdot \frac{a}{2r^3}\sin^2 nt\Big]$$
$$= (b + a\cos nt)\frac{n^2 a}{r}\Big[a(1 - 3\sin^2 nt) + b\cos nt$$
$$+ \frac{1}{2} \cdot \frac{a}{r^3}(b + a\cos nt)^3\sin^2 nt\Big].$$

This proves the required result.

Problem 18. *A mass of liquid of density ρ whose external surface is a long circular cylinder of randius a, which is subject to a constant pressure Π, surrounds a coaxial long circular cylinder of radius b. The internal cylinder is suddenly destroyed. Show that if v is the velocity at the internal surface when its radius is r, then*

$$v^2 = \frac{2\Pi(b^2 - r^2)}{\rho\, r^2 \log[(r^2 + a^2 - b^2)/r^2]}$$ *(Garhwal 2000)*

Solution : Equation of continuity is $xu = F(t)$ and equation of motion is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{and} \quad u = \dot{x}.$$

or $$\frac{F'(t)}{x} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating, $\quad F'(t)\log x + \frac{1}{2}u^2 = -\frac{p}{\rho} + C$

or $$F'(t)\log x + \frac{1}{2}\cdot\frac{F^2}{x^2} = -\frac{p}{\rho} + C \qquad \ldots(1)$$

$$ux = F \Rightarrow x\dot{x} = F \Rightarrow x\,dx = F(t)\,dt$$

Let R and r be external and internal radii at any time t. Since total mass of liquid is constant. Hence mass of the liquid at anytime t = mass of the liquid at time $t = 0$.

i.e., $$(\pi R^2 h - \pi r^2 h)\rho = (\pi a^2 h - \pi b^2 h)\rho$$

or $$R^2 - r^2 = a^2 - b^2 \quad \text{or} \quad R^2 = r^2 + a^2 - b^2 \qquad \ldots(2)$$

Boundary conditions are

(i) when $x = R$, $u = \dot{R}$, $p = \Pi$.

[For external boundary is subjected to a constant pressure Π].

(ii) when $x = r$, $u = \dot{r} = v$, $p = 0$.

(For pressure vanishes on the internal boundary).

(iii) when $r = b$, $u = b = 0$, i.e., $F(t) = 0$, $p = 0$.

Subjecting (1) to (i) and (ii),

$$F'(t)\log R + \frac{1}{2}\cdot\frac{F^2}{R^2} = -\frac{\Pi}{\rho} + C$$

$$F'(t)\log r + \frac{1}{2}\cdot\frac{F^2}{r^2} = 0 + C$$

upon subtracting, $(\log R - \log r)F' + \frac{1}{2}F^2\left\{\frac{1}{R^2} - \frac{1}{r^2}\right\} = \frac{-\Pi}{\rho}$. $\qquad \ldots(3)$

Multiplying (3) by $2F\,dt = 2r\,dr = 2R\,dR$,

$$2FF'\,dt\,.(\log R - \log r) + F^2\left\{\frac{dR}{R} - \frac{dr}{r}\right\} = -\frac{\Pi}{\rho}\,.\,2r\,dr$$

or $$d\,[(\log R - \log r)F^2] = -2r\frac{\Pi}{\rho}\,.\,dr$$

Integrating, $$(\log R - \log r)F^2 = -\frac{\Pi}{\rho}r^2 + C_1$$

By (2), this $\Rightarrow \left[\log\frac{(r^2 + a^2 - b^2)^{1/2}}{r}\right]F^2 = -\frac{r^2\Pi}{\rho} + C_1$.

In view of (iii), this $\Rightarrow 0 = -b^2\frac{\Pi}{\rho} + C_1$

$$\left[\log\frac{(r^2 + a^2 - b^2)^{1/2}}{r}\right]F^2 = (b^2 - r^2)\frac{\Pi}{\rho}$$

or $$\left[\log\left(\frac{r^2 + a^2 - b^2}{r^2}\right)\right](rv)^2 = 2(b^2 - r^2)\frac{\Pi}{\rho}$$

or $$v^2 = \frac{2\Pi(b^2 - r^2)}{\rho r^2}\cdot\frac{1}{\log[(r^2 + a^2 - b^2)/r^2]}$$

Alternate method : Equation of continuity is $xu = F(t)$, where $\dot{x} = u$. Let R and r be external and internal radii. Since total mass of the liquid is constant hence

$$(\pi R^2 h - \pi r^2 h)\rho = (\pi a^2 h - \pi b^2 h)\rho$$

or $$R^2 - r^2 = a^2 - b^2 \quad \text{or} \quad R^2 = r^2 + a^2 - b^2. \qquad \ldots(1)$$

$$\text{K.E. of the liquid} = \frac{1}{2}\int_r^R (2\pi x\,dx)\,\rho\,u^2 = \pi\rho\int_r^R x\,dx\,.\frac{F^2}{x^2}$$

$$= \pi\rho F^2\log\left(\frac{R}{r}\right)$$

$$\text{Work done by outer pressure} = \int_a^R -p\,dV = \int_a^R -2\pi x\,dx\,.\,\Pi$$

$$= \pi\Pi(a^2 - R^2) = \pi\Pi(b^2 - r^2), \text{ by (1)}$$

Work done = K.E.

$$\pi\Pi(b^2 - r^2) = \pi\rho F^2\log\left(\frac{R}{r}\right) = \left(\frac{\pi}{2}\right)\rho r^2 v^2\log\left(\frac{R^2}{r^2}\right)$$

or $$v^2 = \frac{2\Pi(b^2 - r^2)}{\rho r^2}\cdot\frac{1}{\log[(r^2 + a^2 - b^2)/r^2]}$$

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Problem 19. *Liquid is contained between two parallel planes; the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated. Prove that if Π be the pressure at the outer surface, the initial pressure at any point of the liquid, distant r from the cenre, is*

$$\Pi\left[\frac{\log r - \log b}{\log a - \log b}\right].$$

(Kanpur 2000, Garhwal 2004)

Solution : The equation of continuity is $xv = F(t)$ and equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{F'(t)}{x} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = \frac{\partial}{\partial x}\left(-\frac{p}{\rho}\right).$$

Integrating $\quad F'(t)\log x + \frac{1}{2}v^2 = -\frac{p}{\rho} + C \qquad \ldots(1)$

Note that initially (i.e., at $t = 0$) the liquid is at rest.

Boundary conditions are

(i) when $x = a$, $v = \dot{x} = 0$, $p = \Pi$, $t = 0$.

[Since the outer surface is subjected to a constant pressure Π].

(ii) when $x = b$, $v = \dot{x} = 0$, $p = 0$, $t = 0$.

[Since pressure vanishes on the surface of annihilated sphere].

(iii) when $x = r$, $t = 0$, $p = p_0$ say.

We have to prove that $\quad p_0 = \Pi\left[\frac{\log r - \log b}{\log a - \log b}\right].$

Subjecting (1) to (i) and (ii),

$$F'(0)\log a = -\frac{\Pi}{\rho} + C$$

and

$$F'(0)\log b = 0 + C.$$

This $\Rightarrow$

$$F'(0)\log a = -\frac{\Pi}{\rho} + F'(0)\log b$$

or

$$F'(0)\log\left(\frac{a}{b}\right) = \left(-\frac{\Pi}{\rho}\right) \qquad \ldots (2)$$

Also, by (2),

$$C = F'(0)\log b = -\frac{\Pi}{\rho}\frac{\log b}{\log(a/b)} \qquad \ldots (3)$$

In view of (iii), (1) gives $F'(0)\log r = -\dfrac{p_0}{\rho} + C$

or

$$-\frac{\Pi}{\rho}\frac{\log r}{\log(a/b)} = -\frac{p_0}{\rho} - \frac{\Pi}{\rho}\frac{\log b}{\log(a/b)}, \quad \text{by (2)}$$

or

$$p_0 = \frac{\Pi(\log r - \log b)}{\log(a/b)} = \Pi\left(\frac{\log r - \log b}{\log a - \log b}\right)$$

Problem 20. *An infinite mass of homogeneous incompressible fluid is at rest subject to a uniform pressure Π and contains a spherical cavity of radius a, filled with a gas at a pressure $m\Pi$; prove that if the inertia of the gas be neglected, and Boyle's law be supposed to hold throughout the ensuring motion, the radius of the sphere will oscillate between the value a and na, where n is determined by the equation*

$$1 + 3m\log n - n^3 = 0.$$

If m be nearly equal to 1, the time of an oscillation will be $2\pi\left(\dfrac{a^2\rho}{3\Pi}\right)^{1/2}$, ρ being the density of the fluid.

Solution : Equation of continuity is $x^2 v = F(t)$ so that $\dot{x} = v$, $\dfrac{\partial v}{\partial t} = \dfrac{F'(t)}{x^2}$.

Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or

$$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = \frac{\partial}{\partial x}\left(-\frac{p}{\rho}\right)$$

Integrating w.r.t. x,

$$-\frac{F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C. \qquad \ldots (1)$$

Boundary conditions are

(i) when $x = \infty$, $p = \Pi$, $v = 0$.

[Since the infinite mass is at rest subjected to a constant pressure Π]

Let r be the radius of cavity at any time t, then $r < a$ and p_1 the pressure there.

Since the gas within cavity obeys Boyle's law

$$P_1V_1 = P_2V_2 = \text{const., i.e., } \frac{4}{3}\pi r^3 p_1 = m\Pi \cdot \frac{4}{3}\pi a^3$$

$$p_1 = m\Pi\frac{a^3}{r^3}. \qquad \ldots (2)$$

(ii) when $x = r$, $p = p_1$, $v = \dot{r} = u$ say.

(iii) when $r = a$, $p = m\Pi$, $v = \dot{r} = 0$.

Subjecting (1) to (i),

$$0 = -\frac{\Pi}{\rho} + C$$

Now (1) becomes

$$-\frac{F'(t)}{x} + \frac{1}{2}\cdot\frac{F^2}{x^4} = \frac{\Pi - p}{\rho}.$$

Multiplying by $2F\,dt = 2x^2\,dx$ [as $x^2\dot{x} = F(t)$], we get

$$-\frac{2FF'}{x}\,dt + \frac{F^2}{x^2}\,dx = \frac{\Pi - p}{\rho}\cdot 2x^2\,dx \qquad \ldots (3)$$

Now we can not integrate this equation w.r.t. x as p is not constant due to the fact that cavity contains gas at varying pressure. So we subject this equation to the condition (ii) and using (2),

$$-\frac{2FF'}{r}\,dt + \frac{F^2}{r^2}\,dr = \frac{1}{\rho}\left(\Pi - m\Pi\frac{a^3}{r^3}\right)2r^2\,dr$$

or

$$d\left(-\frac{F^2}{r}\right) = \frac{2\Pi}{\rho}\left(r^2 - \frac{ma^3}{r}\right)dr$$

Integrating,

$$-\frac{F^2}{r} = \frac{2\Pi}{\rho}\left(\frac{1}{3}r^3 - ma^3\log r\right) + C_2$$

or

$$r^3u^2 = \frac{2\Pi}{\rho}\left(ma^3\log r - \frac{r^3}{3}\right) + C_3.$$

By (iii), this

$$\Rightarrow 0 = \frac{2\Pi}{\rho}\left[ma^3\log a - \frac{a^3}{3}\right] + C_3.$$

Upon subtraction, we get

$$r^3u^2 = \frac{2\Pi}{\rho}\left[ma^3\log\left(\frac{r}{a}\right) - \left(\frac{r^3 - a^3}{3}\right)\right] \qquad \ldots (4)$$

Since radius oscillates between a and na hence we put $r = na$, $v = \dot{r} = n\dot{a} = 0$.

Hence we get

$$0 = \frac{2\Pi}{\rho}\left[ma^3\log n + \frac{1}{3}(a^3 - n^3a^3)\right]$$

or

$$3m\log n + 1 - n^3 = 0$$

or

$$1 + 3m\log n - n^3 = 0. \qquad \text{Hence the result I.}$$

Second part : When $m = 1$ (approximately).

Let $r = a + y$, y being small $u = \dot{r} = \dot{y}$.

Now (4) gives

$$(a+y)^3\dot{y}^2 = \frac{2\Pi}{\rho}\left[a^3\log\left(\frac{a+y}{a}\right)+\frac{1}{3}\{a^3-(a+y)^3\}\right]$$

or

$$\dot{y}^2 = \frac{2\Pi}{3\rho}\left[3\log\left(1+\frac{y}{a}\right)+1-\left(1+\frac{y}{a}\right)^3\right]\left(1+\frac{y}{a}\right)^{-3}$$

Expanding upto second degree terms,

$$\dot{y}^2 = \frac{2\Pi}{3\rho}\left(1-\frac{3y}{a}+\dots\right)\left[3\left(\frac{y}{a}-\frac{y^2}{2a^2}\right)+1-\left(1+\frac{3y}{a}+\frac{3.2}{2}\cdot\frac{y^2}{a^2}\right)\right]$$

$$= \frac{2\Pi}{\rho}\left(1-\frac{3y}{a}+\dots\right)\left(-\frac{9y^2}{2a^2}\right) = \frac{2\Pi}{3\rho}\left(-\frac{9y^2}{2a^2}\right)$$

$$= -\frac{3\Pi}{\rho a^2}y^2.$$

Differentiating w.r.t. t, $2\dot{y}\ddot{y} = -\frac{3\Pi}{\rho a^2}2y\dot{y}$

or $\ddot{y} = -\mu y$ where $\mu = \frac{3\Pi}{\rho a^2}$.

It is the equation for S.H.M.

Hence time period $= T = \frac{2\pi}{\sqrt{\mu}} = 2\pi\left(\frac{a^2\rho}{3\Pi}\right)^{1/2}$.

Problem 21. *A solid sphere of radius a is surrounded by a mass of liquid whose volume is $\frac{4}{3}\pi c^3$, and its centre is attracted by a force μx^3. If the solid sphere be suddenly annihilated, show that velocity of inner surface, when its radius is x, is given by*

$$\dot{x}^2x^2\,[(x^3+c^3)^{1/3}-x] = \left(\frac{2\Pi}{3\rho}+\frac{2\mu c^3}{9}\right)(a^3-x^3)\,(c^3+x^3)^{1/3}$$

where ρ is the density, Π the external pressure and μ the distance.

Solution : The force $F = -\mu x^2$ as μx^2 is a force directed towards the origin, *i.e.*, in the negative direction. Equation of continuity is $x^2 v = F(t)$ so that

$$\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}.$$ Equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x} = -\mu x^2 - \frac{1}{2}\frac{\partial p}{\partial x}$$

$$\frac{F'(t)}{x^2}+v\frac{\partial v}{\partial x} = -\mu x^2 - \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating w.r.t. x, $-\frac{F'(t)}{x}+\frac{1}{2}v^2 = -\frac{\mu x^3}{3}-\frac{p}{\rho}+C$ ... (1)

Let r and R be internal and external radii respectively at any time t. Since the total mass of the liquid is constant hence

$$\left(\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3\right)\rho = \frac{4}{3}\pi c^3\rho$$

or $R^3 - r^3 = c^3$ ... (2)

First we shall prove that

$$\dot{r}^2r^3\,[(r^3+c^3)^{1/3}-r] = \left(\frac{2\Pi}{3\rho}+\frac{2}{9}\mu c^3\right)(a^3-r^3)\,(c^3+r^3)^{1/3} \quad \dots (3)$$

This equation is obtained by putting $r = x$ in the given result. Boundary conditions are

(i) when $x = R$, $v = \dot{R} = U$ say, $p = \Pi$.

(ii) when $x = r$, $v = \dot{r} = u$ say, $p = 0$.

Since pressure vanishes on the surface of inner sphere.

(iii) when $r = a$, $v = 0$ so that $F(t) = 0$.

Here also we have $x^2v = R^2U = r^2u = F(t)$.

Subjecting (1) to (i) and (ii),

$$-\frac{F'(t)}{R}+\frac{1}{2}U^2 = -\frac{\mu}{3}R^3-\frac{\Pi}{\rho}+C$$

$$-\frac{F'(t)}{r}+\frac{1}{2}u^2 = -\frac{\mu}{3}r^3+C$$

Upon subtraction,

$$\left\{\frac{1}{r}-\frac{1}{R}\right\}F'(t)+\frac{1}{2}\cdot F^2\left\{\frac{1}{R^4}-\frac{1}{r^4}\right\} = \frac{\mu}{3}(r^3-R^3)-\frac{\Pi}{\rho}$$

[as $r^2u = F(t) = R^2U$]

or

$$\left\{\frac{1}{r}-\frac{1}{R}\right\}F'(t)+\frac{1}{2}F^2\left\{\frac{1}{R^4}-\frac{1}{r^4}\right\} = -\frac{\mu}{3}c^3-\frac{\Pi}{\rho}.$$

Multiplying by $2F\,dt = 2R^2\,dR = 2r^2\,dr$,

$$\left\{\frac{1}{r}-\frac{1}{R}\right\}2FF'\,dt+F^2\left\{\frac{dR}{R^2}-\frac{dr}{r^2}\right\} = -\frac{\mu c^3}{3}\cdot 2r^2\,dr - 2r^2\,dr\cdot\frac{\Pi}{\rho}$$

or

$$d\left[\left(\frac{1}{r}-\frac{1}{R}\right)F^2\right] = \left(-\frac{\mu c^3}{3}-\frac{\Pi}{\rho}\right)2r^2\,dr.$$

Integrating,

$$\left(\frac{1}{r}-\frac{1}{R}\right)F^2 = \left(-\frac{\mu c^3}{3}-\frac{\Pi}{\rho}\right)\frac{2}{3}r^3+C_2. \quad \dots (4)$$

By (iii), this gives

$$0 = -\frac{\mu c^3}{9}\cdot 2a^3-\frac{2\Pi}{3\rho}\cdot a^3+C_2 \quad \dots (5)$$

(4) − (5) gives

$$\left(\frac{1}{r}-\frac{1}{R}\right)(r^2u)^2 = \frac{2\mu c^3}{9}(a^3-r^3)+\frac{2\Pi}{3\rho}(a^3-r^3)$$

or

$$(R-r)\,r^3u^2 = \left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-r^3)\,R$$

or

$$[(c^3+r^3)^{1/3}-r]\,r^3\dot{r}^2 = \left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-r^3)\,(c^2+r^3)^{1/3}$$

Replacing r by x, we get the required result to be established.

Problem 22. *A mass of liquid of density ρ and volume $\frac{4}{3}\pi c^3$, is in the form of a spherical shell; a constant pressure Π is exerted on the external surface of the shell; there is no pressure on the internal surface and no other forces act on the liquid; initially the liquid is at rest and the internal radius of the shell is 2c; prove that the velocity of the internal surface, when its radius is c, is*

$$\left(\frac{14\Pi}{3\rho}\cdot\frac{2^{1/3}}{2^{1/3}-1}\right)^{1/2}.$$

Solution : Equations of continuity and motion are

$$x^2 v = F(t)$$

and $$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Hence $$\frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating $$\frac{-F'(t)}{x}+\frac{1}{2}v^2=-\frac{p}{\rho}+C. \quad \ldots(1)$$

Let r and R be internal and external radii respectively. Since the total mass of the liquid is constant therefore

$$\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3=\frac{4}{3}\pi c^3 \quad \text{or} \quad R^3-r^3=c^3. \quad \ldots(2)$$

Boundary conditions are

(i) when $x=R$, $v=\dot{R}=U$, $p=\Pi$.

Since external surface is subjected to a constant pressure Π.

(ii) when $x=r$, $v=\dot{r}=u$ say $p=0$

Since there is no pressure on the internal surface.

(iii) when $t=0$ and $r=2c$, $v=0$ so that $F(t)=0$.

For internal radius of the shell is 2c.

We want to prove that

$$(u)_{r=c}=\left[\frac{14\Pi}{3\rho}\cdot\frac{2^{1/3}}{2^{1/3}-1}\right]^{1/2}.$$

Subjecting (1) to the conditions (i) and (ii),

$$\frac{-F'(t)}{R}+\frac{1}{2}U^2=-\frac{\Pi}{\rho}+C$$

$$\frac{-F'(t)}{r}+\frac{1}{2}u^2=0+C$$

upon subtraction, $$F'(t)\left[\frac{1}{r}-\frac{1}{R}\right]+\frac{1}{2}(U^2-u^2)=-\frac{\Pi}{\rho}$$

or $$\left(\frac{1}{r}-\frac{1}{R}\right)F'(t)+\frac{F^2}{2}\left\{\frac{1}{R^4}-\frac{1}{r^4}\right\}=-\frac{\Pi}{\rho}.$$

[For $R^2U=F(t)=r^2u$].

Multiply by $2F\,dt$ or its equivalent $2R^2\,dR=2r^2\,dr$,

$$\left(\frac{1}{r}-\frac{1}{R}\right)2F\,F'\,dt+F^2\left\{\frac{dR}{R^2}-\frac{dr}{r^2}\right\}=-\frac{\Pi}{\rho}\cdot 2r^2\,dr$$

or $$d\left[\left(\frac{1}{r}-\frac{1}{R}\right)F^2\right]=-\frac{\Pi}{\rho}\cdot 2r^2\,dr.$$

Integrating, $$\left(\frac{1}{r}-\frac{1}{R}\right)F^2=-\frac{2\Pi}{3\rho}r^3+C_1. \quad \ldots(3)$$

In view of (iii), $$0=-\frac{2\Pi}{3\rho}\cdot 8c^3+C_1 \quad \ldots(4)$$

(3) – (4) $$\Rightarrow \left(\frac{1}{r}-\frac{1}{R}\right)F^2(t)=-\frac{2\Pi}{3\rho}(r^3-8c^3)$$

or $$\left[\frac{1}{r}-\frac{1}{(c^3+r^3)^{1/3}}\right](r^2u)^2=\frac{2\Pi}{3\rho}(8c^3-r^3), \quad \text{using (2)}$$

Putting $r=c$, $$\left[\frac{1}{c}-\frac{1}{c\,2^{1/3}}\right]c^4(u^2)_{r=c}=\frac{2\Pi}{3\rho}(8c^3-c^3)$$

or $$(u)_{r=c}=\left[\frac{14\Pi}{\rho}\cdot\frac{2^{1/3}}{2^{1/3}-1}\right]^{1/2} \quad \textbf{Proved.}$$

Problem 23. *A mass of gravitating fluid is at rest under its own attraction only, the free surface being a sphere of radius b and the inner surface a rigid concentric shell of radius a. Show that if this shell suddenly disappears, the initial pressure at any point of the fluid at distance r from the centre is*

$$\frac{2}{3}\pi\rho^2(b-r)(r-a)\left(\frac{a+b}{r}+1\right).$$

(Kanpur 1991)

Solution : Let r be the radius of inner surface at any time t. The force F of attraction at a distance x from the centre of the liquid is

$$\frac{4}{3}\pi\rho\frac{\gamma(x^3-r^3)}{x^2}. \quad \left[\text{For } F=\frac{\gamma m_1 m_2}{d^2}\right].$$

Equations of continuity and motion are

$$x^2v=F(t) \text{ and } \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{4}{3}\pi\rho\gamma\frac{(x^3-r^3)}{x^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

[as the force is directed towards the origin]

or $$\frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right)+\frac{4}{3}\pi\rho\gamma\left(x-\frac{r^3}{x^2}\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating, $$\frac{-F'(t)}{x}+\frac{1}{2}v^2=-\frac{4}{3}\pi\rho\gamma\left(\frac{x^2}{2}+\frac{r^3}{x}\right)-\frac{p}{\rho}+C. \quad \ldots(1)$$

Boundary conditions are

(i) when $t=0$, $v=0$, $r=a$, $p=p_0$ are

[For initially the radius of the inner surface is a and also this surface contains gravitating mass and so there will be pressure on it].

(ii) when $t=0$, $x=a$, $v=0$, $p_0=0$.

For pressure vanishes on the annihilated surface.

(iii) when $t = 0, x = b; p_0 = 0, v = 0.$

[Since there exists no outer pressure].

We want to determine the value of initial pressure,

Subjecting (1) to (i), $\dfrac{-F'(0)}{x} = -\dfrac{4}{3}\pi\rho\gamma\left(\dfrac{x^2}{2}+\dfrac{a^3}{x}\right)-\dfrac{p_0}{\rho}+C.$ ... (2)

Subjecting this to (ii) and (iii),

$$\frac{-F'(0)}{a} = -\frac{4}{3}\pi\rho\gamma\left(\frac{a^2}{2}+\frac{a^3}{a}\right)+C \quad \text{... (3)}$$

$$\frac{-F'(0)}{b} = -\frac{4}{3}\pi\rho\gamma\left(\frac{b^2}{2}+\frac{a^3}{b}\right)+C$$

Upon subtraction,

$$\left\{\frac{1}{b}-\frac{1}{a}\right\}F'(0) = -\frac{4}{3}\pi\rho\gamma\left[\frac{a^2-b^2}{2}+a^3\left(\frac{1}{a}-\frac{1}{b}\right)\right]$$

or, $$F'(0) = -\frac{4}{3}\pi\rho\gamma ab\left[\frac{a+b}{2}-\frac{a^2}{b}\right]$$

or $$F'(0) = -\frac{2}{3}\pi\rho\gamma a\,[b(a+b)-2a^2] \quad \text{... (4)}$$

(2) – (3) gives

$$F'(0)\left\{\frac{1}{a}-\frac{1}{x}\right\} = -\frac{4}{3}\pi\rho\gamma\left[\frac{x^2-a^2}{2}+a^3\left\{\frac{1}{x}-\frac{1}{a}\right\}\right]-\frac{p_0}{\rho}$$

or $$p_0 = -\frac{4}{3}\pi\rho^2\gamma(x-a)\left[\frac{x+a}{2}-\frac{a^2}{x}\right]-F'(0)\frac{(x-a)}{xa}\rho$$

$$= -\frac{2}{3}\pi\rho^2\gamma(x-a)\left[2\left(\frac{x+a}{2}-\frac{a^2}{x}\right)+\frac{F'(0)}{xa\,.\,(2/3)\,\pi\rho\gamma}\right]$$

$$= -\frac{2}{3}\pi\rho^2\gamma(x-a)\left[\frac{(x+a)x-2a^2}{x}-\frac{a}{xa}\,.\,\{b(a+b)-2a^2\}\right], \text{ by (4).}$$

$$= -\frac{2}{3}\pi\rho^2\gamma(x-a)\left[\frac{x^2-b^2+ax-ab}{x}\right]$$

$$= \frac{2}{3}\pi\rho^2\gamma(x-a)(b-x)\left[1+\frac{a+b}{x}\right]$$

Replacing x by r we get the required result.

Problem 24. *A volume $\frac{4}{3}\pi c^3$ of gravitating liquid, of density ρ, is initially in the form of a spherical shell of infinitely great radius. If the liquid shell contracts under the influence of its own attraction, there being no external or internal pressure, show that when the radius of the inner spherical surface is x, its velocity will be given by*

$$V^2 = \frac{4\pi\rho\gamma z}{15x^2}\,[2z^4+2z^3x+2a^2x^2-3zx^3-3x^4]$$

where γ is constant of gravitation and $z^3 = x^3 + c^3$.

Solution : Let r be the radius of inner surface at any time t. The force F of attraction at a distance x from the centre of the liquid is

$$\frac{4}{3}\pi\rho\gamma\frac{(x^3-r^3)}{x^2} \qquad \left[\text{For } F = \frac{m_1 m_2}{d^2}\right]$$

Equations of continuity and motion are

$$x^2 v = F(t) \quad \text{and} \quad \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x} = -\frac{4}{3}\pi\rho\gamma\frac{(x^3-r^3)}{x^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{4}{3}\pi\rho\gamma\left(x-\frac{r^3}{x^2}\right)-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating, $$-\frac{F'(t)}{x}+\frac{1}{2}v^2 = -\frac{4}{3}\pi\rho\gamma\left(\frac{x^2}{2}+\frac{r^3}{x}\right)-\frac{p}{\rho}+C \quad \text{... (1)}$$

Let R be the external radius at any time t. Since the total mass of the liquid is constant,

$$\frac{4}{3}\pi R^3\rho-\frac{4}{3}\pi r^3\rho = \frac{4}{3}\pi c^3\rho \quad \text{or} \quad R^3-r^3 = c^3. \quad \text{... (2)}$$

Boundary conditions are

(i) when $x = R, v = \dot{R} = U$ say, $p = 0$.

Since there being no external pressure,

(ii) when $x = r, v = \dot{r} = u$ say $p = 0$.

Subjecting (1) and (i) and (ii),

$$-\frac{F'(t)}{R}+\frac{1}{2}U^2 = -\frac{4}{3}\pi\rho\gamma\left(\frac{R^2}{2}+\frac{r^3}{R}\right)+C$$

$$-\frac{F'(t)}{r}+\frac{1}{2}u^2 = -\frac{4}{3}\pi\rho\gamma\left(\frac{r^2}{2}+\frac{r^3}{r}\right)+C$$

Upon subtracting,

$$\left\{\frac{1}{r}-\frac{1}{R}\right\}F'(t)+\frac{1}{2}(U^2-u^2) = -\frac{4}{3}\pi\rho\gamma\left[\frac{R^2-r^2}{2}+r^3\left(\frac{1}{R}-\frac{1}{r}\right)\right]$$

Since $$r^2u = F = R^2U.$$

Hence,

$$F'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}+\frac{F^2}{2}\left\{\frac{1}{R^4}-\frac{1}{r^4}\right\} = -\frac{4}{3}\pi\rho\gamma\left[\frac{R^2-r^2}{2}+r^3\left(\frac{1}{R}-\frac{1}{r}\right)\right]$$

Multiplying by $2F\,dt = 2r^2\,dr = R^2\,dR$,

$$2FF'\left\{\frac{1}{r}-\frac{1}{R}\right\}+\left\{\frac{dR}{R^2}-\frac{dr}{r^2}\right\}F^2 = -\frac{4}{3}\pi\rho\gamma\,[R^4\,dR-r^4\,dr+r^3\,2R\,dR-2r^4\,dr]$$

or $$d\left[\left(\frac{1}{r}-\frac{1}{R}\right)F^2\right] = -\frac{4}{3}\pi\rho\gamma\,\{(R^4\,dR-r^4\,dr)+2r^3(R\,dR-r\,dr)\}$$

$$= -\frac{4}{3}\pi\rho\gamma\,\{(R^4\,dR-r^4\,dr)+2(R^3-c^3)R\,dR-2r^4\,dr\}.$$

Integrating, we get

$$\left(\frac{1}{r}-\frac{1}{R}\right)F^2=-\frac{4}{3}\pi\rho\gamma\left[\frac{R^5-r^5-2r^5}{5}+\frac{2R^5}{5}-\frac{2c^3R^2}{2}\right],$$

neglecting constant of integration. But $r^2u=F(t)$.

$$\therefore \quad u^2=-\frac{4}{15}\pi\rho\gamma\,[3(R^5-r^5)-5c^3R^2]\cdot\frac{R}{r^3(R-r)}$$

$$=\frac{4}{15}\pi\rho\gamma\cdot\frac{R}{r^3}\left[\frac{3(r^5-R^5)+5R^2(R^3-r^3)}{R-r}\right], \text{ by (2)}$$

$$=\frac{4}{15}\pi\rho\gamma\frac{R}{r^3}\,[2R^4+2R^3r^2+2R^2r^2-3Rr^3-3r^4].$$

Replacing R by z, r by x and u by V, we get the required result.

Problem 25. *A mass of perfectly incompressible fluid of density ρ, is bounded by concentric surfaces. The outer surface is contained by a flexible envelope which exerts continuously a uniform pressure Π and contracts from radius R_1 to radius R_2. The hollow is filled with a gas obeying Boyle's law, its radius contracts c_1 to c_2 and the pressure of the gas is initially p_1. Initially the whole mass is at rest. Prove that, neglecting the mass of the gas, the velocity v of the inner surface when the configuration (R_2, c_2) is reached, is given by*

$$\frac{1}{2}v^2=\frac{c_1^3}{c_2^3}\left[\frac{1}{3}\left(1-\frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho}-\frac{p_1}{\rho}\log\frac{c_1}{c_2}\right]\Big/\left(1-\frac{c_2}{R_2}\right)$$

Solution : The equations of continuity and motion are

$$x^2v=F(t) \quad\text{and}\quad \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Hence $\quad \dfrac{F'(t)}{x^2}+\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}v^2\right)=-\dfrac{\partial}{\partial x}\left(\dfrac{p}{\rho}\right)$

Integrating,

$$-\frac{F'(t)}{x}+\frac{1}{2}v^2=-\frac{p}{\rho}+C$$

or
$$-\frac{F'(t)}{x}+\frac{1}{2}\cdot\frac{F^2}{x^4}=-\frac{p}{\rho}+C \qquad \ldots(1)$$

Let r and R be internal and external radii at any time t. Let P be the pressure at a distance r as cavity contains gas. By Boyle's law,

$$\frac{4}{3}\pi r^3\cdot P=\frac{4}{3}\pi c_1^3 p_1 \quad\text{or}\quad P=\frac{c_1^3}{r^3}\cdot p_1$$

Boundary conditions are

(i) when $x=R$, $v=\dot{R}=U$, say, $p=\Pi$.

For the outer surface exerts a uniform pressure Π.

(ii) when $x=r$, $v=\dot{r}=u$, say, $p=P$.

(iii) when $r=c_1$, $v=0$ so that $F(t)=0$.

Here $r^2u=F(t)=R^2U$ so that $U^2=\dfrac{F^2}{R^4}$, $u^2=\dfrac{F^2}{r^4}$.

Subject (1) to (i) and (ii),

$$-\frac{F'(t)}{R}+\frac{1}{2}\frac{F^2(t)}{R^4}=-\frac{\Pi}{\rho}+C$$

$$-\frac{F'(t)}{r}+\frac{1}{2}\frac{F^2(t)}{r^4}=-\frac{P}{\rho}+C.$$

Upon subtraction,

$$\left(\frac{1}{r}-\frac{1}{R}\right)F'(t)+\left(\frac{1}{R^4}-\frac{1}{r^4}\right)\frac{F^2}{2}=-\frac{\Pi}{\rho}+\frac{c_1^3}{r^3}\cdot\frac{p_1}{\rho}$$

Multiplying by $2F\,dt=2r^2\,dr=2R^2\,dR$, we get

$$2FF'\left\{\frac{1}{r}-\frac{1}{R}\right\}+F^2\left\{\frac{dR}{R^2}-\frac{dr}{r^2}\right\}=\frac{1}{\rho}\left[\frac{c_1^3p_1}{r^3}-\Pi\right]2r^2\,dr$$

or
$$d\left[\left\{\frac{1}{r}-\frac{1}{R}\right\}F^2\right]=\frac{1}{\rho}\left[\frac{c_1^3p_1}{r^3}-\Pi\right]2r^2\,dr.$$

Integrating,

$$\left(\frac{1}{r}-\frac{1}{R}\right)F^2(t)=\frac{2}{\rho}\left[c_1^3p_1\log r-\frac{\Pi}{3}r^3\right]+A$$

In view of (iii), this $\Rightarrow$

$$0=\frac{2}{\rho}\left[c_1^3p_1\log c_1-\frac{\Pi}{3}c_1^3\right]+A$$

Upon subtraction,

$$u^2=\frac{2}{\rho}\cdot\frac{R}{(R-r)\,r^3}\left[c_1^3p_1\log\left(\frac{r}{c_1}\right)-\frac{\Pi}{3}(r^3-c_1^3)\right]$$

For configuration (R_2, c_2), *i.e.*, when $R=R_2$, $r=c_2$, the velocity v is given by

$$\frac{1}{2}v^2=\frac{1}{2}(u^2)_{(R_2,c_2)}=\frac{1}{\rho}\cdot\frac{R_2}{(R_2-c_2)\,c_2^3}\left[c_1^3p_1\log\left(\frac{c_2}{c_1}\right)-\frac{\Pi}{3}(c_2^3-c_1^3)\right]$$

or
$$\frac{1}{2}v^2=\frac{c_1^3}{c_2^3}\left[-\frac{p_1}{\rho}\log\left(\frac{c_1}{c_2}\right)+\frac{\Pi}{3\rho}\left(1-\frac{c_2^3}{c_1^3}\right)\right]\Big/\left(1-\frac{c_2}{R_2}\right).$$

Problem 26. *A sphere of radius a is alone in an unbounded liquid which is at rest at a great distance from the sphere and is subject to no external force. The sphere is forced to vibrate radially keeping its spherical shape, the radius r at any time being given by $r=a+b\cos nt$. Show that if Π is the pressure in the liuid at a great distance from the sphere, the least pressure (assumed positive at the surface of the sphere during the motion) is $\Pi-n^2\rho b\,(a+b)$.*

Solution : Equations of continuity and motion are

$$x^2v=F(t) \quad\text{and}\quad \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

Hence $\dfrac{F'(t)}{x^2}+\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}v^2\right)=-\dfrac{\partial}{\partial x}\left(\dfrac{p}{\rho}\right)$.

Integrating,

$$-\frac{F'(t)}{x}+\frac{1}{2}v^2=-\frac{p}{\rho}+C, \quad \ldots (1)$$

Subjecting it to the boundary conditon,

when $x=\infty$, $v=0$, $p=\Pi$, we get $0=-\dfrac{\Pi}{\rho}+C$.

$$\therefore \quad -\frac{F'(t)}{x}+\frac{1}{2}v^2=\frac{\Pi-p}{\rho}$$

when $x=r$, let $p=p_1$. Then $v=u=\dot{r}$ so that

$$-\frac{F'(t)}{r}+\frac{1}{2}u^2=\frac{\Pi-p_1}{\rho}. \quad \ldots (2)$$

Given $r=a+b\cos nt$.

Hence $\dot{r}=u=-bn\sin nt$

$$F(t)=r^2u=(a+b\cos nt)^2.(-bn\sin nt)$$

$$F'(t)=2(a+b\cos nt)(b^2n^2\sin^2 nt)-bn^2\cos nt\,(a+b\cos nt)^2$$

or $$\frac{F'(t)}{r}=n^2b\,[2b\sin^2 nt-\cos nt\,(a+b\cos nt)].$$

This $\Rightarrow -\dfrac{2F'(t)}{r}+u^2=2n^2b\,[-2b\sin^2 nt+(a+b\cos nt)\cos nt]+b^2n^2\sin^2 nt$

$$=n^2b\,[-3b\sin^2 nt+2b\cos^2 nt+2a\cos nt]$$

Using this in (2),

$$2(p_1-\Pi)=n^2\rho b\,[3b\sin^2 nt-2b\cos^2 nt-2a\cos nt]. \quad \ldots (3)$$

In order that p_1 is least, we must have $t=0$.

$$2(p_1-\Pi)=n^2\rho b\,[-2b-2a]$$

or $$p_1=\Pi-n^2\rho b\,(a+b).$$

Problem 27. *A centre of force attracting inversely as the square of the distance is at the centre of a spherical cavity within an infinite mass of incompressible fluid, the pressure which at an infinite distance is Π, and is such that the work done by this pressure on a unit area through a unit length is one half the work done by the attractive force on a unit volume of the fluid from infinity to the initial boundary of the cavity; prove that the time of filling up the cavity will be ;*

$$\pi a\left(\frac{\rho}{\Pi}\right)^{1/2}\left[2-\left(\frac{3}{2}\right)^{3/2}\right]$$

a being the initial radius of the cavity and ρ the density of the fluid.

Solution : Equation of continuity is $x^2v=F(t)$ and equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{\mu}{x^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

This $\Rightarrow \dfrac{F'(t)}{x^2}+\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}v^2\right)=-\dfrac{\mu}{x^2}-\dfrac{\partial}{\partial x}\left(\dfrac{p}{\rho}\right)$.

Integrating, $$-\frac{F'(t)}{x}+\frac{1}{2}v^2=\frac{\mu}{x}-\frac{p}{\rho}+C. \quad \ldots (1)$$

Boundary conditions are

(i) when $x=\infty$, $v=0$, $p=\Pi$.

Let r be the radius of cavity at any time t. Then

(ii) when $x=r$, $v=\dot{r}=u$ say, $p=0$.

Since pressure vanishes on the surface of cavity.

(iii) when $r=a$, $v=u=0$ so that $F(t)=0$,

Subjecting (1) to (i) and (ii), $0=-\dfrac{\Pi}{\rho}+C$

and $$-\frac{F'(t)}{r}+\frac{1}{2}u^2=\frac{\mu}{r}+C$$

These two equations $\Rightarrow -\dfrac{F'(t)}{r}+\dfrac{1}{2}\dfrac{F^2}{r^4}=\dfrac{\mu}{r}+\dfrac{\Pi}{\rho}$ as $r^2u=F(t)$

Multiply by $2F\,dt=2r^2\,dr$,

$$-\frac{2FF'}{r}dt+\frac{F^2}{r^2}dr=\left(\frac{\mu}{r}+\frac{\Pi}{\rho}\right)2r^2\,dr$$

or $$d\left[-\frac{F^2}{r}\right]=2\mu r\,dr+\frac{\Pi}{\rho}.2r^2\,dr$$

Integration yields, $$-\frac{F^2}{r}=\mu r^2+\frac{2}{3}\frac{\Pi}{\rho}r^3+A \quad \ldots (2)$$

In view of (iii), this $\Rightarrow$ $$0=\mu a^2+\frac{2}{3}\frac{\Pi}{\rho}a^3+A$$

Upon subtraction,

$$-\frac{F^2}{r}=\mu(r^2-a^2)+\frac{2}{3}\frac{\Pi}{\rho}.(r^3-a^3)$$

or $$r^3u^2=\mu(a^2-r^2)+\frac{2}{3}\frac{\Pi}{\rho}(a^3-r^3). \quad \ldots (3)$$

It is given that

Work done by Π on unit area through a unit length

$=\dfrac{1}{2}$. work done by $-\dfrac{\mu}{x^2}$ on a unit volume of fluid from $x=\infty$ to $x=a$.

Hence $$\Pi.1.1=\frac{1}{2}\int_{\infty}^{a}-\frac{\mu}{x^2}x\,dx=\frac{\mu\rho}{2a}.$$

This $\Rightarrow$ $$\mu=2a\frac{\Pi}{\rho}$$

$$\frac{p}{\Pi} = \frac{3r^4 x^2 + (a^3 - 4x^3) r^2 - x^3 (a^3 - x^3)}{3r^4 x^2}$$

Problem 29. *A spehre is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being P. If the radius R of the sphere varies in such a way that $R = a + b \cos nt$, where $b < a$, show that pressure at the surface of the sphere at any time is $P + \frac{bn^2\rho}{4}(b - 4a \cos nt - 5b \cos nt)$.*

Solution : For the sake of convenience we write $P = \Pi$.

Prove as in problem 26 that

$$2(p_1 - \Pi) = n^2 \rho b \,[3b \sin^2 nt - 2b \cos^2 nt - 2a \cos nt].$$

[This is the equation (3) of Problem 26].

$$= n^2 \rho b \left[\frac{b}{2}\{3(1 - \cos 2nt) - 2(1 + \cos 2nt)\} - 2a \cos nt\right]$$

$$= \frac{n^2 \rho b}{2}[b - 4a \cos nt - 5b \cos^2 nt]$$

or
$$p_1 = \Pi + \frac{n^2 \rho b}{4}[b - 4a \cos nt - 5b \cos^2 nt].$$

Problem 30. *A mass of uniform liquid is in the form of a thick spherical shell bounded by concentric spheres of radii a and b ($a < b$). The cavity is filled with gas, the pressure of which varies according to Boyle's law, and is intially equal to atmospheric pressure Π and the mass of which may be neglected. The outer surface of the shell is exposed to atmospheric pressure. Prove that if the system is symmetrically disturbed, so that particle moves along a line joining it to the centre, the time of small oscillation is $2\pi a\left[\rho \cdot \frac{b-a}{3\Pi b}\right]^{1/2}$, where ρ is the density of the liquid.*

Solution : Equation of continuity is $x^2 v = F(t)$ and equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

This
$$\Rightarrow \frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right).$$

Integrating,
$$\frac{-F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C. \quad \ldots (1)$$

Let r and R be internal and external radii of the shell at any time t. Since the shell contains gas hence there will be pressure on the inner surface. Let $p = p_1$ when $x = r$. Since the total mass of the liquid is constant,

$$\left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3\right)\rho = \left(\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3\right)\rho$$

or
$$R^3 - r^3 = b^3 - a^3 \quad \ldots (2)$$

By Boyle's law,
$$\frac{4}{3}\pi r^3 p_1 = \frac{4}{3}\pi a^3 . \Pi$$

[$\because$ the initial pressure of the gas is equal to atmospheric pressure Π].

Boundary conditions are

(i) when $x = R$, $v = \dot{R} = U$ say, $p = \Pi$

(Since the outer surface is exposed to atmospheric pressure Π).

(ii) when $x = r$, $v = \dot{r} = u$ say, $p = p_1 = \frac{a^3 \Pi}{r^3}$.

We want to determine an equation of form $\ddot{x} = -\mu x$.

Subjecting (1) to (i) and (ii),

$$\frac{-F'(t)}{R} + \frac{1}{2}U^2 = -\frac{\Pi}{\rho} + C$$

$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = -\frac{a^3 \Pi}{\rho r^3} + C$$

Upon subtraction,
$$\left\{\frac{1}{r} - \frac{1}{R}\right\}F'(t) + \frac{1}{2}(U^2 - u^2) = \frac{\Pi}{\rho}\left(\frac{a^3}{r^3} - 1\right).$$

For small oscillations, U^2 and u^2 are small quantities and hence neglected.

$$\therefore \quad F'(t) = \frac{\Pi}{\rho}\left(\frac{a^3 - r^3}{r^2}\right) \cdot \frac{R}{R - r}$$

$F(t) = r^2 u \Rightarrow F'(t) = 2ru^2 + r^2\dot{u} = r^2\dot{u}$ as u^2 is neglected

$$\therefore \quad r^2\dot{u} = r^2\ddot{r} = \frac{\Pi}{\rho}\left(\frac{a^3 - r^3}{r^2}\right) \cdot \frac{R}{R - r}.$$

Since the displacement is small, let $r = a + x$, $R = b + x'$. Then

$$(a + x)^2 \ddot{x} = \frac{\Pi}{\rho}\frac{[a^3 - (a + x)^3]}{(a + x)^2} \cdot \frac{b + x'}{b + x' - a - x}$$

or
$$\ddot{x} = \frac{\Pi}{a\rho}\frac{[1 - (1 + x/a)^3](b + x')}{\left(1 + \frac{x}{a}\right)^4 (x' - x + b - a)}$$

$$= \frac{\Pi}{a\rho}\frac{(-3x/a)(b + x')}{(1 + 4x/a)(x' - x + b - a)} \text{ approx.} \quad \ldots (3)$$

$$(2) \Rightarrow (b + x')^3 - (a + x)^3 = b^3 - a^3$$

$$\Rightarrow b^3\left(1 + \frac{3x'}{b}\right) - a^3\left(1 + \frac{3x}{a}\right) = b^3 - a^3$$

$$\Rightarrow x' = \frac{a^2 x}{b^2}.$$

$$N^r \text{ of } (3) = \left(-\frac{3x}{a}\right)(b + x') = -\frac{3x}{a}\left(b + \frac{a^2 x}{b^2}\right) = -\frac{3xb}{a}, \quad x^2 \text{ is neglected.}$$

or
$$\left(-\frac{3x}{a}\right)(b + x') = -\frac{3xb}{a}$$

$$D^r \text{ of } (3) = \left(1 + \frac{4x}{a}\right)(x' - x + b - a) = \left(1 + \frac{4x}{a}\right)\left(\frac{a^2 x}{b^2} - x + b - a\right)$$

Now (3) becomes

$$r^3 u^2 = \frac{2a\,\Pi}{\rho}(a^2 - r^2) + \frac{2}{3}\frac{\Pi}{\rho}(a^3 - r^3)$$

$$\frac{dr}{dt} = -\left(\frac{2\Pi}{3\rho}\right)^{1/2}\left[\frac{3a(a^2 - r^2) + (a^3 - r^3)}{r^3}\right]^{1/2}$$

[Negative sign is taken before the radical sign because r decreases when t increases].

Let T be the required time. Then

$$\int_0^T dt = -\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_a^0 \frac{r^{3/2}\,dr}{[3a(a^2 - r^2) + (a^3 - r^3)]^{1/2}}$$

or

$$T = \left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^a \frac{r^{3/2}\,dr}{[(a - r)(2a + r)^2]^{1/2}}$$

$$= \left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^a \frac{r^{3/2}\,dr}{(a - r)^{1/2}(2a + r)}$$

Put $r = a\sin^2\theta$.

$$T = \left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^{\pi/2} \frac{a^{3/2}\sin^3\theta\,.\,2a\sin\theta\cos\theta\,d\theta}{a^{1/2}\cos\theta\,.\,a(2 + \sin^2\theta)}$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^{\pi/2}\left(\sin^2\theta - 2 + \frac{4}{2 + \sin^2\theta}\right)d\theta$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\left[\frac{\pi}{4} - 2\,.\,\frac{\pi}{2} + 4\int_0^{\pi/2}\frac{\sec^2\theta\,d\theta}{2 + 3\tan^2\theta}\right]$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\left[-\frac{3\pi}{4} + 4\int_0^{\infty}\frac{du}{2 + 3u^2}\right],\quad \tan\theta = u$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\left[-\frac{3\pi}{4} + \frac{4}{3}\,.\,\frac{\pi}{2}\,.\,\sqrt{\frac{3}{2}}\right]$$

$$= \pi a\left(\frac{\rho}{\Pi}\right)^{1/2}\left[2 - \left(\frac{3}{2}\right)^{3/2}\right]$$

For integral $\frac{4}{3}\,.\left[\frac{1}{\sqrt{2/3}}\tan^{-1}\frac{u}{\sqrt{(2/3)}}\right]_0^{\infty} = \frac{4}{3}\,.\,\frac{\pi}{2}\left(\frac{3}{2}\right)^{1/2} = \pi\left(\frac{2}{3}\right)^{1/2}$

Problem 28. *A spherical hollow of radius a initially exists in an infinite fluid subject to a constant pressure at infinity. Show that the pressure at distance r from the centre when the radius of the cavity is x, is to the pressure at infinity as*

$$3x^2r^4 + (a^3 - 4x^3)r^3 - (a^3 - x^3)x^3 : 3x^2r^4.$$

(Garhwal 2001)

Solution : Equation of continuity is $x^2 v = F(t)$ and equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

This $\Rightarrow \dfrac{F'(t)}{x^2} + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}v^2\right) = -\dfrac{\partial}{\partial x}\left(\dfrac{p}{\rho}\right).$

Integrating, $\dfrac{-F'(t)}{x} + \dfrac{1}{2}v^2 = -\dfrac{p}{\rho} + C.$ ... (1)

Let r be the radius of cavity at any time t. Boundary conditions are

(i) when $x = \infty$, $v = 0$, $p = \Pi$.

(ii) when $x = r$, $v = u = \dot{r}$, $p = 0$.

Since pressure vanishes on the surface of cavity.

(iii) when $r = a$, $v = u = 0$ so that $F(t) = 0$.

Subjecting (1) to (i) and (ii),

$$0 = -\frac{\Pi}{\rho} + C \quad\text{and}\quad \frac{-F'(t)}{r} + \frac{1}{2}u^2 = 0 + C.$$

This $\Rightarrow \dfrac{-F'(t)}{r} + \dfrac{1}{2}\dfrac{F^2}{r^4} = \dfrac{\Pi}{\rho}$ ... (2) [as $r^2 u = F(t)$]

Multiply by $2F\,dt$ ($= 2r^2\,dr$), we get

$$\frac{-2FF'\,dt}{r} + \frac{F^2}{r^4}\,.\,r^2\,dr = \frac{\Pi}{\rho}\,.\,2r^2\,dr$$

or

$$d\left[\frac{-F^2}{r}\right] = \frac{2\Pi}{\rho}\,.\,r^2\,dr.$$

Integrating,

$$\frac{-F^2}{r} = \frac{2}{3}\frac{\Pi}{\rho}r^3 + A$$

In view of (iii), this gives $0 = \frac{2}{3}\frac{\Pi}{\rho}a^3 + A.$

$$\frac{-F^2}{r} = \frac{2}{3}\frac{\Pi}{\rho}(r^3 - a^3)$$

or

$$F^2(t) = \frac{2}{3}\frac{\Pi r}{\rho}(a^3 - r^3) \quad \text{... (3)}$$

Using this in (2), we get

$$-\frac{F'(t)}{r} = \frac{\Pi}{\rho} - \frac{1}{2}\frac{F^2}{r^4} = \frac{\Pi}{\rho} - \frac{\Pi}{3\rho r^3}(a^3 - r^3)$$

or

$$F'(t) = \frac{\Pi}{3\rho r^2}(a^3 - 4r^3). \quad \text{... (4)}$$

Writing (1) with the help of (3) and (4),

$$-\frac{\Pi}{3\rho}\,.\,\frac{1}{r^2}\,.\,\frac{(a^3 - 4r^3)}{x} + \frac{1}{2}\,.\,\frac{1}{x^4}\,.\,\frac{2}{3}\frac{\Pi r}{\rho}(a^3 - r^3) = -\frac{p}{\rho} + \frac{\Pi}{\rho}$$

or

$$\frac{p}{\Pi} = 1 + \left(\frac{a^3 - 4r^3}{3r^2x}\right) - \frac{r(a^3 - r^3)}{3x^4}$$

$$= \frac{3x^4r^2 + (a^3 - 4r^3)x^3 - r^3(a^3 - r^3)}{3x^4r^2}$$

This gives the pressure at a distance x, when r is the radius of cavity. In order to get the pressure at a distance r when the radius of cavity is x, we replace r by x and x by r. Thus

$$= \frac{a^2 x}{b^2} - x + b - a + \frac{4xb}{a} - 4x$$

$$= x\left(\frac{a^2}{b^2} - 5 + \frac{4b}{a}\right) + b - a$$

Therefore $\left(-\frac{3x}{a}\right)(b + x) \Big/ \left[\left(1 + \frac{4x}{a}\right)(x - x + b - a)\right]$

$$= -\frac{3xb}{a} \cdot \frac{1}{b - a}\left[1 + \left(\frac{a^2}{b^2} - 5 + \frac{4b}{a}\right)\frac{x}{b - a}\right]^{-1}$$

$$= -\frac{3xb}{a(b - a)},$$

using this in (3)

$$\ddot{x} = -\frac{3xb}{a(b - a)} \cdot \frac{\Pi}{a\rho} = -\mu x \qquad \text{when } \mu = \frac{3b\Pi}{a^2 \rho (b - a)}$$

Time of small oscillation is

$$\frac{2\pi}{\sqrt{\mu}} = 2\pi a \left[\frac{\rho(b - a)}{3b\Pi}\right]^{1/2}$$

Problem 30. *A velocity field is given by*

$$q = \frac{(-iy + jx)}{x^2 + y^2}.$$

Determine whether the flow is irrotational. Calculate the circulation round (a) square with corners at (1, 0), (2, 0), (2, 1), (1, 1); (b) unit circle with centre at the origin.

Solution : $q = \dfrac{-iy + jx}{(x^2 + y^2)} = ui + vj$

(i) To determine the nature of motion

$$\text{Curl } q = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} & 0 \end{vmatrix}$$

$$= i\,(0) + j\,(0) + k\left[\frac{\partial}{\partial x}\left\{\frac{x}{x^2 + y^2}\right\} + \frac{\partial}{\partial y}\left\{\frac{y}{x^2 + y^2}\right\}\right]$$

$$= k\left[\frac{(y^2 - x^2)}{(x^2 + y^2)} + \frac{(x^2 - y^2)}{(x^2 + y^2)^2}\right] = 0$$

Motion is irrotational.

(ii) Let Γ denote circulation. Then

$\Gamma = \int_c q \cdot dr$, where c is closed path.

Applying Stoke's theorem

$$\left[\int_c F \cdot dr = \int_S \text{curl } F \cdot \hat{n}\, dS\right]$$

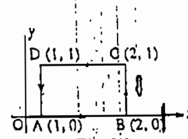


Fig. 2.8

we get $\qquad \Gamma = \int_S \text{curl } q \cdot \hat{n}\, dS. \qquad \ldots (1)$

Hence q must be continuous differentiable over S. In present case q is not continuous at the origin but origin does not lie inside the rectangle so that Stoke's theorem is applicable. By part (i), curl q = 0.

Now (1) gives $\quad \Gamma = 0$ **Ans.**

(b) Equation of path c is $x^2 + y^2 = 1$.

This circle c contains origin, the point of singularity. Hence Stoke's theorem is not applicable.

$$\Gamma = \int_d q \cdot dr = \int_c \left(\frac{-y}{x^3 - y^2}\,dx + \frac{x\,dy}{x^3 + y^3}\right)$$

$$= \int_c (M dx + N dy), \text{ say.} \qquad \ldots (2)$$

$$\frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x}$$

$\therefore$ $M dx + N dy$ is exact.

$$\int (M dx + N dy) = \int \frac{-y}{x^2 + y^2}\,dx + \int 0\,dy = -y \int \frac{dx}{x^2 + y^2}$$

$$= -\frac{y}{y}\tan^{-1}\left(\frac{x}{y}\right) = -\tan^{-1}\left(\frac{x}{y}\right)$$

Now (2) becomes

$$\Gamma = \int_c q \cdot dr = -\left[\tan^{-1}\frac{x}{y}\right]_c = -\left[\tan^{-1}\left(\frac{r\cos\theta}{r\sin\theta}\right)\right]_c$$

$$= -[\tan^{-1}(\cot\theta)]_c = -\left[\tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \theta\right)\right\}\right]_c$$

$$= -\left[\left(\frac{\pi}{2} - \theta\right)\right]_0^{2\pi} = -\left[\left(\frac{\pi}{2} - 2\pi\right) - \left(\frac{\pi}{2} - 0\right)\right]$$

$\Gamma = 2\pi.$ **Ans.**

Problem 31. *Show that if* $\phi = -\frac{1}{2}(ax^2 + by^2 + cz^2)$, $V = \frac{1}{2}(lx^2 + my^2 + nz^2)$, *where a, b, c, l, m, n are functions of time and* $a + b + c = 0$, *irrotational motion is possible with a free surface of equipressure if* $(l + a^2 + \dot{a})\,e^{2\int a\,dt}$, $(m + b^2 + \dot{b})\,e^{2\int b\,dt}$, $(n + c^2 + \dot{c})\,e^{2\int c\,dt}$ *are constants.*

Solution : $\qquad \phi = -\frac{1}{2}(ax^2 + by^2 + cz^2)$

(i) Motion is irrotational if $\nabla^2\phi = 0$

$$0 = \nabla^2\phi = \Sigma \frac{\partial^2 \phi}{\partial x^2} = \Sigma \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = \Sigma \frac{\partial}{\partial x}(-ax)$$

or $\quad \Sigma a = 0$

or $\quad a + b + c = 0$

(ii) Bernoulli's pressure equation for unsteady motion is

$$\frac{p}{\rho} + \frac{1}{2} q^2 - \frac{\partial \phi}{\partial t} + V = F(t) \qquad \ldots (2)$$

$$(1) \Rightarrow \frac{\partial \phi}{\partial t} = -\Sigma \frac{1}{2} \dot{a} x^2 \qquad \ldots (3)$$

$$q^2 = (\nabla\phi) \cdot (\nabla\phi) = (\nabla\phi)^2 = \Sigma \left(\frac{\partial \phi}{\partial x}\right)^2 = \Sigma (ax)^2$$

Putting the values in (2),

$$\frac{p}{\rho} + \frac{1}{2} \Sigma a^2 x^2 + \frac{1}{2} \Sigma \dot{a} x^2 + \frac{1}{2} \Sigma l x^2 = F(t)$$

$$-\frac{2p}{\rho} = \Sigma x^2 (l + a^2) + \Sigma \dot{a} x^2 - 2F(t) \qquad \ldots (3)$$

For a free surface of equipressure :

$$\frac{dp}{dt} = 0$$

or, $\quad \dfrac{\partial p}{\partial t} + u \dfrac{\partial p}{\partial x} + v \dfrac{\partial p}{\partial y} + w \dfrac{\partial p}{\partial z} = 0$

or, $\quad \dfrac{\partial p}{\partial t} + \Sigma u \dfrac{\partial p}{\partial x} = 0 \qquad \ldots (4)$

By (3), $\quad \dfrac{-2}{\rho} \dfrac{\partial p}{\partial t} = \Sigma x^2 (\dot{l} + 2a\dot{a}) + \Sigma \ddot{a} x^2 - 2F'(t)$

or $\quad \dfrac{-2}{\rho} \dfrac{\partial p}{\partial t} = \Sigma x^2 (\dot{l} + 2a\dot{a} + \ddot{a}) - 2F'(t)$

By (3), $\quad \dfrac{-2}{\rho} \dfrac{\partial p}{\partial x} = \Sigma\, 2x (l + a^2) + \Sigma\, 2\dot{a}x$

or $\quad \dfrac{-2}{\rho} \dfrac{\partial p}{\partial x} = 2\Sigma\, [l + a^2 + \dot{a}]\, x$

$$u = -\frac{\partial \phi}{\partial x} = ax$$

Putting these values in (4),

$$\Sigma x^2 (\dot{l} + 2a\dot{a} + \ddot{a}) - 2F'(t) + \Sigma\, 2ax^2 (l + a^2 + \dot{a}) = 0$$

or $\quad \Sigma x^2 [(\dot{l} + 2a\dot{a} + \ddot{a}) + 2a (l + a^2 + \dot{a})] - 2F'(t) = 0$

It is identity. Hence each coefficient of x^2, y^2, z^2 vanishes identically.

$$(\dot{l} + 2a\dot{a} + \ddot{a}) + 2a (l + a^2 + \dot{a}) = 0 \text{ etc.} \qquad \ldots (5)$$

and $\quad F'(t) = 0 \qquad \ldots (6)$

Integrating (6), we get $F(t) = c$ = constant.

By (5), $\quad \displaystyle\int \left(\frac{\dot{l} + 2a\dot{a} + \ddot{a}}{l + a^2 + \dot{a}}\right) dt + \int 2a\, dt = 0$

or $\quad \log (l + a^2 + \dot{a}) + 2\int a\, dt = \log c_1$

or $\quad (l + a^2 + \dot{a})\, e^{2\int a\, dt} = c_1$

Similarly $\quad (m + b^2 + \dot{b})\, e^{2\int b\, dt} = c_2$

$$(n + c^2 + \dot{c})\, e^{2\int c\, dt} = c_3$$

Problem 32. *Fluid is coming out from a small hole of cross-section σ_1 in a tank. If the minimum cross-section of the stream coming out of the hole is σ_2, then show that*

$$\frac{\sigma_2}{\sigma_1} = \frac{1}{2}.$$

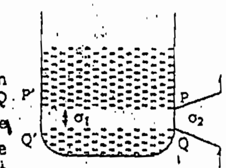


Fig. 2.9

Solution : Let PQ be the hole and $P'Q'$ be its image on the opposite wall of the tank. Let p_1 be the pressure at PQ when the hole is closed. Let p_2 be the pressure and q_2 be the velocity at the minimum cross-section. The velocity of the fluid coming out from minimum cross-section is at right angles to the hole and the direction of velocity will be horizontal there.

Equation of motion is

$$\sigma_1 (p_1 - p_2) = \sigma_2 \rho q_2^2$$

$$\Rightarrow \quad (p_1 - p_2) = \frac{\sigma_2}{\sigma_1} \rho q_2^2 \qquad \ldots (1)$$

Bernoulli's equation for the steam line connecting a point of $P'Q'$ and a point of minimum cross-section of the jet, becomes

$$\frac{p_1}{\rho} = \frac{p_2}{\rho} + \frac{1}{2} q_2^2 \Rightarrow p_1 - p_2 = \frac{1}{2} \rho q_2^2 \qquad \ldots (2)$$

From (1) and (2), we have

$$\frac{\sigma_2}{\sigma_1} = \frac{1}{2}. \qquad \text{Proved.}$$

Problem 33. *A horizontal straight pipe gradually reduces in diameter from 24 in. to 12 in. Determine the total longitudinal thrust exerted on the pipe if the pressure at the larger end is 60 lbf/in^2 and the velocity of the water is 8 ft/sec.*

Solution : Let A_1 and A_2 be the cross-section of the larger and the smaller end. Let q_1 and q_2 be the velocity and p_1 and p_2 be the pressure at the larger and the smaller end of the pipe. From the equation of continuity, we have

$$A_1 q_1 = A_2 q_2$$

or $$\pi(12)^2 q_1 = \pi(6)^2 q_2 \Rightarrow 4q_1 = q_2 \qquad \ldots (1)$$

$$q_1 = 8 \text{ ft/second} = 8 \times 12 \text{ inches/second}$$

$$= 96 \text{ inches/second}$$

By Bernoulli's equation, we have

$$\frac{p_1}{\rho} + \frac{1}{2} q_1^2 = \frac{p_2}{\rho} + \frac{1}{2} q_2^2$$

or $$p_1 - p_2 = \frac{1}{2}\rho(q_2^2 - q_1^2) = \frac{1}{2}\rho \times 15 \times (96)^2. \quad \text{by (1)}. \qquad (2)$$

To longitudinal thrust exerted on the pipe

$$= p_1 A_1 - p_2 A_2$$

$$= \pi(12)^2 p_1 - \pi(6)^2 p_2$$

$$= 36\pi(4p_1 - p_2) \qquad \ldots (3)$$

From (2), we have

$$p_2 = p_1 - \frac{1}{2}\rho \times 15 \times (96)^2$$

From (3) and (4), we have

$$\text{Total thrust} = 36\pi\left(p_1 + \frac{1}{2}\rho \times 15 \times 96 \times 96\right)$$

$$= 36\pi\left(150 - \frac{1}{2} \times \frac{62 \cdot 4 \times 15 \times 96 \times 96}{12 \times 12 \times 12}\right)$$

$$= 36 \times 2640\pi. \qquad \text{Ans.}$$

Problem 34. *Liquid is discharged at the rate of 3·86 ft³/sec from a siphon in the reservoir. The siphon has a diameter of 6 in. Find the elevation z and the fluid pressure at the top of the siphon.*

Solution : Bernoulli's equation for the three points on the same streamline can be written as

$$\frac{q_0^2}{2g} + \frac{p_0}{\rho g} + z_0 = \frac{q_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = \frac{q_2^2}{2g} + \frac{p_2}{\rho g} + z_2$$

Here $q_0 = 0, p_1 = p_2, z = z_0 - z_2$ (let)

$$q_2 = \frac{3 \cdot 86}{\pi\left(\frac{1}{4}\right)^2}$$

$$= \frac{3 \cdot 86 \times 16 \times 7}{22}$$

$$= 19 \cdot 62 \text{ ft./sec.}$$

and $$q_2^2 = 2gz$$

$$\Rightarrow \quad z = \frac{19 \cdot 62 \times 19 \cdot 62}{2 \times 32}$$

$$= 6 \text{ ft. (app.)}$$

Since the velocity at the top is the same as that at the bottom, Bernoulli's equation written between these two levels gives

$$\frac{p_1}{\rho g} = -8 \text{ ft. of liquid.}$$

i.e., below the atmospheric pressure.

Problem 35. *A conical pipe has diameters of 10 cm. and 15 cm. at the two ends. If the velocity at the smaller end is 2m/sec, what is the velocity at the other end and the discharge through the pipe ?*

Solution : Let q_1 and q_2 be the velocity at the smaller and larger end. From continuity equation, we have

$$q_1 A_1 = q_2 A_2.$$

Here $q_2 = 2$ m/sec, $A_1 = (\pi/4)(0 \cdot 1)^2$, $A_2 = (\pi/4)(0 \cdot 15)^2$,

$$q_2 = q_1 \frac{A_1}{A_2}$$

$$= 2\frac{(0 \cdot 1)^2}{(0 \cdot 15)^2}$$

$$= 0 \cdot 89 \text{ m/sec.} \qquad \text{Ans.}$$

Discharge through the pipe

$$Q = q_1 A_1$$

$$= 2\left(\frac{\pi}{4}\right)(0 \cdot 1)^2$$

$$= 0 \cdot 0157 \text{ m}^2\text{/sec.} \qquad \text{Ans.}$$

Problem 36. *A horizontal conical pipe has diameter 25 cm and 40 cm at the two ends. (a) Calculate the pressure at the larger end if the pressure at the smaller end is 5 m. of water and rate of flow is 0·3 m²/sec. (b) Calculate the discharge through the pipe if the manometer connected between the two ends reads 10 cm. of mercury.*

Solution : Let q_1, q_2 be the velocities and p_1, p_2 be the pressure at the larger and smaller ends of the conical pipe. Let Q be the discharge through the pipe, then

$$Q = A_1 q_1$$

$$q_1 = \frac{Q}{A_1} = \frac{0 \cdot 3}{(\pi/4)(0 \cdot 4)^2} = 2 \cdot 38 \text{ m/sec.} \qquad \ldots (1)$$

From the continuity equation, we have

$$A_1 q_1 = A_2 q_2$$

or $$q_2 = \frac{A_1}{A_2} q_1 = \frac{(0 \cdot 4)^2}{(0 \cdot 25)^2} \times 2 \cdot 38 = 6 \cdot 10 \text{ m/sec.} \qquad \ldots (2)$$

(a) Using Bernoulli's equation, we have

$$\frac{p_1}{\rho} + \frac{q_1^2}{2g} = \frac{p_2}{\rho} + \frac{q_2^2}{2g} \qquad (\text{Here } z_1 = z_2)$$

or $$5+\frac{(2.38)^2}{2\times 9.81}=\frac{p_2}{\rho}+\frac{(6.10)^2}{2\times 9.81}$$

or $$\frac{p_2}{\rho}=5+\frac{(2.38)^2-(6.10)^2}{2\times 9.81}$$
$$=3.4 \text{ m} = 0.34 \text{ kg/cm}^2$$

Pressure at the larger end = 0.34 kg/cm². Ans.

(b) From manometer, we have

$$\frac{p_1}{\rho}-\frac{p_2}{\rho}=10\,(13.6-1)=126 \text{ cm.} = 1.26 \text{ m.}$$

From continuity equation, we have

$$A_1q_1=A_2q_2$$

$$\Rightarrow \quad q_2=\frac{A_1}{A_2}q_1=\frac{(0.4)^2}{(0.25)^2}q_1=2.56q_1.$$

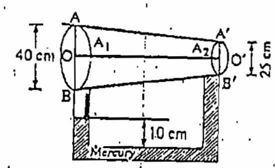


Fig. 2.11

Using Bernoulli's equation, we have

$$\frac{p_1}{\rho}+\frac{q_1^2}{2g}=\frac{p_2}{\rho}+\frac{q_2^2}{2g}$$

or $$\frac{q_1^2}{2g}\left(\frac{q_2^2}{q_1^2}-1\right)=\frac{p_1-p_2}{\rho}$$

$$\Rightarrow \quad \frac{q_1^2}{2g}[(2.56)^2-1]\;1.26$$

$$\therefore \quad q_1=\sqrt{1.26\times 2\times\frac{9.81}{5.55}}=2.11 \text{ m/sec.}$$

Hence discharge through the pipe is

$$Q=A_1q_1$$

$$Q=\frac{\pi}{4}\times(0.4)^2\times 2.11=2.65 \text{ m}^3\text{/sec.}$$ Ans.

Problem 37. *A pipe of 10 cm diameter is suddenly enlarged to 20 cm diameter. Find the loss of head when 50 litre/sec of water is flowing.*

Solution : Let q_1 and q_2 be the velocities at the smaller and larger end of the pipe, then

$$Q=A_1q_1=A_2q_2$$

or $$q_1=\frac{Q}{A_1},\quad q_2=\frac{Q}{A_2}$$

or $$q_1=\frac{0.05}{(\pi/4)\,(0.1)},\quad q_2=\frac{0.05}{(\pi/4)\,(0.2)^2}$$

or $$q_1=6.36 \text{ m/sec.},\quad q_2=1.59 \text{ m/sec.}$$

Loss of head due to sudden enlargement

$$=\frac{(q_1-q_2)^2}{2g}=\frac{(6.36-1.59)^2}{2\times 9.81}=1.16 \text{ m}$$ Ans.

3 SOURCES, SINKS & DOUBLETS (Motion in two Dimensions)

3.1. Motion in two dimensions :

If the lines of motion are parallel to a fixed plane (say, xy plane), and if the velocity at corresponding points of all planes has the same magnitude and direction, then motion is said to be two dimensional. Evidently, in this case $w = 0$ and

$$u = u(x, y, t),\ v = v(x, y, t).$$

In the diagram, a normal is drawn through P which meets $x'y'$. plane in P'. The points P and P' are *corresponding points*.

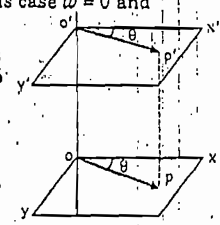


Fig. 3.1.

3.2. Lagrange's stream function :

(*i.e. current function*).

Suppose the motion is two-dimensional so that $w = 0$. The differential equations of stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v},\ i.e.,$$

$$v\,dx - u\,dy = 0\ (= M dx + N dy) \quad \ldots (1)$$

The equation of continuity for incompressible fluid in two dimensions is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{This} \Rightarrow \frac{\partial(-u)}{\partial x} = \frac{\partial v}{\partial y}\ \left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$$

This declares that (1) is an exact differential say $d\psi$ *i.e.*

$$v\,dx - u\,dy = d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy.$$

$$\ldots\ v = \frac{\partial \psi}{\partial x},\ -u = \frac{\partial \psi}{\partial y}.$$

[illegible] $dy = 0$. $\quad \ldots (2)$

[illegible] the stream *function or current function*.

[illegible] by (1), *i.e.*, ψ = const. It follows that stream function

[illegible]

Remark. (1) It is clear that the existence of a stream function is a consequence of stream lines and equation of continuity for incompressible fluid. (2) Stream function exists for all types of two dimensional motion—rotational or irrotational. (3) The necessary conditions for the existence of ψ are

(i) the flow must be continuous

(ii) the flow must be incompressible.

3.3. The difference of the values of ψ at the two points represents the flux of a fluid across any curve joining the two points. *(Kanpur 2005)*

Proof : Suppose ds is a line element at a point $P(x, y)$ of a curve AB. Let the tangent PT make an angle θ with x-axis. Let PN be normal at P, and (u, v) the velocity components of the fluid at P. Direction cosines of the normal PN are

$$\cos(90 + \theta),\ \cos\theta,\ \cos 90,$$

i.e., $\quad -\sin\theta,\ \cos\theta,\ 0.$

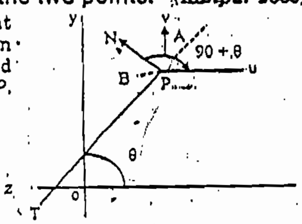


Fig. 3.2.

For PN makes angles $90 + \theta, \theta, 90$ with x, y, z axes respectively

Inward normal velocity $= \hat{n}.q$, in usual notation

$$= u(-\sin\theta) + v(\cos\theta) + (0).0$$

$$= -u\sin\theta + v\cos\theta$$

Flux across the curve AB from right to left

= density. normal velo. area of the cross section

$$= \int_{AB} \rho(\hat{n}.q)\,ds = \int_{AB} \rho(-u\sin\theta + v\cos\theta)\,ds$$

$$= \rho\int_{AB}\left[-u\frac{dy}{ds} + v\frac{dx}{ds}\right]ds \quad \text{as } \tan\theta = \frac{dy}{dx}$$

$$= \rho\int_{AB}\left[\left(\frac{\partial\psi}{\partial y}\right)dy + \left(\frac{\partial\psi}{\partial x}\right)dx\right] = \rho\int_{AB} d\psi = \rho\,[\psi_2 - \psi_1]$$

where ψ_1 and ψ_2 are the values of ψ at A and B respectively.

Flux across AB is $\rho\,[\psi_2 - \psi_1]$.

This proves the required result.

3.4. Irrotational motion in two dimensions :

To show that in two-dimensional irrotational motion, stream function and velocity potential both satisfy Laplace's equation. *(Agra 2001)*

Proof : Let the fluid motion be irrotational so that $\exists$ velocity potential ϕ s.t. $q = -\nabla\phi$, this $\Rightarrow$

$$u = -\frac{\partial\phi}{\partial x},\ v = -\frac{\partial\phi}{\partial y}. \quad \ldots (1)$$

(Here the component w does not exist).

... stream function, then

$$u = -\frac{\partial \psi}{\partial y},\ v = \frac{\partial \psi}{\partial x}. \qquad \dots (2)$$

Step I : From (1) and (2),

$$-\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y},\ -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}.$$

This $\Rightarrow \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0.$$

This $\Rightarrow \nabla^2 \psi = 0$. Hence the result.

Step II : To show that ϕ satisfies Laplace's equation. *(Meerut 1991)*

Solution : We know that

$$u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y}.$$

By equation of continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

i.e.,
$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) = 0$$

or
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 \phi = 0.$$

Hence the result.

Note the following points :

(1) The stream function ψ exists whether the motion is irrotational or not.
(2) The velocity potential ϕ exists only when the motion is irrotational.
(3) When motion is irrotational, ϕ exists.
(4) ϕ and ψ both satisfy Laplace' equation, *i.e.*,

$$\nabla^2 \phi = 0 = \nabla^2 \psi$$

Also
$$\phi_x = \psi_y,\quad \phi_y = -\psi_x$$

i.e.,
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y},\ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Problem 1. *To show that the family of curves $\phi(x, y) = \text{const.}$ and $\psi(x, y) = \text{const}$ cut orthogonally at their point of intersection.*

Or,

***To show** that the curves of constant potential and constant stream functions cut **orthogonally** at their point of intersection.*

Solution. Curve of constant potential is given by

$$\phi = \text{const, this} \Rightarrow d\phi = 0 \Rightarrow \frac{d\phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = 0$$

or
$$-\frac{\phi_x}{\phi_y} = \frac{dy}{dx} = m_1, \text{ say, where } \phi_x = \frac{\partial \phi}{\partial x},\ \phi_y = \frac{\partial \phi}{\partial y}.$$

Here m_1 is the gradient of tangent to the curve ϕ = const.

Similarly, $\psi = \text{const} \Rightarrow -\dfrac{\psi_x}{\psi_y} = \dfrac{dy}{dx} = m_2$, say

Then
$$m_1 m_2 = \left(-\frac{\phi_x}{\phi_y}\right)\left(-\frac{\psi_x}{\psi_y}\right) = \frac{\phi_x \cdot \psi_x}{\phi_y \cdot \psi_y} = \frac{(-u)\,v}{(-v)\,(-u)} = -1$$

or $m_1 m_2 = -1$, this proves the required result.

Problem 2. *If $\phi = A(x^2 - y^2)$ represents a possible flow phenomenon, determine the stream function.*

Solution : Here $\phi = A(x^2 - y^2)$.

Since
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \Rightarrow \frac{\partial \psi}{\partial y} = 2Ax$$
$$\Rightarrow \psi = 2Axy + C,$$

where C is an integration constant, which is the required stream function. *Ans.*

Problem 3. *The velocity potentials $\phi_1 = x^2 - y^2$ and $\phi_2 = \sqrt{r}\cos(\theta/2)$ are solutions of the Laplace equation. Prove that the velocity potential $\phi_3 = (x^2 - y^2) + \sqrt{r}\cos(\theta/2)$ satisfies $\nabla^2 \phi_3 = 0$.*

Solution : Here $\phi_1 = x^2 - y^2$ and $\phi_2 = \sqrt{r}\cos(\theta/2)$

The Laplace's equation in cartesian coordinates and cylindrical polar coordinates is given as

$$\nabla^2 \phi_1 = \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 2 - 2 = 0,$$

and
$$\nabla^2 \phi_2 = \frac{\partial^2 \phi_2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi_2}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi_2}{\partial r}$$

or
$$\nabla^2 \phi_2 = -\frac{1}{4r^{3/2}}\cos(\theta/2) - \frac{1}{4r^{3/2}}\cos(\theta/2) + \frac{1}{2r^{3/2}}\cos(\theta/2) = 0$$

$\Rightarrow$ that ϕ_1 and ϕ_2 satisfy Laplace's equation.

Thus $\quad \nabla^2 \phi_1 = 0,\ \nabla^2 \phi_2 = 0$

Adding $\quad \nabla^2(\phi_1 - \phi_2) = 0$

But $\quad \phi_1 + \phi_2 = \phi_3$

$$\nabla^2 \phi_3 = 0.$$

Problem 4. *Show that $u = 2Axy$, $v = A(a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine the stream function.*

Solution : Here $u = 2Axy$, $v = A(a^2 + x^2 - y^2)$.

If ψ is a stream function, then

$$u = -\frac{\partial \psi}{\partial y},\ v = \frac{\partial \psi}{\partial x}. \quad \ldots (2)$$

Step I : From (1) and (2),

$$-\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y},\ -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}.$$

This $\Rightarrow \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0.$$

This $\Rightarrow \nabla^2 \psi = 0$. Hence the result.

Step II : To show that ϕ satisfies Laplace's equation. *(Meerut 1991)*

Solution : We know that

$$u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y}$$

By equation of continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

i.e., $$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) = 0$$

or $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 \phi = 0.$$

Hence the result.

Note the following points :

(1) The stream function ψ exists whether the motion is irrotational or not.
(2) The velocity potential ϕ exists only when the motion is irrotational.
(3) When motion is irrotational, ϕ exists.
(4) ϕ and ψ both satisfy Laplace' equation, *i.e.,*

$$\nabla^2 \phi = 0 = \nabla^2 \psi$$

Also $\phi_x = \psi_y,\quad \phi_y = -\psi_x$

i.e., $$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y},\ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Problem 1. *To show that the family of curves $\phi(x, y) = const.$ and $\psi(x, y) = const$ cut orthogonally at their point of intersection.*

Or,

***To show** that the curves of constant potential and constant stream functions cut **orthogonally** at their point of intersection.*

Solution. Curve of constant potential is given by

$$\phi = \text{const, this} \Rightarrow d\phi = 0 \Rightarrow \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

or $$-\frac{\phi_x}{\phi_y} = \frac{dy}{dx} = m_1, \text{ say, where } \phi_x = \frac{\partial \phi}{\partial x},\ \phi_y = \frac{\partial \phi}{\partial y}.$$

Here m_1 is the gradient of tangent to the curve ϕ = const.

Similarly, $\psi = \text{const} \Rightarrow -\dfrac{\psi_x}{\psi_y} = \dfrac{dy}{dx} = m_2$, say

$$\text{Then } m_1 m_2 = \left(-\frac{\phi_x}{\phi_y}\right)\left(-\frac{\psi_x}{\psi_y}\right) = \frac{\phi_x \cdot \psi_x}{\phi_y \cdot \psi_y} = \frac{(-u)\,v}{(-v)\,(-u)} = -1,$$

or $m_1 m_2 = -1$, this proves the required result.

Problem 2. *If $\phi = A(x^2 - y^2)$ represents a possible flow phenomenon, determine the stream function.*

Solution : Here $\phi = A(x^2 - y^2)$.

Since $$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \Rightarrow \frac{\partial \psi}{\partial y} = 2Ax$$

$$\Rightarrow \psi = 2Axy + C,$$

where C is an integration constant, which is the required stream function. *Ans.*

Problem 3. *The velocity potentials $\phi_1 = x^2 - y^2$ and $\phi_2 = \sqrt{r}\cos(\theta/2)$ are solutions of the Laplace equation. Prove that the velocity potential $\phi_3 = (x^2 - y^2) + \sqrt{r}\cos(\theta/2)$ satisfies $\nabla^2 \phi_3 = 0$.*

Solution : Here $\phi_1 = x^2 - y^2$ and $\phi_2 = \sqrt{r}\cos(\theta/2)$

The Laplace's equation in cartesian coordinates and cylindrical polar coordinates is given as

$$\nabla^2 \phi_1 = \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 2 - 2 = 0.$$

and $$\nabla^2 \phi_2 = \frac{\partial^2 \phi_2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi_2}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi_2}{\partial r}$$

or $$\nabla^2 \phi_2 = -\frac{1}{4r^{3/2}}\cos(\theta/2) - \frac{1}{4r^{3/2}}\cos(\theta/2) + \frac{1}{2r^{3/2}}\cos(\theta/2) = 0$$

$\Rightarrow$ that ϕ_1 and ϕ_2 satisfy Laplace's equation.

Thus $\nabla^2 \phi_1 = 0,\ \nabla^2 \phi_2 = 0$

Adding $\nabla^2(\phi_1 + \phi_2) = 0$

But $\phi_1 + \phi_2 = \phi_3$

$\therefore$ $\nabla^2 \phi_3 = 0.$

Problem 4. *Show that $u = 2Axy$, $v = A(a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine the stream function.*

Solution : Here $u = 2Axy$, $v = A(a^2 + x^2 - y^2)$.

i.e., This will be a possible fluid motion if it satisfies the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \Rightarrow 2Ay-2Ay=0,$$

which is true. Therefore, the given velocity components constitute a possible fluid motion.

We know that $u=-\dfrac{\partial \psi}{\partial y}$ and $v=\dfrac{\partial \psi}{\partial x}$.

So $\dfrac{\partial \psi}{\partial y}=-2Axy$, and $\dfrac{\partial \psi}{\partial x}=A\,(a^2+x^2-y^2)$. ... (1)

By integrating, we have

$$\psi=-Axy^2+f(x,t)$$ ... (2)

Differentiating (2), we have

$$\frac{\partial \psi}{\partial x}=-Ay^2+\frac{\partial f}{\partial x}$$ ... (3)

From (1) and (3), we have

$$-Ay^2+\frac{\partial f}{\partial x}=A\,(a^2+x^2-y^2)\Rightarrow \frac{\partial f}{\partial x}=A\,(a^2+x^2)$$

By integrating, we have

$$f(x,t)=A\left(a^2x+\frac{1}{3}x^3\right)+g(t).$$

Substituting the value of $f(x,t)$ in (2), we have

$$\psi=A\left(a^2x-xy^2+\frac{1}{3}x^3\right)+g(t),$$

which is the required stream function. Ans.

Problem 5. *Find the stream function ψ for the given velocity potential $\phi=cx$, where c is constant. Also, draw a set of streamlines and equipotential lines.*

Solution : The velocity potential $\phi=cx$ represents fluid flow because it satisfies Laplace equation $\nabla^2\phi=0$.

Since $-\dfrac{\partial \phi}{\partial x}=-c=u$ and $u=-\dfrac{\partial \psi}{\partial y}$.

Therefore $\dfrac{\partial \psi}{\partial y}=c\Rightarrow \psi=cy+f(x).$ ... (1)

Differentiating with regard to x, we have

$$\frac{\partial \psi}{\partial x}=f'(x)$$

But $\dfrac{\partial \psi}{\partial x}=v=-\dfrac{\partial \phi}{\partial y}\Rightarrow \dfrac{\partial \psi}{\partial x}=0$, as $\dfrac{\partial \phi}{\partial y}=0$

$\Rightarrow$ $f'(x)=0, \;\Rightarrow f(x)=$ const.

The stream function ψ is given as

$$\psi=\text{const.}+cy,$$ ... (2)

which represents parallel flow in which stream lines are parallel to X-axis.

The corresponding stream lines and equipotential lines are represented as follows (Fig. 3.3).

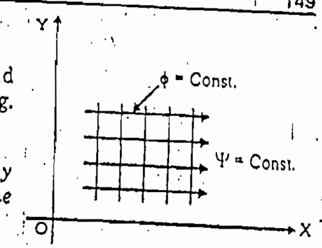


Fig. 3.3.

Problem 6. *A velocity field is given by $\mathbf{q}=-x\mathbf{i}+(y+t)\mathbf{j}$. Find the stream function and the stream lines for this field at $t=2$.*

Solution. Here $\mathbf{q}=u\mathbf{i}+v\mathbf{j}=-x\mathbf{i}+(y+t)\mathbf{j}$

$$\Rightarrow\quad u=-x,\; v=y+t$$

We know that

$$-\frac{\partial \psi}{\partial y}=u=-x \text{ and } \frac{\partial \psi}{\partial x}=v=y+t.$$ ... (1, 2)

By integrating (1) with regard to y, we have

$$\psi=xy+f(x,t),$$ ... (3)

where $f(x,t)$ is an integration constant.

or $$\frac{\partial \psi}{\partial x}=y+\frac{\partial f}{\partial x}$$ ... (4)

From (2) and (4), we have

$$y+\frac{\partial f}{\partial x}=y+t\Rightarrow \frac{\partial f}{\partial x}=t$$

$\Rightarrow$ $$f(x,t)=xt+g(t).$$ ... (5)

From (3) and (5), we have

$$\psi=xy+xt+g(t).$$

At $t=2$, $\psi=x\,(y+2)+g(2).$

The stream lines are given by $\psi=$ const., therefore

$$x\,(y+2)=\text{const.},$$

which represent rectangular hyperbolas. Ans.

Problem 7. *Prove that for the complex potential $\tan^{-1}z$ the stream lines and equi-potentials are circles. Find the velocity at any point and examine the singularities at $z=\pm i$.*

Solution. The complex potential is given by

$$w=\phi+i\psi=\tan^{-1}z,$$ ... (1)

Also $$\overline{w}=\phi-i\psi=\tan^{-1}\overline{z}.$$ ... (2)

By subtracting (1) and (2), we have

$$2i\psi=\tan^{-1}z-\tan^{-1}\overline{z}=\tan^{-1}\frac{z-\overline{z}}{1+z\overline{z}}$$

or $$\tan 2i\psi=\frac{2iy}{1+x^2+y^2}\Rightarrow x^2+y^2+1=2y\coth 2\psi.$$

The stream lines $\psi=$ constant represent the circles

$$x^2+y^2+1=2y\coth 2\psi.$$ ... (3)

Similarly, by adding (1) and (2), we have

$$2\phi = \tan^{-1} z + \tan^{-1} \bar{z} = \tan^{-1} \frac{z + \bar{z}}{1 - z\bar{z}} = \tan^{-1}\left(\frac{2x}{1 - x^2 - y^2}\right)$$

or $\quad 1 - x^2 - y^2 = 2x \cot 2\phi.$ ...(4)

The equi-potentials ϕ = const. also represent circles which are orthogonal to the streamlines ψ = const. and form a co-axial system with limit points at $z = \pm i$. The velocity component (u, v) is given by

$$\frac{dw}{dz} = -u + iv = \frac{1}{z^2+1}\frac{1}{i}, \text{ by (1)} \qquad \text{...(5)}$$

the denominator vanishes at $z = \pm i$, therefore, it represents the singularities at these points.

At $z = +i$, substitute $z = i + z_1$, where $|z_1|$ is very small

$$-u + iv = \frac{dw}{dz} = \frac{dw}{dz_1} = \frac{1}{1 + (-1 + 2iz_1)} = \frac{1}{2iz_1}.$$

by integrating, we have

$$w = -\frac{1}{2} i \log z_1$$

$\Rightarrow$ that the singularity at $z = i$ is a vortex of strength $k = -\frac{1}{2}$ with circulation $-\pi k$.

Similarly, the singularity at $z = -i$ is a vortex of strength $k = \frac{1}{2}$ with cirulation πk.

3.5. Complex Potential.

Suppose ϕ and ψ represent velocity potential and stream function of a two dimensional irrotational motion of a prefect fluid. Let $w = \phi + i\psi$. Then w is defined as complex potential of the fluid motion. Since $\phi = \phi(x, y)$, $\psi = \psi(x, y)$ and so $w = \phi + i\psi$ can be expressed as function of z. Hence $w = f(z) = \phi + i\psi$ where $z = x + iy$.

Also we know that

$$-\frac{\partial \phi}{\partial x} = u = -\frac{\partial \psi}{\partial y}, -\frac{\partial \phi}{\partial y} = v = \frac{\partial \psi}{\partial x}$$

i.e., $$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

which are Cauchy-Riemann equations. Thus Cauchy-Riemann equations are satisfied so that w is analytic function of z.

Conversely, if w is analytic function, then its real and imaginary, *i.e.*, ϕ and ψ give the velocity potential and stream function for a possible two dimensional irrotational fluid motion.

Theorem 1. To prove that any relation of the form $w = f(z)$ where $w = \phi + i\psi$ and $z = x + iy$, represents a two dimensional irrotational motion, in which the magnitude of velocity is given by

$$\left|\frac{dw}{dz}\right|.$$

Proof. $w = \phi + i\psi$, $w = f(z)$.

Differentiating w.r.t. x,

$$\frac{dw}{dz} \cdot \frac{\partial z}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = -u + iv$$

or $$\frac{dw}{dz} = -u + iv \quad \text{as } z = x + iy \Rightarrow \partial z/\partial x = 1.$$

This $\Rightarrow \left|\frac{dw}{dz}\right| = \sqrt{(u^2 + v^2)} =$ magnitude of velocity,

Hence $\left|\frac{dw}{dz}\right|$ represents magnitude of velocity.

Remark : The points, where velocity is zero, are called *stagnation points.*

Thus for stagnation points, $\frac{dw}{dz} = 0.$

3.6. Cauchy-Riemann equations in polar form.

$$w = f(z),\ w = \phi + i\psi, \qquad z = re^{i\theta}$$

Hence $\quad \phi + i\psi = f(re^{i\theta})$

Differentiating w.r.t. r and θ, respectively,

$$\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$$

$$\frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = f'(re^{i\theta}) \cdot ri\, e^{i\theta}$$

Combining these two equations,

$$ri\left[\left(\frac{\partial \phi}{\partial r}\right) + i\frac{\partial \psi}{\partial r}\right] = \frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta}$$

Equating real and imaginary parts,

$$-r\frac{\partial \psi}{\partial r} = \frac{\partial \phi}{\partial \theta}, r\frac{\partial \phi}{\partial r} = \frac{\partial \psi}{\partial \theta}$$

This $\Rightarrow \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$

These two equations are known as polar form of Cauchy-Riemann equations.

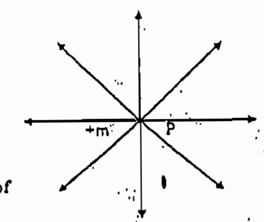


Fig. 3.4.

3.7. Two dimensional sources, sinks.

(i) Sources : *(Garhwal 2004, Kanpur 2002)*

A source (two dimensional simple source) is a point from which liquid is emitted radially and symmetrically in all directions in xy-plane.

(ii) **Sink :** A point to which fluid is flowing in symmetrically and radially in all directions is called sink. This sink is a negative of source.

Difference between source and sink.

Source is a point at which liquid is continuously created and sink is a point at which liquid is continuously annihilated. Really speaking, source and sink are purely abstract conceptions which do not occur in nature.

(iii) **Strength : Strength of a source** is defined as total volume of flow per unit time from it.

Thus, if $2\pi m$ is the total volume of flow across any small circle surrounding the source, then m is called strength of the source. Sink is a source of strength $-m$.

3.8. Complex potential due to a source :

To find the complex potential for a two dimensional source of strength m placed at the origin. *(Garhwal 2002, 2004)*

Proof : Consider a source of strength m at the origin. We are required to determine complex potential w at a point $P(r, \theta)$ due to this source. The velocity at P due to the source is purely radial, let this velocity be q_r. Flux across a circle of radius r surrounding the source at O is $2\pi r q_r$. By definition of strength,

$$2\pi r q_r = 2\pi m, \text{ hence } q_r = m/r.$$

Then $\quad u = q_r \cos\theta = \frac{m}{r}\cos\theta$

$$v = q_r \sin\theta = \frac{m}{r}\sin\theta.$$

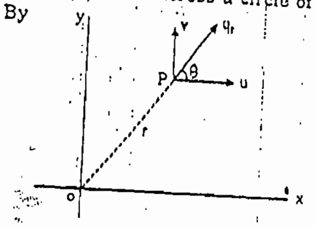


Fig. 3.5.

We know that
$$\frac{dw}{dz} = -u + iv$$
$$= \frac{m}{r}[-\cos\theta + i\sin\theta]$$

or $\quad \frac{dw}{dz} = -\frac{m}{r}e^{-i\theta} = -\frac{m}{re^{i\theta}}$

$$= -\frac{m}{z} \quad \text{or} \quad dw = -m\frac{dz}{z}$$

Integrating, $\quad w = -m \log z \qquad$ ... (1)

(neglecting constant of integration)

(1) is the required expression. Ans.

Deductions : (i) If the source $+m$ is at a point $z = z_1$ in place of $z = 0$, then by shifting the origin, we have
$$w = -m \log(z - z_1).$$

This is the required expression for w in this case.

(ii) *To find the complex potential w at any point z due to sources of strength $m_1, m_2, m_3, \ldots$ situated at $a_1, a_2, a_3, \ldots$.*

Proof : *Step I.* To determine w due to a source $+m$ at the point $z = 0$. (Here prove as in § 3.8 that $w = -m \log z$.

Step II. If a source of strength $+m_1$ is at $z = z_1$, then
$$w = -m_1 \log(z - z_1), \text{ by step I.}$$

The required complex potential is given by
$$w = -m_1 \log(z - a_1) - m_2 \log(z - a_2) - \ldots$$

$$= -\sum_{n=1}^{\infty} m_n \log(z - a_n).$$ Ans.

Hence $\quad \phi = -\sum_{n=1}^{\infty} m_n \log r_n, \quad \psi = -\sum_{n=1}^{\infty} m_n \theta_n$

where $\quad z - a_n = r_n e^{i\theta_n}$.

3.9. Two dimensional doublet. *(Garhwal 2004)*

A doublet is defined as a combination of source $+m$ and sink $-m$ at a small distance δs apart s.t. the product $m\delta s$ is finite. (Sink $-m$ means sink of strength $-m$).

Strength of doublet. If $m\delta s = \mu =$ finite where $m \to \infty$, $\delta s \to 0$, then μ is called strength of the doublet and line δs is called the axis of the doublet and its direction is taken from sink to source.

3.10. Complex potential for a doublet : *(Garhwal 2003, Kanpur 2000, 2005)*

Let a doublet AB of strength μ be formed by a sink $-m$ at A $(z = a)$ and source $+m$ at B $(z = a + \delta a)$.

Then $\quad \mu = m.AB,$

$$\delta a = AB\, e^{i\alpha}, \qquad \ldots (1)$$

[For $z = re^{i\theta}$]

as α is the inclination of the axis of the doublet with x-axis. The complex potential w due to this doublet at any point $P(z)$ is given by
$$w = +m\log(z - a) - m\log[z - (a + \delta a)]$$
$$= -m\log\left(\frac{z - a - \delta a}{z - a}\right)$$
$$= -m\log\left(1 - \frac{\delta a}{z - a}\right)$$
$$= m.\left(\frac{\delta a}{z - a}\right) \text{ upto first approximation,}$$

as $\quad -\log(1 - z) = z + \frac{z^2}{2} + \frac{z^2}{3} + \ldots$

or, $\quad w = \frac{m.AB\, e^{i\alpha}}{z - a}$, by (1) $= \frac{\mu e^{i\alpha}}{z - a}$

$\therefore \quad w = \frac{\mu e^{i\alpha}}{z - a}$ is the required expression.

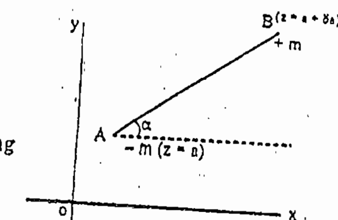


Fig. 3.6.

Deductions (i) If the axis of doublet is along x-axis, then
$$\alpha = 0 \text{ so that } w = \frac{\mu e^{i.0}}{z - a} = \frac{\mu}{z - a}.$$

(ii) If the axis of doublet is along x-axis and the doublet is at the origin, then $\alpha = 0$, $a = 0$ so that $w = \frac{\mu}{z}$.

(Kanpur 1976)

(iii) If a system consists of doublets of strength $\mu_1, \mu_2, \dots$ placed at $z = a_1, a_2, \dots$, then w due to this system is given by

$$w = \sum_{n=1}^{\infty} \frac{\mu_n e^{i\alpha_n}}{z - a_n}$$

where α_n is the inclination of the axis of the doublet of strength μ_n with x-axis.

3.11. Image.

If there exists a curve C in the xy-plane in a fluid s.t. there is no flow across it, then the system of sources, sinks and doublets on one side of C is said to be the images of the sources, sinks and doublets on the other side of C.

Significance of Image

A two dimensional irrotational motion when confined to rigid boundaries is regarded to have been caused by the presence of sources and sinks. If we take the set of sources and sinks (imagining) to be on either side of the rigid boundaries, the velocity normal to these boundaries will be zero. As sch these boundaries can be taken as stream lines. This is due to the property of stream lines that the velocity prependicular to stream lines is zero. This set of sources and sinks on either side is called the image. Thus the motion is no longer constrained by boundaries so that it is possible to predict the nature of the velocity and pressure at each point of the fluid.

3.12. *To find the image of a simple source w.r.t a plane (straight line) and show that the image of a doublet w.r.t. a plane is an equal doublet symmertically placed.*

(Kanpur 2002, 2003, 2004; Meerut 2002)

Proof: (i) *To find the image of a source w.r.t a straight line (plane).* We are to determine the image of a source $+m$ at $A\ (a, 0)$ w.r.t the straight line OY. Place a source $+m$ at $B\ (-a, 0)$. The complex potential at P due to this system is given by

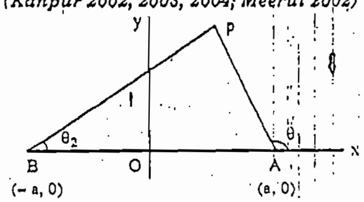


Fig. 3.7.

by $\quad w = -m \log (z - a) - m \log (z + a)$

$\quad = -m \log (z - a)(z + a)$

$\quad = -m \log (r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2})$

$\quad = -m \log [r_1 r_2 e^{i(\theta_1 + \theta_2)}]$ [where $PA = r_1$, $PB = r_2$]

or $\quad \phi + i\psi = -m\ [\log (r_1 r_2) + i(\theta_1 + \theta_2)]$

This $\Rightarrow \psi = -m\ (\theta_1 + \theta_2)$. ... (1)

If P lies on y-axis, then $PA = PB$ so that $\angle PAB = \angle PBA$,

i.e., $\quad \pi - \theta_1 = \theta_2,\ \pi = \theta_1 + \theta_2$. ... (2)

By (1) and (2),

$\quad \psi = -m\pi$ or $\psi =$ const.

It means that y-axis is stream line. Hence the image of a source $+m$ at $A\ (a, 0)$ is a source $+m$ at $B\ (-a, 0)$. That is to say, image of a source w.r.t. a line is

a source of the same strength situated on the opposite side of the line at an equal distance.

(ii) *Image of a doublet w.r.t. a plane.* We are to find the image of the doublet AA' w.r.t. y-axes. Treat the doublet AA' as a combination of source $+m$ at A' and sink $-m$ at A with its axis AA' inclined at an angle α with x-axis. The images of $-m$ at A and $+m$ at A' w.r.t. y-axis are respectively $-m$ at B and $+m$ at B' s.t. $BL = LA$, $B'M = MA'$. Hence the image is a doublet BB' of the same strength with its axis anti-parallel to AA'.

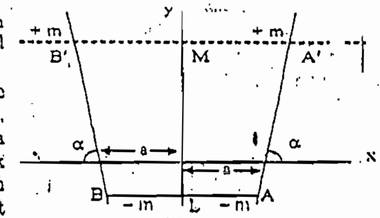


Fig. 3.8.

3.13 Image of a source in a circle.

(Meerut 1992, 91)

We are required to find the image of a source $+m$ at A w.r.t the circle whose centre is O. Let B be the inverse point of A w.r.t the circle. Let P be any current point on the circle at which ψ is to be determined.

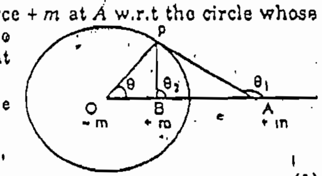


Fig. 3.9.

Place a source $+m$ at B and sink $-m$ at O. The value of ψ due to this system is given by

$\quad \psi = -m\theta_1 - m\theta_2 + m\theta$

or $\quad \psi = -m\ (\theta_1 + \theta_2 - \theta)$. ... (1)

Since B is the inverse point of A,

$\quad OB \cdot OA = (\text{radius})^2 = OP^2$

or $\quad \dfrac{OB}{OP} = \dfrac{OP}{OA}$ also $\angle BOP = \angle POA$.

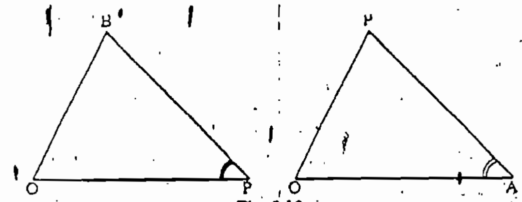


Fig. 3.10.

Hence ΔOPB and ΔOPA are similar. Therefore

$\quad \angle OPB = \angle OAP$, *i.e.*, $\theta_2 - \theta = \pi - \theta_1$ or $\theta_2 + \theta_1 - \theta = \pi$.

Now (1) becomes $\psi = -m\pi$ or $\psi =$ const.

This declares that circle is a stream line so that there exists no flux across the boundary. It means that :

Image of source $+m$ at A is a source $+m$ at B, the inverse point of A and sink $-m$ at the centre.

3.14. Image of a doublet relative to a circle

(Kanpur 2003)

We are required to find the image of the doublet AA' w.r.t the circle whose centre is O. The axis of the doublet is inclined at an angle α with Ox. Let $OA = f$ and μ the strength of the doublet. Treat this doublet as a combination of sink $-m$ at A and source $+m$ at A' so that $\mu = m \,.\, AA'$ where $m \to \infty$, and length of $AA' \to 0$. The image of sink $-m$ at A' at A is a sink $-m$ at B, the inverse point of A and source $+m$ at O. The image of source $+m$ at A' is a source $+m$ at B'; the inverse point of A' and sink $-m$ at O. Compounding these, we find that source $+m$ and sink $-m$ both at O cancel each other and there remains a doublet of strength $\mu = m \,.\, BB'$ at B, the inverse of A.

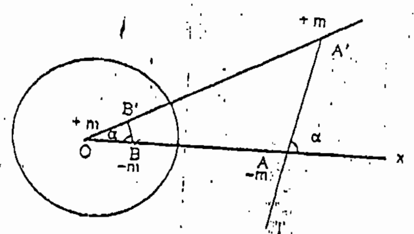


Fig. 3.11.

For $A' \to A \Rightarrow B' \to B$.

Here we have $OB \,.\, OA = a^2 = OB' \,.\, OA'$,

a = radius of the circle.

Hence $\dfrac{OB}{OB'} = \dfrac{OA'}{OA}$. Also $\angle BOB' = \angle A'OA$.

This $\Rightarrow \Delta OBB'$ and $\Delta OAA'$ are similar.

Hence $\dfrac{BB'}{A'A} = \dfrac{OB}{OA'}$

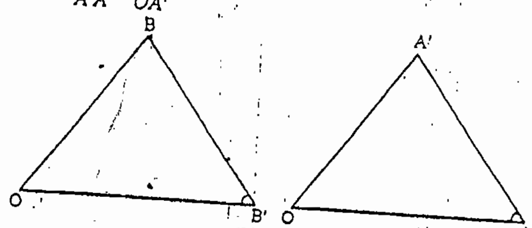


Fig. 3.12.

This $\Rightarrow m \,.\, BB' = m \,.\, AA' \,.\, \dfrac{OB}{OA'}$

$m \,.\, BB' = (m \,.\, AA') \dfrac{OB \,.\, OA}{OA' \,.\, OA}$

Taking limits as $A' \to A$, so that $B' \to B$, we get

$$\mu' = \mu \frac{a^2}{f^2}$$

Also, by similarity of triangles,

$$\angle OBB' = \angle OA'A = \angle OAT \text{ (in limit)} = \alpha.$$

Thus the image of a doublet of strength μ at A (where $OA = f$) relative to a circle is a doublet of strength $\mu' = \mu a^2/f^2$ at B, the inverse point of A, the axis of the doublet makes supplementary angle, *i.e.*, $\pi - \alpha$ with the radius OA.

3.15. Circle Theorem of Milne-Thomson

(Kanpur 2000, Garhwal 2003)

Suppose $f(z)$ is the complex potential of a two dimensional irrotational motion of an incompressible liquid with no rigid boundaries. Then, if a circular cylinder $|z| = a$ is inserted in the flow field, the complex potential of the resulting motion is given by

$$w = f(z) + \bar{f}(a^2/z) \text{ for } |z| \geq a$$

provided $f(z)$ has no singularity inside $|z| = a$.

Proof : Let C be the cross-section of the circular cylinder $|z| = a$. Then on C, $z\bar{z} = a^2$ or $\bar{z} = a^2/z$ so that $f(\bar{z}) = f(a^2/z)$ and so $\bar{f}(\bar{z}) = \bar{f}(a^2/z)$. *(Meerut 1992)*

This $\Rightarrow w = f(z) + \bar{f}(a^2/z) = f(z) + \bar{f}(\bar{z}) = f(z) + f(z)$

$\Rightarrow w$ = real quantity.

Equating imaginary parts on both sides, $\psi = 0$ or ψ = const. for 0 is also a constant. Thus ψ is constant along the boundary C. It means that C is a stream line in the new flow.

We know that if z lies outside C, then the point a^2/z is inside C. Also it is given that $f(z)$ has singularities outside C. Consequently, $f(a^2/z)$ and therefore $\bar{f}(a^2/z)$ has singularities inside C. It means that the additional term $\bar{f}(a^2/z)$ introduces no new singularity outside C. In particular $\bar{f}(a^2/z)$ has no singularity at $z = \infty$ as $f(z)$ has no singularity at $z = 0$.

Since the motion is irrotational and fluid is incompressible, the function $f(z)$ will satisfy Laplace's equation and therefore w will satisfy Laplace's equation for two dimensional irrotational flow of liquid with C inserted as does the function $f(z)$ in the absence of C.

Remark : The Milne-Thomson circle theorem provides a conventional [illegible] for finding the image system of a given two dimensional system which [illegible] a circular boundary. For, if $w = f(z)$ represents the given system in the [illegible] the circular boundary $|z| = a$, then $w = f(a^2/z)$ represents the image [illegible]

3.16. Image of source w.r.t. a circle of radius a. (i.e. alternative [illegible] of 3.13).

Consider a source of strength $+m$ at $z = f$ so that the complex [illegible] this source is

$$f(z) = -m \log (z - f).$$

Let a circular cylinder $|z| = a$ [illegible] the complex potential is given by [illegible]

$$\ldots = -m\log(z-f) - m\log\left(\frac{a^2}{z}-f\right)$$

$$= -m\log(z-f) - m\log\left(\frac{a^2-fz}{z}\right)$$

$$= -m\log(z-f) - m\log\left(\frac{-f}{z}\right)\left(z-\frac{a^2}{f}\right)$$

$$= -m\log(z-f) - m\log\left(z-\frac{a^2}{f}\right) - m\log(-f) + m\log z$$

Ignoring the constant term $-m\log(-f)$, we get

$$w = -m\log(z-f) + m\log z - m\log\left(z-\frac{a^2}{f}\right) \quad \ldots(1)$$

This is the complex potential due to

(i) source $+m$ at $z = f$,

(ii) sink $-m$ at $z = 0$,

(iii) source $+m$ at $z = a^2/f$.

For this complex potential, circle is a stream line and hence the image system for a source $+m$ outside the circle consists of a source $+m$ at the inverse point and sink $-m$ at the origin, the centre of the centre. Since f and a^2/f both are inverse points w.r.t the circle $|z| = a$.

3.17. Alternative method for the image of a doublet relative to a circle.

The complex potential $f(z)$ due to a doublet of strength μ at $z = f$ with its axis inclined at an angle α, is given by

$$f(z) = \frac{\mu e^{i\alpha}}{z-f}.$$

When a circular cylinder $|z| = a$ where $a < f$, is inserted in the flow of motion, then the complex potential is given by

$$w = f(z) + \bar{f}(a^2/z), \text{ by circle theorem}$$

$$= \frac{\mu e^{i\alpha}}{z-f} + \left[\left(\frac{\mu e^{i\alpha}}{(a^2/z)-f}\right)\right] = \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu e^{-i\alpha}}{(a^2/z)-f}$$

$$= \frac{\mu e^{i\alpha}}{z-f} - \frac{\mu z\, e^{i(\pi-\alpha)}}{a^2-fz} = \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu z\, e^{i(\pi-\alpha)}}{f[z-(a^2/f)]}$$

$$= \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu e^{i(\pi-\alpha)}\left(z-\frac{a^2}{f}+\frac{a^2}{f}\right)}{f[z-(a^2/f)]}$$

$$= \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu}{f}e^{i(\pi-\alpha)} + \frac{\mu a^2}{f^2}\cdot\frac{e^{i(\pi-\alpha)}}{\left(z-\frac{a^2}{f}\right)}$$

Ignoring the constant term $\mu e^{i(\pi-\alpha)}/f$, we get

$$w = \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu a^2}{f^2}\cdot\frac{e^{i(\pi-\alpha)}}{\left(z-\frac{a^2}{f}\right)}$$

This is the complex potential due to :

(i) doublet of strength μ at $z = f$ with its axis inclined at an angle α.

(ii) doublet of strength $\mu a^2/f^2$ at $z = a^2/f$, the inverse point of $z = f$, its axis is inclined $\pi - \alpha$.

For this complex potential circle is a stream line and hence the image system for a doublet of strength μ at $z = f$ (outside the circle) is a doublet of strength $\mu' = \mu a^2/f^2$ and its axis inclined at an angle $\pi - \alpha$.

3.18. Blasius Theorem :

In steady two dimensional motion given by the complex potential $w = f(z) = \phi + i\psi$, if the pressure thrusts on the fixed cylinder of any shape are represented by a force (X, Y) and a couple of moment N about the origin of coordinates, then neglecting external forces.

$$X - iY = \frac{i\rho}{2}\int_c \left(\frac{dw}{dz}\right)^2 dz;$$

and $n =$ real part of $\left[-\frac{1}{2}\rho\int_c \left(\frac{dw}{dz}\right)^2 z\, dz\right]$

where ρ is the density and integrals are taken round the contour c of the cylinder.

(Kanpur 2001; Meerut 1991; Agra 2000, 2004)

Proof: Consider an element ds of arc surrounding the point $P(x, y)$ of the fixed cylinder, c dennotes the boundary of the cylinder of any shape and size. Let the tangent at P make an angle θ with X-axis, so that the inward normal at P make angle $90° + \theta$ with X-axis. The thrust pds at P acts along inward normal, its components along x and y axes are respectively

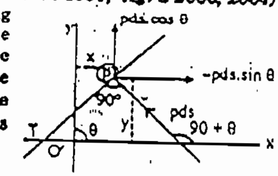


Fig. 3.13.

$pds\cos(90° + \theta)$, $pds\,.\sin(90° + \theta)$

i.e., $-pds\,.\sin\theta$, $pds\,.\cos\theta$.

Hence $\quad X = \int_c -pds\sin\theta,\ Y = \int_c pds.\cos\theta$

This $\quad \Rightarrow X - iY = \int_c p(-\sin\theta - i\cos\theta)\, ds$

$$= -i\int_c p(\cos\theta - i\sin\theta)\, ds.$$

Bernoulli's equation for steady motion gives

$$\frac{p}{\rho} + \frac{1}{2}q^2 = A = \text{const.}$$

or $\quad p = \left(A - \frac{1}{2}q^2\right)\rho. \quad \ldots(1)$

$\therefore \quad X - iY = -i\rho\int_c \left(A - \frac{1}{2}q^2\right)(\cos\theta - i\sin\theta)\, ds$

$$= \frac{i\rho}{2}\int_c q^2 e^{-i\theta} ds - i\rho A \int_c (\cos\theta - i \sin\theta)\, ds$$

But $\frac{dx}{ds} = \cos\theta,\ \frac{dy}{ds} = \sin\theta$ as $\tan\theta = dy/dx$.

$$X - iY = \frac{i\rho}{2}\int_c q^2 e^{-i\theta} ds - i\rho A \int_c (dx - i\,dy).$$

But $\int_c (dx - i\,dy) = \int_c dz = 0$, by Cauchy's theorem.

Hence $$X - iY = \frac{i\rho}{2}\int_c q^2 e^{-i\theta} ds. \qquad \ldots (2)$$

Let u and v be velocity components. Then we know that

$$\frac{dw}{dz} = -u + iv = -q\cos\theta + iq\sin\theta = -q(\cos\theta - i\sin\theta)$$

or $$\frac{dw}{dz} = -qe^{-i\theta}, \text{ or } \left(\frac{dw}{dz}\right)^2 dz = q^2 e^{-2i\theta}\,.\,(dx + i\,dy)$$

or $$\left(\frac{dw}{dz}\right)^2 dz = q^2 e^{-2i\theta}\,.\,(\cos\theta + i\sin\theta)\, ds = q^2 e^{-i\theta} ds. \qquad \ldots (3)$$

Using this in (2) we get the first required result, namely

$$X - iY = \frac{i\rho}{2}\int_c \left(\frac{dw}{dz}\right)^2 dz.$$

We consider anticlockwise moments as positive.

The moment of the thrust $p\,ds$ about the origin is

$$N = \int_c [-(-p\,ds \sin\theta)\, y + (p\,ds \cos\theta)\, x]$$

$$= \int_c p\,(y \sin\theta + x\cos\theta)\, ds = \int_c \left(A - \frac{1}{2}q^2\right)\rho\,(y\sin\theta + x\cos\theta)\, ds$$

$$= A\rho \int_c (y \sin\theta + x\cos\theta)\, ds - \frac{\rho}{2}\int_c q^2 (y\sin\theta + x\cos\theta)\, ds$$

$$= A\rho \int_c (y\,dy + x\,dx) - \frac{\rho}{2}\int_c q^2 (y\sin\theta + x\cos\theta)\, ds$$

But $A\rho \int_c (y\,dy + x\,dy) = A\rho \int_c y\,dy + A\rho\int_c x\,dx = 0 + 0$, by Cauchy's theorem.

Hence $$N = \text{Real part of}\left[-\frac{\rho}{2}\int_c q^2 z e^{-i\theta} ds\right]$$

$$= \text{Real part of}\left[-\frac{\rho}{2}\int_c z\left(\frac{dw}{dz}\right)^2 dz\right], \text{ by (3).}$$

This proves the second required result.

Solved Problems

Problem 1. *A line source is in the presence of an infinite plane on which is placed a semi-circular cylindrical boss, the direction of the source is parallel to the axis of boss, the source is at a distance c from the plane and the axis of boss, whose radius is a. Show that the radius to the point on the boss at which the velocity is a maximum makes an angle θ with the radius to the source, where*

$$\theta = \cos^{-1}\frac{a^2 + c^2}{[2(a^4 + c^4)]^{1/2}}$$

Or

If the axis of y and the circle $x^2 + y^2 = a^2$, are fixed boundaries and there is a two-dimensional source at the point (c, 0), where c > a, show that the radius drawn from the origin to the point on the circle, where the velocity is a maximum, makes with the axis of x an angle

$$\cos^{-1}\left[\frac{a^2 + c^2}{[2(a^4 + c^4)]^{1/2}}\right]$$

When c = 2a, show that the required angle is $\cos^{-1}[5/\sqrt{(34)}]$.

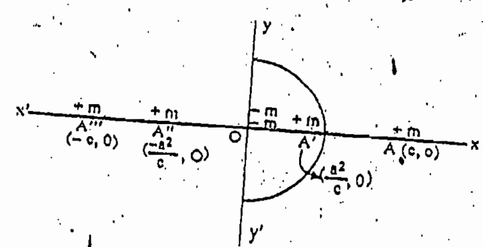


Fig. 3.14.

Solution: The object system consists of source $+m$ at A $(c, 0)$ with semi-circular boundary and parts of y-axis lying outside. Image system consists of (a) (i) source $+m$ at A', the inverse point of A so that $OA' \cdot OA' = a^2$ or $OA' = a^2/c$

(ii) sink $-m$ at O, the centre (origin). It is due to circle.

(b) Above system now gives its own images as

(i) source $+m$ at A'' $(z = -a^2/c)$

(This is the image of A' relative to y-axis)

(ii) source $+m$ at A''' $(z = -c)$

This is the image of A relative to y-axis.

(iii) sink $-m$ at O.

or $$w = -m \log \frac{(z+a)(z-a)}{z} = -m \log \frac{z^2 - a^2}{z}$$

or $$w = -m \log \left(\frac{z^2 - a^2}{z} \right)$$

Second Part : We have $w = -m \log \left(\frac{z^2 - a^2}{z} \right)$

or $$\phi + i\psi = -m \log (r^2 e^{2i\theta} - a^2) + m \log re^{i\theta}.$$

Equating imaginary parts,

$$\psi = -m \tan^{-1} \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - a^2} \right) + m \tan^{-1} \left(\frac{r \sin \theta}{r \cos \theta} \right)$$

$$= -m \left[\tan^{-1} \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - a^2} \right) - \tan^{-1} \left(\frac{\sin \theta}{\cos \theta} \right) \right]$$

$$= -m \tan^{-1} \left[\frac{r^2 (\sin 2\theta \cos \theta - \sin \theta \cos 2\theta) + a^2 \sin \theta}{(r^2 \cos 2\theta - a^2) \cos \theta + r^2 \sin 2\theta \sin \theta} \right]$$

$$\left[\begin{array}{l} \text{For } \tan^{-1} a - \tan^{-1} b = \tan^{-1} \dfrac{a-b}{1+ab} \\ \text{and } \log (x + iy) = \dfrac{1}{2} \log (x^2 + y^2) + i \tan^{-1} \dfrac{y}{x} \end{array} \right]$$

or $$\psi = -m \tan^{-1} \left[\frac{r^2 \sin (2\theta - \theta) + a^2 \sin \theta}{r^2 \cos (2\theta - \theta) - a^2 \cos \theta} \right]$$

or $$\psi = -m \tan^{-1} \left[\frac{(r^2 + a^2) \sin \theta}{(r^2 - a^2) \cos \theta} \right] \quad \ldots (1)$$

$\psi = -m (\pi - \alpha)$ gives the stream lines which make angle α at A. By (1) and (2),

$$-m (\pi - \alpha) = -m \tan^{-1} \left[\frac{(r^2 + a^2) \sin \theta}{(r^2 - a^2) \cos \theta} \right]$$

or $$-\tan \alpha = \frac{(r^2 + a^2) \sin \theta}{(r^2 - a^2) \cos \theta}$$

or $$-\sin \alpha \cdot \cos \theta \cdot (r^2 - a^2) = (r^2 + a^2) \sin \theta \cdot \cos \alpha$$

or $$r^2 \sin (\alpha + \theta) = a^2 \sin (\alpha - \theta).$$

Remark : To justify the image system of the above problem :

Let OA be a bounding radius. Consider a source $+m$ at A, sink $-m$ at O. Take an image source $+m$ at A' s.t.

$OA = OA' = a$. Then complex potential W is given by

$w = -m \log (z - a) + m \log (z - o) - m \log (z - a)$

this gives

$$\psi = -m \tan^{-1} \left[\frac{(r^2 + a^2) \sin \theta}{(r^2 - a^2) \cos \theta} \right] \text{[By equation (1) of the above solution]}$$

or, $$\psi = -m \tan^{-1} \left[\frac{(r^2 + a^2)}{(r^2 - a^2)} \tan \theta \right] \quad \ldots (:$$

By (1), at $r = a$, $\psi = -m\pi/2$ = const.

and at $\theta = \pi/2$, $\psi = -m\pi/2$ = const

Also when $\theta = 0$, $\psi = 0$ = const.

OA is stream line when $\theta = 0$

OB is stream line when $\theta = \pi/2$

and arc AB is stream line when $r = a$

Thus the image system for the fluid motion bounded by quadrantal ar $OABO$ due to sink $-m$ at O, source $+m$ at A would be a source $+m$ at A'.

Problem 3. *Within a circular boundary of radius a there is two dimensional liquic motion due to a source producing liquid at the rate m, at a distance f from the centr and an equal sink at the centre. Find the velocity potential and show that the resultan of the pressure on the boundary is* $\rho m^2 f^3 / [2a^2 \pi (a^2 - f^2)]$. *Deduce as a limit, th velocity potential due to a doublet at the centre.* *(Agra 2000; Kanpur 90)*

Solution : Liquid is generated due to a source at the rate m at the point A wher $OA = f$. Let k be the strength of the source, then by def. $2\pi k = m$ or $k = m/2\pi$. the object system consists of (i) a source $+k$ at A (ii) sink $-k$ at O. The image system consists of (i)' source $+k$ at A', the inverse of A so that $OA' \cdot OA = a^2$ or $OA' = f' = a^2/f$ and a sink $-k$ at O.

(ii)' sink k at infinity, the inverse point O and a source $+k$ at O.

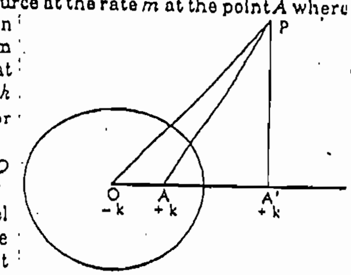


Fig. 3.10.

Source $+k$ and sink $-k$ both at O cancel each other. Finally, the object and its image system consists of source $+k$ at A, source $+k$ at A', sink $-k$ at O. Sink at infinity is neglected, since it has no effect on fluid motion.

The complex potential due to object system with rigid boundary is equivalent to complex potential due to object sysem and its image system with no rigid boundary. Hence w is given by

$$w = -k \log (z - f) - k \log (z - f') + k \log z. \quad \ldots (1)$$

Equating real parts from both sides,

$$\phi = -k \log |z - f| - k \log |z - f'| + k \log |z|$$

$$= -k \log AP - k \log A'P + k \log OP$$

or $$\phi = -k \log \frac{AP \cdot A'P}{OP}.$$

Second Part : By (1), $$\frac{dw}{dz} = -\frac{k}{z - f} - \frac{k}{z - f'} + \frac{k}{z}$$

$$\frac{1}{k^2} \left(\frac{dw}{dz} \right)^2 = \frac{1}{(z - f)^2} + \frac{1}{(z - f')^2} + \frac{1}{z^2} + 2 \left[\frac{1}{(z - f)(z - f')} - \frac{1}{z (z - f)} - \frac{1}{z (z - \ldots} \right.$$

... $-m$ at O relative to y-axis).
... potential due to object system with rigid boundary is equivalent ... and its image system without rigid boundary. Now complex ... by

$$w = -m \log (z - c) - m \log \left(z - \frac{a^2}{c} \right) + 2m \log (z - 0)$$
$$- m \log (z + c) - m \log \left(z + \frac{a^2}{c} \right)$$

$$w = -m \log (z^2 - c^2) - m \log \left(z^2 - \frac{a^4}{c^2} \right) + 2m \log z$$

$$\frac{dw}{dz} = -2m \left[\frac{z}{z^2 - c^2} + \frac{z}{z^2 - \frac{a^4}{c^2}} - \frac{1}{z} \right]$$

or
$$\frac{dw}{dz} = -\frac{2m (z^4 - a^4)}{z (z^2 - c^2) \left(z^2 - \frac{a^4}{c^2} \right)}$$

If q is velocity at $z = ae^{i\theta}$, then

$$q = \left| \frac{dw}{dz} \right| = \frac{2m \mid a^4 e^{i4\theta} - a^4 \mid}{|ae^{i\theta}| \cdot |a^2 e^{i2\theta} - c^2| \cdot \left| a^2 e^{i2\theta} - \frac{a^4}{c^2} \right|}$$

or
$$q = \frac{2mac^2 \mid e^{i4\theta} - 1 \mid}{|a^2 e^{i2\theta} - c^2| \cdot |a^2 e^{i2\theta} - a^2|} \quad \text{...(1)}$$

But
$$| e^{i4\theta} - 1 |^2 = (\cos 4\theta - 1)^2 + \sin^2 4\theta$$
$$= 2 - 2 \cos 4\theta = 4 (\sin 2\theta)^2$$
$$| e^{i4\theta} - 1 | = 2 \sin 2\theta \quad \text{...(2)}$$
$$| c^2 e^{i2\theta} - a^2 |^2 = (c^2 \cos 2\theta - a^2)^2 + (c^2 \sin 2\theta)^2$$
$$= c^4 + a^4 - 2a^2c^2 \cos 2\theta \quad \text{...(3)}$$
$$|a^2 e^{i2\theta} - c^2|^2 = (a^2 \cos 2\theta - c^2)^2 + (a^2 \sin 2\theta)^2$$
$$= a^4 + c^4 - 2a^2c^2 \cos 2\theta \quad \text{...(4)}$$

Writing (1) with the help of (2), (3), (4),

$$q = \frac{4mac^2 \sin 2\theta}{(a^4 + c^4 - 2a^2c^2 \cos 2\theta)} \quad \text{...(5)}$$

q is maximum if $\frac{d}{d\theta} \left[\frac{\sin 2\theta}{a^4 + c^4 - 2a^2c^2 \cos 2\theta} \right] = 0$

This gives,

$$2 \cos 2\theta (a^4 + c^4 - 2a^2c^2 \cos 2\theta) - \sin 2\theta (4a^2c^2 \sin 2\theta) = 0$$
$$2 (a^4 + c^4) \cos 2\theta - 4a^2c^2 = 0$$

$$\cos 2\theta = \frac{2a^2c^2}{a^4 + c^4} \quad \text{...(6)}$$

or
$$\theta = \frac{1}{2} \cos^{-1} \left[\frac{2a^2c^2}{a^4 + c^4} \right]$$

This gives the position of the point where pressure is minimum as :

$$\frac{p}{\rho} + \frac{1}{2} q^2 = c$$

suggests that p is minimum if q is maximum.

By (6), $2 \cos^2 \theta - 1 = \frac{2a^2c^2}{a^4 + c^4}$

or
$$2 \cos^2 \theta = \frac{(a^2 + c^2)^2}{a^4 + c^4} \quad \text{or} \quad \cos^2 \theta = \frac{(a^2 + c^2)^2}{2(a^4 + c^4)}$$

or
$$\cos \theta = \frac{a^2 + c^2}{[2(a^4 + c^4)]^{1/2}} \quad \text{...(7)}$$

If $c = 2a$, $\cos \theta = \frac{(1 + 4) a^2}{[2(17a^4)]^{1/2}} = \frac{5}{\sqrt{(34)}}$

Similar Problem : *In a two dimensional motion of an infinite liquid there is a rigid boundary consisting of that part of the circle $x^2 + y^2 = a^2$ which lies in the first and fourth quadrants and the parts of y-axis which lie outside the circle. A simple source of strength m is placed at the point (f, 0) where $f > a$. Prove that the speed of the fluid at the point $(a \cos \theta, a \sin \theta)$ of the semicircular boundary is $4amf^2 \sin 2\theta / [a^4 + f^4 - 2a^2f^2 \cos 2\theta]$.*

Find at what speed of the boundary the pressure is least.

Hint : Put $c = f$ in the above problem and refer equations (5) and (7).

Problem 2. *A region is bounded by a fixed quadrantal arc and its radii, with a source and an equal sink at the ends of one of the bounding radii. Whow that the motion is given by*

$$w = -m \log \left(\frac{z^2 - a^2}{z} \right),$$

and prove that the stream line leaving either the source or the sink at an angle α with the radius is $r^2 \sin (\alpha + \theta) = a^2 \sin (\alpha - \theta)$.

Solution : The object system and its image system consists of (i) source $+m$ at A $(z = a)$, (ii) sink $-m$ at $z = 0$, (iii) source $+m$ at A' $(z = -a)$.

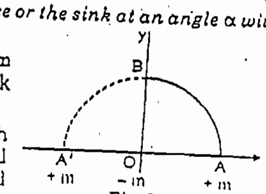


Fig. 3.16.

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to object system and its image system with no rigid boundary, hence complex potential is given by

$$w = -m \log (z + a) + m \log (z - 0) - m \log (z - a)$$

The poles inside the boundary c of the circle are $z = 0$ and $z = f$. Hence the sum of the residues of the function

$$\frac{1}{k^2}\left(\frac{dw}{dz}\right) \text{ at } z = 0$$

and $z = f$ is obtained by adding the coefficients of $\frac{1}{z}$ and $\frac{1}{z-f}$

Sum of residues $= \frac{2}{f-f'} - \frac{2}{f'} + \frac{2}{f} + \frac{2}{f} + \frac{2}{f'} = \frac{2f}{(f-f')f'}$

By Cauchy's residues theorem,

$$\int_c \frac{1}{k^2}\left(\frac{dw}{dz}\right)^2 dz = 2\pi i \text{ [Sum of residues]}$$

or
$$\int_c \left(\frac{dw}{dz}\right)^2 dz = \frac{4\pi f k^2 i}{(f-f')f'}.$$

By Blausius theorem,

$$X - iY = \frac{i\rho}{2}\int_c \left(\frac{dw}{dz}\right)^2 dz = \frac{i\rho}{2} \cdot \frac{4\pi f k^2 i}{(f-f')f'}$$

$$= \frac{2\pi\rho f k^2}{\frac{a^2}{f}\left(\frac{a^2}{f} - f\right)} = \frac{2\pi\rho f^3}{a^2(a^2 - f^2)} \cdot \frac{m^2}{4\pi^2}$$

or
$$X - iY = \rho f^3 m^2/2a^2\pi(a^2 - f^2).$$

Equating real and imaginary parts, we get

$$X = \rho f^3 m^2/2\pi a^2 (a^2 - f^2),\ Y = 0.$$

Resultant pressure on the boundary

$$= (x^2 + Y^2)^{1/2} = \rho m^2 f^3/2\pi a^2 (a^2 - f^2).$$

Third Part: To deduce velocity potential due to a doublet at O as a limit.

If we take limit as $f \to \infty$, then $A \to \infty$ and hence neglected. Also A' comes near the point O. We have already a sink $-k$ at O and we have brought a source near it. This combination forms a doublet of strength μ where $\mu = k.\ (a^2/f)$ as $f \to \infty$.

Now w becomes

$$w = -k \log(z - f') + k \log z \text{ as source } + k \text{ at } A \text{ is neglected}.$$

or
$$w = -k \log \frac{1}{z}\left(z - \frac{a^2}{f}\right) = -k \log\left(1 - \frac{a^2}{fz}\right)$$

$$= k\left[\frac{a^2}{fz} + \frac{1}{2}\left(\frac{a^2}{fz}\right)^2 + \ldots\right] \text{ For } -\log(1-x) = x + \frac{x^2}{2} + \ldots$$

$$= \frac{ka^2}{fz} \text{ neglecting higher degree terms}$$

or
$$\phi + i\psi = \frac{ma^2}{2\pi f r e^{i\theta}} = \frac{ma^2 e^{-i\theta}}{2\pi r f}$$

This $\Rightarrow \phi = \frac{ma^2}{2\pi fr}\cos\theta.$ **Ans.**

This is the required velocity potential.

Remark. By (1),

or
$$w = -k\log\left(1 - \frac{f}{z}\right) - k\log\left(z - \frac{a^2}{f}\right)$$

$$= -k\log\left(1 - \frac{f}{z}\right) - k\log\left(-\frac{f}{a^2}\right)\left(1 - \frac{fz}{a^2}\right)$$

$$= -k\log\left(1 - \frac{f}{z}\right) - k\log\left(1 - \frac{fz}{a^2}\right), \text{ neglecting constant.}$$

$$= k\left[-\log\left(1 - \frac{f}{z}\right) - \log\left(1 - \frac{fz}{a^2}\right)\right]$$

$$= k\left[\left(\frac{f}{z} + \ldots\right) + \left(\frac{fz}{a^2} + \ldots\right)\right]$$

or
$$w = k\left[\frac{f}{z} + \frac{fz}{a^2}\right]$$

If we make $f \to 0$ so that $\frac{a^2}{f} \to \infty$, then we get a doublet at the centre and its strength $\mu = kf$. Then $w = \frac{\mu}{z} + \frac{\mu z}{a^2}$.

Equating real parts, $\phi = \mu\left(\frac{1}{r} + \frac{r}{a^2}\right)\cos\theta.$ **Ans.**

Thus we get two answers for the two limits namely $f \to 0$ and $f \to \infty$.

Problem 4. *A source of fluid situated in space of two dimensions is of such strength that $2\pi\rho\mu$ represents the mass of fluid of density ρ emitted per unit of time. Show that the force necessary to hold a circular disc at rest in the plane of source is*

$$2\pi\rho\mu^2 a^2/r(r^2 - a^2),$$

where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure? *(Kanpur 2006)*

Solution: Let X and Y be the components of the required force. Then we have to prove that

$$\sqrt{(X^2 + Y^2)} = \frac{2\pi\rho\mu^2 a^2}{r(r^2 - a^2)},$$

This $\Rightarrow r > a$. By Blausius theorem,

$$X - iY = \frac{i\rho}{2}\int_c \left(\frac{dw}{dz}\right)^2 dz,$$

where c represents the boundary of the disc. Since $2\pi\rho\mu$ represents the mass of the fluid emitted at A hence strength of the

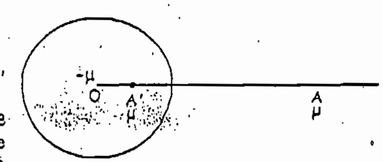


Fig. 3.17.

sources is μ. The image of source $+\mu$ at A $(OA = r)$ is a source $+\mu$ at the inverse point A' s.t. $OA . OA' = a^2$ and sink $-\mu$ at O.

Then $\qquad OA' = a^2/r = r'$, say.

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to the object system and its image system with no rigid boundary. Hence

$$w = -\mu \log (z - r) - \mu \log (z - r') + \mu \log (z - 0)$$

$$\frac{dw}{dz} = -\mu \left[\frac{1}{z-r} + \frac{1}{z-r'} - \frac{1}{z} \right]$$

$$\therefore \frac{1}{\mu^2} \left(\frac{dw}{dz} \right)^2 = \frac{1}{(z-r)^2} + \frac{1}{(z-r')^2} + \frac{1}{z^2} + \frac{2}{(z-r)(z-r')} - \frac{2}{z(z-r')} - \frac{2}{z(z-r)}.$$

The function $\frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2$ has poles $z = 0$ and $z = r'$ within c. Residue at $z = 0$ is the sum of coefficients of $\frac{1}{z}$ which is equal to

$$\left[-\frac{2}{z-r'} - \frac{2}{z-r} \right]_{z=0} = 2\left(\frac{1}{r'} + \frac{1}{r}\right).$$

Residue at $z = r'$ = sum of coefficients of $1/(z - r')$

$$= \left[\frac{2}{z-r} - \frac{2}{z} \right]_{z=r'} = \frac{2}{r'-r} - \frac{2}{r'}.$$

Sum of residues at $z = 0$ and $z = r'$

$$= \frac{2}{r'-r} - \frac{2}{r'} + \frac{2}{r'} + \frac{2}{r} = \frac{2r}{(r'-r)\,r} = \frac{2a^2}{(a^2 - r^2)\,r}.$$

By Cauchy's residues theorem,

$$\int_c \frac{1}{\mu^2} \left(\frac{dw}{dz} \right)^2 dz = 2\pi i. \text{ Sum of residues within } C$$

$$= 2\pi i. \frac{2a^2}{(a^2 - r^2)\,r}.$$

We have seen that

$$X - iY = \frac{i\rho}{2} \int_c \left(\frac{dw}{dz} \right)^2 dz$$

$$= \frac{i\rho}{2} . \frac{2\pi i \, 2a^2}{(a^2 - r^2)} \frac{\mu^2}{r} = \frac{2a^2 \pi \mu^2 \rho}{r(r^2 - a^2)}.$$

This $\Rightarrow X = \dfrac{2a^2 \pi \mu^2 \rho}{r(r^2 - a^2)}, \; Y = 0$

$$\Rightarrow \quad \sqrt{(X^2 + Y^2)} = \frac{2\pi a^2 \mu^2 \rho}{r(r^2 - a^2)}$$

This also declares that the force is purely along $\vec{OA}$, the disc will be urged to move along OA. Also the cylinder is attracted towards the source, and sketch of the stream lines reveals that the pressure is greater on the opposite side of the disc than that of the source.

Remark : The above problem can be expressed as :

Show that the force per unit length exerted on a circular cylinder, radius a, due to a source of strength m, at a distance c from the axis is

$$2\pi \rho m^2 a^2 / c\,(c^2 - a^2).$$

(Kanpur 1993, 91; Meerut 91)

Problem 5. *What arrangement of sources and sinks will give rise to the function $w = \log\left(z - \frac{a^2}{z}\right)$? Draw a rough sketch of the stream lines in this case and prove that two of them sub-divide into the circle $r = a$ and axis of y.*

(Kanpur 2003, 2004, 2005; Garhwal 2002, 2004)

Solution : Given $w = \log\left(z - \frac{a^2}{z}\right)$.

This $\Rightarrow w = \log\left(\frac{z^2 - a^2}{z}\right) = \log \frac{(z-a)(z+a)}{z}$

or $\qquad w = \log (z - a) + \log (z + a) - \log (z - 0). \qquad \ldots (1)$

This shows that the given arrangement consists of two sinks each of strength -1 at $z = a$ and $z = -a$, and a source of strength $+1$ at the origin. **Ans.**

Second Part : To determine stream lines.

By (1),

$$\phi + i\psi = \log (x - a + iy) + \log (x + a + iy) - \log (x + iy).$$

Equating imaginary parts,

$$\psi = \tan^{-1} \frac{y}{x-a} + \tan^{-1} \frac{y}{x+a} - \tan^{-1} \frac{y}{x}$$

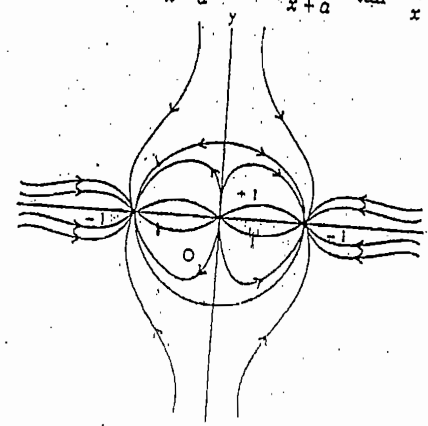


Fig. 3.10.

$$= \tan^{-1}\left[\frac{y/(x-a)+y/(x+a)}{1-y/(x-a).y/(x+a)}\right]-\tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\frac{2xy}{x^2-a^2-y^2}-\tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left[\frac{2xy/(x^2-a^2-y^2)-(y/x)}{1+(y/x).2xy/(x^2-a^2-y^2)}\right]$$

$$= \tan^{-1}\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}$$

Stream lines are given by ψ = const., *i.e.*,

$$\tan^{-1}\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = \text{const.,}$$

or
$$\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = \text{const.,} \quad \ldots(2)$$

If const. = 0, then (2) $\Rightarrow y(x^2+y^2+a^2)=0$

$\Rightarrow y = 0$. for $x^2+y^2+a^2 \neq 0$.

If const. = ∞, then (2) $\Rightarrow x(x^2+y^2-a^2)=0 \Rightarrow x=0, r^2=a^2$

$\Rightarrow x = 0, r = a$

But $x = 0$ represents y-axis and $r = a$ represents circle with radius a and centre at the origin. Thus we see that particular stream lines are y-axis and the circle $r = a$.

A rough sketch of the stream lines is as given in figure 3.18.

Similar Problem. *What arrangement of sources and sinks will give rise to the function* $w = \log\left(z - \frac{1}{z}\right)$? *Draw a rough sketch of stream lines in this case and prove that two of them subdivide into the circle $r = 1$ and axis of y.* (Meerut 2003)

Hint. On replacing a by 1 in the above problem, we get this problem.

Problem 6. *In the case of two dimensional fluid motion produced by a source of strength μ placed at a point S outside a rigid circular disc of radius a whose centre is O, show that velocity of slip of the fluid in contact with the disc is greatest at the points where the lines joining S to the ends of the diameter at right angles to OS cut the circle, and prove that its magnitude at these points is.*

$2\mu r/(r^2-a^2)$, where $OS = r$.

Solution. Let S' be the inverse point of S w.r.t. the circle so that $OS.OS' = a^2$ or $OS' = a^2/r = r'$.

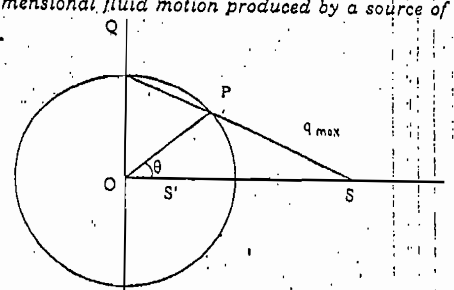


Fig. 3.19.

The image system consists of source $+\mu$ at S' and sink $-\mu$ at O. Take O as origin and OS as real axis; then the equation of complex potential is given by

$$w = -\mu \log(z-r) - \mu \log(z-r') + \mu \log(z-0)$$

$$\frac{dw}{dz} = -\frac{\mu}{z-r} - \frac{\mu}{z-r'} + \frac{\mu}{z}$$

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{-\mu}{z-r} - \frac{\mu}{z-r} + \frac{\mu}{z}\right| = \mu\left|\frac{z^2-r'r}{(z-r)(z-r')z}\right|$$

$$= \mu\left|\frac{z^2-a^2}{(z-r)[z-(a^2/r)]z}\right|.$$

In order to determine velocity at any point on the boundary of the disc, we shall put $z = a e^{i\theta}$.

Then $$q = \mu\left|\frac{a^2 e^{2i\theta}-a^2}{(ae^{i\theta}-r)[(ae^{i\theta}-(a^2/r)]a e^{i\theta}}\right|$$

or $$q = \mu r\left\{\frac{(\cos 2\theta - 1)^2 + (\sin 2\theta)^2}{[(a\cos\theta - r)^2 + a^2\sin^2\theta][(r\cos\theta - a)^2 + r^2\sin^2\theta]}\right\}^{1/2}$$

or $$q = \mu\frac{2r\sin\theta}{(a^2+r^2-2ar\cos\theta)^{1/2}(a^2+r^2-2ra\cos\theta)^{1/2}}$$

or $$q = \frac{2\mu r\sin\theta}{a^2+r^2-2ar\cos\theta}. \quad \ldots(1)$$

For q to be maximum, $\frac{dq}{d\theta} = 0$, this $\Rightarrow$

$$2\mu r\left\{\frac{\cos\theta(a^2+r^2-2ar\cos\theta)-2ar\sin^2\theta}{(a^2+r^2-2ar\cos\theta)^2}\right\} = 0$$

or $$(a^2+r^2-2ar\cos\theta)\cos\theta - 2ar\sin^2\theta = 0$$

or $$\cos\theta = 2ar/(a^2+r^2) \quad \ldots(2)$$

The value of θ, given by (2), gives maximum velocity.

$$(2) \Rightarrow \sin\theta = (r^2-a^2)/(r^2+a^2)$$

By (1), $$q_{max} = 2\mu r\left\{\frac{1}{a^2+r^2-2ar\,[2ar/(a^2+r^2)]}\cdot\frac{(r^2-a^2)}{(r^2+a^2)}\right\}$$

$$= \frac{2\mu r(r^2-a^2)}{(r^2-a^2)^2} = \frac{2\mu r}{r^2-a^2}$$

or $$q_{max} = 2\mu r/(r^2-a^2).$$

The velocity will be along the direction of tangent to the boundary and will be equal to the velocity of slip as the boundary of the disc is a stream line.

Remark. This result is also expressible as

$$q_{max} = \frac{2\mu\,.\,OS}{OS^2-a^2}.$$

Problem 7. *Between the fixed boundaries $\theta = \pi/4$ and $\theta = -\pi/4$, there is a two-dimensional fluid motion due to a source of strength m at the point $(r = a, \theta = 0)$ and an equal sink at the $(r = b, \theta = 0)$. Show that the stream function is*

$$-m \tan^{-1}\left[\frac{r^4 (a^4 - b^4) \sin 4\theta}{r^8 - r^4 (a^4 - b^4) \cos 4\theta + a^4 b^4}\right]$$

and that the velocity at (r, θ) is

$$\frac{4m (a^4 - b^4) r^3}{(r^8 - 2a^4 r^4 \cos 4\theta + a^8)^{1/2} (r^8 - 2b^4 r^4 \cos 4\theta + b^8)^{1/2}}.$$

(Meerut 90)

Solution. Consider the transformation $\zeta = z^2$ which maps points from z-plane to ζ-plane. Let $z = r e^{i\theta}$ and $\zeta = R e^{i\delta}$, then

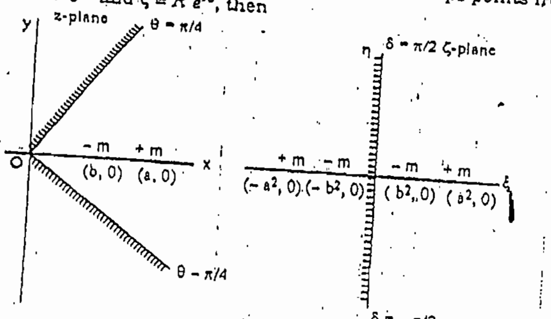


Fig. 3.20.

$$\zeta = z^2 \Rightarrow R e^{i\delta} = r^2 e^{i2\theta} \Rightarrow R = r^2, \ \delta = 2\theta$$

Also $\theta = \pm \pi/4$ so that $\delta = \pm \pi/2$, *i.e.*, η-axis.

By this transformation points $(a, 0)$ and $(b, 0)$ in z-plane are mapped on $(a^2, 0)$ and $(b^2, 0)$ in ζ-plane. The images of $+m$ at $(a^2, 0)$ and $-m$ at $(b^2, 0)$ in ζ-plane w.r.t. η-axis are $+m$ at $(-a^2, 0)$ and $-m$ at $(-b^2, 0)$, respectively.

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to object system and its image system without rigid boundary. This $\Rightarrow$

$$w = -m \log (\zeta - a^2) - m \log (\zeta + a^2) + m \log (\zeta - b^2) + m \log (\zeta + b^2)$$

$$= -m \log (\zeta^2 - a^4) + m \log (\zeta^2 - b^4)$$

or

$$w = -\log (z^4 - a^4) + m \log (z^4 - b^4)$$

$$= -m \log (r^4 e^{i4\theta} - a^4) + m \log (r^4 e^{i4\theta} - b^4) \qquad \ldots (1)$$

Equating imaginary parts,

$$\psi = -m \left[\tan^{-1}\left(\frac{r^4 \sin 4\theta}{r^4 \cos 4\theta - a^4}\right) - \tan^{-1}\left(\frac{r^4 \sin 4\theta}{r^4 \cos 4\theta - b^4}\right)\right]$$

Since $\qquad \tan^{-1} x - \tan^{-1} y = \tan^{-1} (x - y)/(1 + xy)$

Hence, $\qquad \psi = -m \tan^{-1}\left[\frac{r^4 (a^4 - b^4) \sin 4\theta}{r^8 - r^4 (a^4 + b^4) \cos 4\theta + a^4 b^4}\right]$

This completes the first part.

By (1),

$$\frac{dw}{dz} = -\frac{m (4z^3)}{z^4 - a^4} + \frac{4mz^3}{z^4 - b^4}$$

$$= -4mz^3 \left[\frac{a^4 - b^4}{(z^4 - a^4)(z^4 - b^4)}\right]$$

$$q = \left|\frac{dw}{dz}\right| = \frac{4mr^3 (a^4 - b^4)}{|(r^4 e^{i4\theta} - a^4)(r^4 e^{i4\theta} - b^4)|}$$

or

$$q = \frac{4m r^3 (a^4 - b^4)}{[(r^8 + a^8 - 2a^4 r^4 \cos 4\theta)(r^8 + b^8 - 2b^4 r^4 \cos 4\theta)]^{1/2}}$$

This completes the problem.

Problem 8. *Between the fixed boundaries $\theta = \pi/6$ and $\theta = -\pi/6$, there is a two dimensional liquid motion due to a source at the point $r = c$, $\theta = \alpha$, and a sink at the origin, absorbing water at the same rate as the source produces it. Find the stream function and show that one of the stream lines is a part of the curve*

$$r^3 \sin 3\alpha = c^3 \sin 3\theta.$$

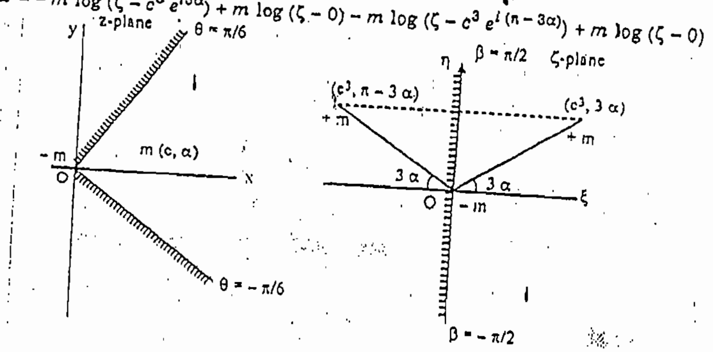
(Kanpur 2000, Garhwal 2003, 2002)

Solution. Consider the map $\zeta = z^3$ from z-plane to se ζ-plane. Let $z = r e^{i\theta}$, $\zeta = R e^{i\beta}$. Then $R e r^{i\beta} = r^3 e^{i3\theta}$, this $\Rightarrow$

$$R = r^3 \qquad \ldots (1), \qquad \beta = 3\theta \qquad \ldots (2)$$

By this map the boundaries $\theta = \pm \pi/6$ are mapped on the boundaries $\beta = \pm \pi/2$, *i.e.*, η-axis is the new boundary in ζ-plane. The points (c, α) and $(0, 0)$ in z-plane are mapped respectively on the points $(c^3, 3\alpha)$ and $(0, 0)$ by virtue of (1) and (2). The object system consists of (i) source $+m$ at $(c^3, 3\alpha)$ and (ii) $-m$ at $(0, 0)$. The image system consists of (i) source $+m$ at $(c^3, \pi - 3\alpha)$ and (ii) sink $-m$ at $(0, 0)$ w.r.t. η-axis. The complex potential is given by

$$w = -m \log (\zeta - c^3 e^{i3\alpha}) + m \log (\zeta - 0) - m \log (\zeta - c^3 e^{i(\pi - 3\alpha)}) + m \log (\zeta - 0)$$



$= 2m \log \zeta - m \log (\zeta - c^3 e^{i3\alpha}) - m \log (\zeta + c^3 e^{-i3\alpha})$

Putting $\zeta = z^3$,

$w = 2m \log z^3 - m \log (z^3 - c^3 e^{i3\alpha})(z^3 + c^3 e^{-i3\alpha})$

$w = 6m \log z - m \log (z^6 - c^6 - 2i z^3 c^3 \sin 3\alpha)$

$= 6m \log r e^{i\theta} - m \log (r^6 e^{i6\theta} - c^6 - 2i c^3 \sin 3\alpha \,.\, r^3 e^{i3\theta})$

$= 6m \log r e^{i\theta} - m \log [(r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta$

$+ i (r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta)]$

Equating imaginary parts on both sides,

$$\psi = 6m \tan^{-1}\left(\frac{r \sin \theta}{r \cos \theta}\right)$$

$$- m \tan^{-1}\left(\frac{r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta}{r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta}\right)$$

Stream lines are given by ψ = const., *i.e.*,

$$6m\theta - m \tan^{-1}\left(\frac{r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta}{r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta}\right) = \text{const.}$$

Taking const. = 0, we get particular stream lines as

$$6m\theta - m \tan^{-1}\left(\frac{r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta}{r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta}\right) = 0$$

or $$6\theta = \tan^{-1}\left(\frac{r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta}{r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta}\right)$$

or $\sin 6\theta \,.\, (r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta)$

$= \cos 6\theta \,(r^6 \sin 6\theta - 2c^3 r^3 \sin 3\alpha \,.\, \cos 3\theta)$

or $-c^6 \sin 6\theta + 2c^3 r^3 \sin 3\alpha \,.\, \cos (6\theta - 3\theta) = 0$

or $2r^3 \sin 3\alpha \,.\, \cos 3\theta - c^3 \sin 6\theta = 0$

or $2\cos 3\theta \,[r^3 \sin 3\alpha - c^3 \sin 3\theta] = 0$

This $\Rightarrow \cos 3\theta = 0,$ ...(3)

$r^3 \sin 3\alpha = c^3 \sin 3\theta.$

By (3), $\theta = \pm \pi/6$ which gives no new stream lines as these are the given stream lines. The other stream line is a part of the curve

$$r^3 \sin 3\alpha = c^3 \sin \theta.$$

Problem 9. *In the case of motion of liquid in a part of a plane bounded by a straight line due to a source in the plane prove that if $m\rho$ is the mass of the liquid (of density ρ) generated at the source per unit of time, the pressure on the length $2l$ of the boundary, immediately opposite to the source is less than that on an equal length at a great distance by*

$$\frac{1}{2} \frac{m^2 \rho}{\pi^2} \left(\frac{1}{c} \tan^{-1} \frac{l}{c} - \frac{l}{l^2 + c^2}\right)$$

where c is the distance of the source from the boundary. (Kanpur 2000)

Solution. Suppose μ is the strength of the source at P where $OP = c$. Then by def. of strength

$$2\pi \mu \rho = m\rho$$

$$m/2\pi = \mu.$$

The boundary is y-axis. The image of a source $+\frac{m}{2\pi}$ at $P(c, 0)$ is a source $+m/2\pi$ at $P'(-c, 0)$.

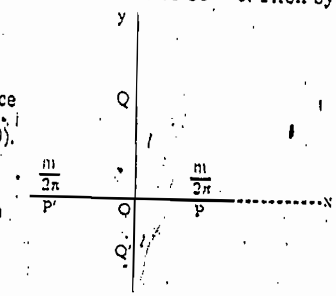


Fig. 3.22.

Now the complex potential is

$$w = -\frac{m}{2\pi} \log (z - c) - \frac{m}{2\pi} \log (z + c)$$

$$= -\frac{m}{2\pi} \log (z^2 - c^2)$$

$$\frac{dw}{dz} = -\frac{m}{2\pi} \cdot \frac{2z}{z^2 - c^2}$$

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{m}{\pi} \cdot \frac{z}{z^2 - c^2}\right|$$

For any point on y-axis, $z = iy$, so that

$$q = \frac{m}{\pi}\left|\frac{z}{z^2 - c^2}\right| = \frac{m}{\pi}\left|\frac{iy}{-y^2 - c^2}\right| = \frac{my}{\pi (y^2 + c^2)}$$

This is the expression for velocity at any point on y-axis. By Bernoulli's equation for steady motion,

$$\frac{p}{\rho} + \frac{1}{2} q^2 = A.$$

Subjecting this to the condition

$p = p_0$ when $y = \infty$, $q = 0$, we get $A = p_0/\rho$.

(Since velocity is negligible at great distance).

Hence $$\frac{p}{\rho} + \frac{1}{2} q^2 = \frac{p_0}{\rho}.$$

Pressure on QQ' $$= \int_{-l}^{l} p \, dy.$$

But $$p - p_0 = -\frac{1}{2} \rho q^2 \Rightarrow \int_{-l}^{l} (p - p_0)\, dy = -\frac{1}{2} \rho \int_{-l}^{l} q^2 \, dy$$

Required difference of pressure

$$= \int_{-l}^{l} (p_0 - p)\, dy = \frac{1}{2} \rho \int_{-l}^{l} \frac{m^2}{\pi^2} \frac{y^2 \, dy}{(y^2 + c^2)^2}$$

$$= \frac{m^2}{\pi^2}\rho \int_0^l \frac{y^2}{(y^2+c^2)^2}\,dy.$$

[Put $y = c\tan\theta,\ dy = c\sec^2\theta\, d\theta$]

$$= \frac{m^2\rho}{\pi^2}\int_0^{\theta_1} \frac{c^2\tan^2\theta \cdot c}{c^4\sec^4\theta}\sec^2\theta\, d\theta = \frac{m^2\rho}{\pi^2 c}\int_0^{\theta_1}\sin^2\theta\, d\theta, \text{ where } \tan\theta_1 = \frac{l}{c}.$$

$$= \frac{m^2\rho}{2\pi^2 c}\int_0^{\theta_1}(1-\cos 2\theta)\,d\theta = \frac{m^2\rho}{2\pi^2 c}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\theta_1}$$

$$= \frac{m^2\rho}{2\pi^2 c}\left[\theta_1 - \sin\theta_1\cos\theta_1\right] = \frac{m^2\rho}{2\pi^2 c}\left[\tan^{-1}\frac{l}{c} - \frac{lc}{l^2+c^2}\right]$$

$$= \frac{m^2\rho}{2\pi^2}\left[\frac{1}{c}\tan^{-1}\frac{l}{c} - \frac{l}{l^2+c^2}\right]$$

Proved.

Problem 10. *Within a rigid boundary in the form of the circle* $(x+\alpha)^2 + (y-4\alpha)^2 = 8\alpha^2$, *there is liquid motion due to doublet of strength* μ *at the point* $(0, 3\alpha)$ *with its axis along the axis of* y. *Show that velocity potential is*

$$\mu\left[\frac{4(x-3\alpha)}{(x-3\alpha)^2+y^2} + \frac{y-3\alpha}{x^2+(y-3\alpha)^2}\right]$$

(Kanpur 1991)

Solution. The rigid boundary is a circle given by

$$(x+\alpha)^2 + (y-4\alpha)^2 = 8\alpha^2.$$

The centre is $(-\alpha, 4\alpha)$ and radius $= \sqrt{(8\alpha^2)}$.

Object doublet is at $P(0, 3\alpha)$ with its axis along y-axis, CM and PN are perpendiculars on x-axis and CM respectively. Produce CP to meet x-axis at Q. Evidently, $CN = NP = \alpha$ so that $\angle NPC = 45°$ and therefore $\angle CQM = 45°$ so that

$$CQ = \sqrt{[(4\alpha)^2 + (4\alpha)^2]} = 4\alpha\sqrt{2}.$$

Hence $CM = MQ = 4\alpha$.

Observe that

$$CP \cdot CQ = \alpha\sqrt{2} \cdot 4\alpha\sqrt{2}$$

$$= 8\alpha^2 = (\text{radius})^2.$$

Hence Q is the inverse point of P w.r.t the circle. The image of the doublet μ at $P(0, 3\alpha)$ w.r.t circle is a doublet μ' at the inverse point $Q(3\alpha, 0)$ with its axis along x-axis. For object and image doublets make supplementary angles with the line CQ.

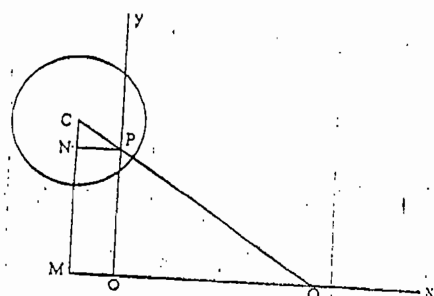


Fig. 3.23.

Here

$$\mu' = \frac{\mu a^2}{f^2} = \mu\frac{8\alpha^2}{CP^2} = \frac{\mu\, 8\alpha^2}{2\alpha^2} = 4\mu.$$

Thus

$$w = \frac{\mu\, e^{i\pi/2}}{z - i\,3\alpha} + \frac{4\mu\, e^{i\,0}}{z - 3\alpha}$$

$$= \frac{\mu i}{x + i(y-3\alpha)} + \frac{4\mu}{(x-3\alpha)+iy}$$

$$= \frac{\mu i\,[x - i(y-3\alpha)]}{x^2+(y-3\alpha)^2} + \frac{4\mu\,[(x-3\alpha)-iy]}{(x-3\alpha)^2+y^2}$$

Equating real parts on both sides,

$$\phi = \mu\left[\frac{y-3x}{x^2+(y-3\alpha)^2} + \frac{4(x-3\alpha)}{(x-3\alpha)^2+y^2}\right].$$

This concludes the problem.

Problem 11. *In the part of an infinite plane bounded by a circular quadrant AB and the production of the radii OA, OB, there is a two dimensional motion due to the production of liquid at A, and its absorption at B, at the uniform rate m. Find the velocity potential of the motion; and show that the fluids which issue from A in the direction making an angle* μ *with OA follows the path whose polar equation is*

$$r = a\sin^{1/2}2\theta\,[\cot\mu + \sqrt{(\cot^2\mu + \operatorname{cosec}^2 2\theta)}]^{1/2},$$

the positive sign being taken for all the square roots.

Solution. The object system consists of source $+m/2\pi$ at A and $-m/2\pi$ at B. The image of $+(m/2\pi)$ at A w.r.t. circular boundary is a source $+m/2\pi$ at A, the inverse point of A and sink $-m/2\pi$ at O. The image of sink $-m/2\pi$ at B, w.r.t. circle is a sink $-m/2\pi$ at B, the inverse point of B and source $+m/2\pi$ at O. The source $+m/2\pi$ and sink $-m/2\pi$ both at O cancel each other.

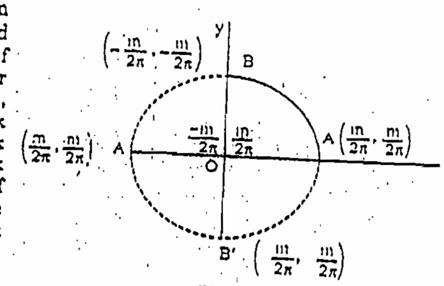


Fig. 3.24.

Image w.r.t. bounding plane.

The image of source $+m/2\pi$ at A is a source $+m/2\pi$ at A' w.r.t. line BB' and image of sink $-m/2\pi$ at B w.r.t. the line AA' is a sink $-m/2\pi$ at B'. Also the images at A and B have their images $+m/2\pi$ and $-m/2\pi$ at A' and B' respectively.

The object and its image system consists of 2 sources of strength $+m/2\pi$ at A, 2 sinks of strength $-m/2\pi$ at B, two sources $+m/2\pi$ at A', two sinks $-m/2\pi$ at B'.

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to object system and its image systems with no rigid boundary. Thus

$$w = -\frac{2m}{2\pi}\log(z-a) - \frac{2m}{2\pi}\log(z+a) + \frac{2m}{2\pi}\log(z-ia) + \frac{2m}{2\pi}\log(z+ia).$$

or
$$w = -\frac{m}{\pi}\log(z-a) - \frac{m}{\pi}\log(z+a) + \frac{m}{\pi}\log(z-ia) + \frac{m}{\pi}\log(z+ia) \qquad \ldots(1)$$

Equating real part on both sides,

$$\phi = -\frac{m}{\pi}[\log|z-a| + \log|z+a| - \log|z-ia| - \log|z+ia|]$$

$$= -\frac{m}{\pi}[\log PA + \log PA' - \log PB - \log PB']$$

or
$$\phi = -\frac{m}{\pi}\log\frac{PA \cdot PA'}{PB \cdot PB'}.$$

This is the required expression for velocity potential. Again by (1),

$$\phi + i\psi = -\frac{m}{\pi}\log(z^2-a^2) + \frac{m}{\pi}\log(z^2+a^2)$$

or
$$\phi + i\psi = -\frac{m}{\pi}[\log(r^2 e^{i2\theta} - a^2) - \log(r^2 e^{i2\theta} + a^2)].$$

Equating imaginary parts,

$$\psi = -\frac{m}{\pi}\left[\tan^{-1}\left(\frac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2}\right) - \tan^{-1}\left(\frac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}\right)\right]$$

$$= -\frac{m}{\pi}\tan^{-1}\left[\frac{2a^2 r^2 \sin 2\theta}{r^4\cos^2 2\theta - a^4 + r^4\sin^2 2\theta}\right] \qquad \ldots(2)$$

For $\tan^{-1}x - \tan^{-1}y = \tan^{-1}[(x-y)/(1+xy)]$.

For a particular streamline which leaves A at an angle μ,

$$\psi = -\frac{m}{\pi}\mu. \qquad \ldots(3)$$

By (2) and (3),

$$-\frac{m}{\pi}\mu = -\frac{m}{\pi}\tan^{-1}\frac{2a^2 r^2\sin 2\theta}{r^4 - a^4}.$$

This $\Rightarrow \tan\mu = \dfrac{2a^2 r^2 \sin 2\theta}{r^4 - a^4}$.

$$\Rightarrow (r^2)^2 - 2a^2 r^2 \sin 2\theta \cot\mu - a^4 = 0.$$

This is quadratic in r^2.

Hence
$$r^2 = \frac{2a^2\sin 2\theta\cot\mu \pm \sqrt{(4a^4\sin^2 2\theta\cot^2\mu + 4a^4)}}{2}$$

Taking positive radical sign,

$$r = [a^2\sin 2\theta\cot\mu + a^2\sqrt{(\sin^2 2\theta\cot^2\mu + 1)}]^{1/2}$$

or
$$r = a(\sin 2\theta)^{1/2}[\cot\mu + \sqrt{(\cot^2\mu + \operatorname{cosec}^2 2\theta)}]^{1/2}.$$

This is the required path.

IAY 2009 **Problem 12.** *Two sources, each of strength m, are placed at the points $(-a, 0)$ and $(a, 0)$ and a sink of strength $2m$ is placed at the origin. Show that the stream lines are curves*

$$(x^2+y^2)^2 = a^2(x^2 - y^2 + \lambda xy), \text{ where } \lambda \text{ is a parameter.}$$

Show also that the fluid speed at any point is $2ma^2/r_1 r_2 r_3$ where r_1, r_2, r_3 are respectively the distances of the point from the source and the sink. (Garhwal 2000)

Solution. The complex potential at any point $P(z)$ is given by

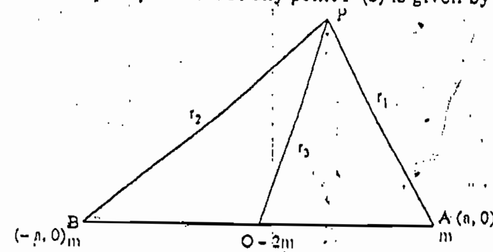


Fig. 3.26.

$$w = -m\log(z-a) - m\log(z+a) + 2m\log(z-0)$$

or
$$w = -m\log(z^2-a^2) + m\log z^2 \qquad \ldots(1)$$

or
$$\phi + i\psi = -m\log(x^2 - a^2 - y^2 + 2ixy) + m\log(x^2 - y^2 + 2ixy).$$

Equating imaginary parts,

$$\psi = -m\tan^{-1}\frac{2xy}{x^2-a^2-y^2} + m\tan^{-1}\frac{2xy}{x^2-y^2}$$

$$= -m\tan^{-1}\frac{2a^2xy}{(x^2-y^2)(x^2-a^2-y^2) + 4x^2y^2}$$

Stream lines are given by ψ = const., *i.e.*,

$$-m\tan^{-1}\frac{2a^2xy}{(x^2-y^2)(x^2-a^2-y^2)+4x^2y^2} = -m\tan^{-1}\left(\frac{2}{\lambda}\right), \text{ say}$$

or
$$\lambda a^2 xy = (x^2-y^2)^2 - a^2(x^2-y^2) + 4x^2y^2$$

or
$$\lambda a^2 xy = (x^2+y^2)^2 - a^2(x^2-y^2)$$

or
$$(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy) \quad \text{where } \lambda \text{ is a variable parameter.}$$

This completes the first part of the problem.

$$\text{Flow speed} = \left|\frac{dw}{dz}\right| = \left|-\frac{2mz}{z^2-a^2} + \frac{2mz}{z^2}\right| = \frac{2ma^2}{|z(z^2-a^2)|}$$

$$= \frac{2ma^2}{|z|\,|z-a|\,|z+a|} = \frac{2ma^2}{r_1 r_2 r_3}.$$

This concludes the problem.

Problem 13. *The space on one side of an infinite plane wall $y = 0$ is filled with inviscid, incompressible fluid, moving at infinity with velocity U in the direction of x-axis. The motion of the fluid is wholly two dimensional in xy-plane. A doublet of strength μ is at a distance a from the wall and the points in the negative direction of x-axis. Show that if $\mu < 4a^2 U$, the pressure of the fluid on the wall is maximum at points distant $a\sqrt{3}$ from O, the foot of the perpendicular from the doublet on the wall and is a minimum also.*

If $\mu = 4a^2 U$, find points where the velocity of the fluid is zero and show that stream lines include the circle.

$$x^2 + (y-a)^2 = 4a^2.$$

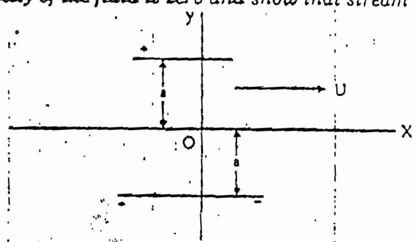


Fig. 3.26.

Solution. Since the points of the doublet are in the negative direction of x-axis so that the doublet makes an angle π with x-axis. Image of the given doublet is an equal doublet similarly oriented at $z = -ia$.

The system consists object doublet, image doublet and stream with velocity U parallel to x-axis. Hence

$$w = \frac{\mu e^{i\pi}}{z - ia} + \frac{\mu e^{i\pi}}{z + ia} - Uz$$

$$= -\frac{\mu}{z - ia} - \frac{\mu}{z + ia} - Uz$$

$$= -\frac{2\mu z}{z^2 + a^2} - Uz$$

$$-\frac{dw}{dz} = U + \frac{2\mu}{(z^2 + a^2)^2}[z^2 + a^2 - 2z^2]$$

or
$$-\frac{dw}{dz} = U + \frac{2\mu(a^2 - z^2)}{(z^2 + a^2)^2} \quad \dots(1)$$

$$\left|-\frac{dw}{dz}\right| = q = \left|U + \frac{2\mu(a^2 - z^2)}{(a^2 + z^2)^2}\right|.$$

For any point on the wall, $z = x$ so that,

$$q = U + \frac{2\mu(a^2 - x^2)}{(x^2 + a^2)^2}$$

This $\Rightarrow q^2 - U^2 = \dfrac{4\mu^2(a^2 - x^2)^2}{(x^2 + a^2)^4} + \dfrac{4\mu U(a^2 - x^2)}{(x^2 + a^2)^2}. \quad \dots(2)$

To determine pressure at any point on the wall. By Bernoulli's equation for steady motion, $\dfrac{p}{\rho} + \dfrac{1}{2}q^2 = C$. Subjecting this to the condition $p = \Pi, q = U$ where $z = \infty$, so that $\dfrac{\Pi}{\rho} + \dfrac{1}{2}U^2 = C$.

Thus $\dfrac{p}{\rho} + \dfrac{1}{2}q^2 = \dfrac{\Pi}{\rho} + \dfrac{1}{2}U^2$ or $\dfrac{1}{2}(q^2 - U^2) = \dfrac{\Pi - p}{\rho}$

or
$$\frac{\Pi - p}{\rho} = \frac{2\mu^2(a^2 - x^2)^2}{(a^2 + x^2)^4} + \frac{2\mu U(a^2 - x^2)}{(a^2 + x^2)^2}, \text{ using (2).}$$

$$-\frac{1}{\rho}\frac{dp}{dx} = -\frac{8\mu^2 x(a^2 - x^2)}{(a^2 + x^2)^4} - \frac{16\mu^2(a^2 - x^2)^2 \cdot x}{(a^2 + x^2)^5}$$

$$+ 2\mu U\left[\frac{-2x}{(a^2 + x^2)^2} - \frac{4x(a^2 - x^2)}{(a^2 + x^2)^3}\right]$$

$$= \frac{8\mu^2 x(a^2 - x^2)}{(a^2 + x^2)^5} a^2 + x^2 + 2(a^2 - x^2) - \frac{4\mu Ux}{(a^2 + x^2)^3} a^2 + x^2 + 2(a^2 - x^2)$$

or
$$\frac{1}{\rho}\frac{dp}{dx} = \frac{4\mu x(3a^2 - x^2)}{(a^2 + x^2)^5} \cdot 2\mu(a^2 - x^2) + U(a^2 + x^2)^2]$$

For extremum values of p, $\dfrac{dp}{dx} = 0$, this $\Rightarrow x(3a^2 - x^2) = 0$ so that $x = 0, \pm a\sqrt{3}$.

Thus, if $\mu < 4a^2 U$, then $\dfrac{d^2 p}{dx^2} < 0$ so that p is maximum where $z = a\sqrt{3}$. Again, if $\mu < 4a^2 U$, then $d^2p/dx^2 > 0$ where $z = 0$ so that p is minimum. Consider the case in which $\mu = 4a^2 U$.

Let the fluid velocity = 0, so that $\dfrac{dw}{dz} = 0$, then (1) $\Rightarrow$

$$U + \frac{2 \cdot 4a^2 U(a^2 - z^2)}{(a^2 + z^2)^2} = 0$$

or
$$(z^2 + a^2)^2 + 8a^2(a^2 - z^2) = 0.$$

On the wall this becomes,

$$(x^2 + a^2)^2 + 8a^2(a^2 - x^2) = 0$$

or
$$x^4 - 6a^2x^2 + 9a^4 = 0 \text{ or } (x^2 - 3a^2)^2 = 0 \text{ or } x = \pm a\sqrt{3}$$

Ans. $(\pm a\sqrt{3}, 0)$ are the points where velocity vanishes.

To determine stream lines.

We have $w = -\dfrac{2\mu z}{z^2 + a^2} - Uz$

or
$$\phi + i\psi = -\frac{2.4a^2 U(x + iy)(x^2 + a^2 - y^2 - 2ixy)}{(x^2 + a^2 - y^2)^2 + 4x^2y^2} - Uz$$

or
$$-\psi = \frac{8a^2 U[-2x^2 y + y(x^2 + a^2 - y^2)]}{(x^2 + a^2 - y^2)^2 + 4x^2y^2} + Uy$$

Stream lines are given by ψ = const. Take const. = 0 Then stream lines are given by $\psi = 0$, i.e. $\dfrac{8a^2 y(-2x^2 + x^2 + a^2 - y^2)}{(x^2 + a^2 - y^2)^2 + 4x^2y^2} + y = 0$

or
$$8a^2[a^2 - (x^2 + y^2)] + (x^2 - y^2)^2 + a^4 + 2a^2(x^2 - y^2) + 4x^2y^2 = 0$$

or $\quad 8a^4 - 8a^2(x^2+y^2) + (x^2+y^2)^2 + 2a^2(x^2-y^2) = 0$

or $\quad (x^2+y^2)^2 - 6a^2x^2 - 10a^2y^2 + 9a^4 = 0$

or $\quad (x^2+y^2-3a^2)^2 - 4a^2y^2 = 0$

or $\quad (x^2+y^2-3a^2-2ay)(x^2+y^2-3a^2+2ay) = 0.$

This includes the circle $x^2+y^2-2ay-3a^2=0.$

i.e. $\quad x^2+(y-a)^2 = 4a^2.$

Problem 14. *Find the lines of flow in two dimensional fluid motion given by*

$$\phi + i\psi = -\frac{n}{2}(x+iy)^2 e^{2int}.$$

Prove or verify that the paths of the particles of the fluid (in polar co-ordinates) may be obtained by eliminating t from the equations

$$r\cos(nt+\theta) - x_0 = r\sin(nt+\theta) - y_0 = nt(x_0-y_0).$$

Solution. Write $x+iy = re^{i\theta}$. It is given that

$$\phi + i\psi = -\frac{n}{2}(x+iy)^2 e^{2int} = -\frac{n}{2}r^2e^{i2\theta}\cdot e^{i2nt}$$

or $$\phi + i\psi = -\frac{1}{2}nr^2 e^{i2(nt+\theta)}.$$

This $\Rightarrow \phi = -\frac{nr^2}{2}\cos 2(nt+\theta),\ \psi = -\frac{nr^2}{2}\sin 2(nt+\theta).$

Lines of low are given by ψ = const., i.e.,

$$-\frac{nr^2}{2}\sin 2(nt+\theta) = \text{const.}$$

i.e., $$r^2\sin 2(nt+\theta) = \text{const} \qquad \text{Ans.}$$

By def., $$\dot r = -\frac{\partial\phi}{\partial r} = nr\cos 2(nt+\theta)$$

$$r\dot\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{1}{r}\left[\frac{nr^2}{2}\sin 2(nt+\theta)\cdot 2\right]$$

or $$\dot\theta = -nr\sin 2(nt+\theta)$$

$$\frac{d}{dt}[r\cos(nt+\theta)] = \dot r\cos(nt+\theta) - (n+\dot\theta)r\sin(nt+\theta)$$

$$= nr\cos 2(nt+\theta)\cos(nt+\theta)[n - n\sin 2(nt+\theta)]\,r\sin(nt+\theta)$$

$$= nr[\cos 2\{(nt+\theta)-(nt+\theta)\} - \sin(nt+\theta)]$$

$$= nr[\cos(nt+\theta) - \sin(nt+\theta)].$$

Similarly we can show that

$$\frac{d}{dt}[r\sin(nt+\theta)] = nr[\cos(nt+\theta) - \sin(nt+\theta)].$$

Hence we see that

$$\frac{d}{dt}[r\cos(nt+\theta)] = \frac{d}{dt}[r\sin(nt+\theta)]$$

$$= nr[\cos(nt+\theta) - \sin(nt+\theta)] \qquad \ldots(1)$$

This $\Rightarrow \frac{d}{dt}[r\cos(nt+\theta)] = \frac{d}{dt}[r\sin(nt+\theta)]$

Integrating, $\quad r\cos(nt+\theta) - r\sin(nt+\theta) = A. \qquad \ldots(2)$

Subjecting this to initial condition, when $t=0$

$$r\cos\theta = x = x_0,\ r\sin\theta = y = y_0. \qquad \ldots(3)$$

we get $\quad \begin{cases} A = x_0 - y_0 \\ r\cos(nt+\theta) - r\sin(nt+\theta) = x_0 - y_0 \end{cases}$

Hence, by (1), $\frac{d}{dt}[\cos(nt+\theta)] = \frac{d}{dt}[r\sin(nt+\theta)] = n(x_0-y_0).$

Integrating, $r\cos(nt+\theta) = r\sin(nt+\theta) = nt(x_0-y_0) + B. \quad \ldots(4)$

This $\quad \Rightarrow r\cos(nt+\theta) = nt(x_0-y_0) + B.$

Subjecting this to (3), $\quad x_0 = 0 + B.$

$\therefore \quad r\cos(nt+\theta) = nt(x_0-y_0) + x_0.$

or $\quad r\cos(nt+\theta) - x_0 = nt(x_0-y_0).$

Again, (4) gives $\ r\sin(nt+\theta) - y_0 = -nt(x_0-y_0)$

Combining the last two equations,

$$r\cos(nt+\theta) - x_0 = r\sin(nt+\theta) - y_0 = nt(x_0-y_0)$$

This concludes the problem.

IFoS 9005 **Problem 15.** *Use the method of images to prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta=0$ and $\theta=\pi/3$, the solution is*

$$\phi + i\psi = -m\log(z^3 - z_0^3)(z^3 - z_0'^3)),$$

where $\quad z_0 = x_0 + iy_0,\ z'_0 = x_0 - iy_0.$ *(Kanpur 2002, 2004)*

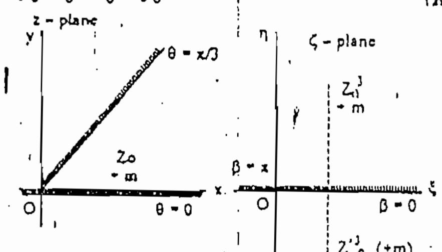


Fig. 3.27.

Solution. Consider the map $\zeta = z^3$ from z-plane to ζ-plane, where $z = re^{i\theta}$, $\zeta = Re^{i\beta}$ so that $Re^{i\beta} = r^3e^{i3\theta}$. This $\Rightarrow R = r^3, \beta = 3\theta$. hence the source $+m$ at z_0 in z- plane is mapped on the source $+m$ at z_0^3 in ζ-plane. Also the boundaries $\theta=0$, $\theta=\pi/3$ in z-plane become $\beta=0$, $\beta=\pi$, i.e. ζ-axis.

The image of source $+m$ at z_0^3 w.r.t. ξ-axis is a source $+m$ at $z_0'^3$, where $z_0' = x_0 - iy_0$.

The complex potential due to object system with rigid boundaries is equivalent to the object and its image system without rigid boundaries. Hence w is given by

$$w = -m \log (\zeta - z_0^3) - m \log (\zeta - z_0'^3)$$

or

$$\phi + i\psi = -m \log (z^3 - z_0^3)(z^3 - z_0'^3).$$

Proved.

Problem 16. *A source S and sink T of equal strength m are situated within the space bounded by a circle whose centre is O. If S and T are at equal distance from O on opposite sides of it and on the same diameter AOB, show that velocity of the liquid at any point P is*

$$2m \cdot \frac{OS^2 + OA^2}{OS} \cdot \frac{PA \cdot PB}{PS \cdot PS' \cdot PT \cdot PT'},$$

where S' and T' are inverse points of S and T w.r.t. the circle.

(Meerut 1993, Kanpur 90)

Solution. Take O as origin and OA as x-axis.

Let $OS = OT = c$, $OA = OB = a$. Then

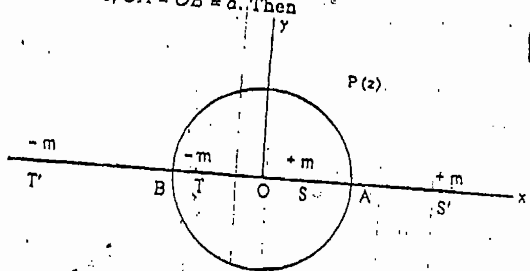


Fig. 3.28.

$$OS \cdot OS' = a^2,\ OT \cdot OT' = a^2.$$

Hence $OS' = OT' = a^2/c$.

The object system consists of

(i) source $+m$ at S $(c, 0)$,

(ii) sink $-m$ at T $(-c, 0)$.

The image system consists of

(i)' source $+m$ at S' $(a^2/c, 0)$ and sink $-m$ at O.

(ii)' sink $-m$ at T' $(-a^2/c, 0)$ and source $+m$ at O.

Source and sink both at O cancel each other.

Hence

$$w = -m \log (z - c) + m \log (z + c) - m \log (z - a^2/c) + m \log (z + a^2/c)).$$

$$-\frac{dw}{dz} = m\left[\frac{1}{z-c} - \frac{1}{z+c} + \frac{1}{z-c'} - \frac{1}{z+c'}\right], \text{ where } c' = a^2/c.$$

$$= m\left[\frac{2c}{z^2 - c^2} + \frac{2(a^2/c)}{z^2 - (a^4/c^2)}\right] = 2m \frac{(z^2 - a^2)(c + a^2/c)}{(z^2 - c^2)(z^2 - c'^2)}$$

$$= 2m \cdot \frac{a^2 + c^2}{c} \cdot \frac{(z-a)(z+a)}{(z-c)(z+c)(z - a^2/c)\left(z + \frac{a^2}{c}\right)}$$

Taking modlus of both sides and noting that fluid velocity $= \left|-\frac{dw}{dz}\right|$, we get

$$\text{velo.} = 2m \cdot \frac{OA^2 + OS^2}{OS} \cdot \frac{PA \cdot PB}{PS \cdot PT \cdot PS' \cdot PT'}.$$

Proved.

Problem 17. *Prove that $u = -\omega y$, $v = -\omega x$, $w = 0$ represents a possible motion of inviscid fluid. Find the stream function and sketch stream lines. What is the basic difference between this motion and one represented by the potential $\phi = A \log r$, $r = (x^2 + y^2)^{1/2}$.*

Solution. I. Consider the motion defined by

$$u = -\omega y,\ v = \omega x,\ w = 0.$$

Evidently it is two dimensional motion.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 + 0 + 0 = 0.$$

This declares that the liquid motion is possible.

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$

But
$$-\frac{\partial \phi}{\partial x} = u = -\frac{\partial \psi}{\partial y},\ -\frac{\partial \phi}{\partial y} = v = \frac{\partial \psi}{\partial x}.$$

$$d\psi = v\,dx - u\,dy = \omega x\,dx + \omega y\,dy = d\left[\frac{\omega}{2}(x^2 + y^2)\right]$$

Integrating, $\psi = \frac{\omega}{2}(x^2 + y^2) + a$

This gives the required stream function. Ans.

Stream lines are given by ψ = const. $= b$, say, so that

$$x^2 + y^2 = \frac{2(b - a)}{\omega} = c \text{ or } x^2 + y^2 = c$$

It means that stream lines are concentric circles with their centres at the origin. Ans.

II. Next, we consider the motion defined by

$$\phi = A \log r = \frac{A}{2} \log (x^2 + y^2).$$

$$\text{This} \Rightarrow \frac{\partial \phi}{\partial x} = \frac{Ax}{x^2 + y^2} \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{A(y^2 - x^2)}{(x^2 + y^2)^2}.$$

$$\Rightarrow \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{A(y^2 - x^2) + A(x^2 - y^2)}{(x^2 + y^2)^2} = 0$$

Hence liquid motion is possible.

III. Difference. The basic difference in these two motions is that velocity potential does not exist in the first case whereas in the second case it exists.

Problem 18. *A two dimensional flow field is given by $\psi = xy$. Show that the flow is irrotational. Find velocity potential, stream lines.* (Garhwal 2000)

Solution. $\psi = xy$

$$u = -\frac{\partial \psi}{\partial y} = -x, \qquad v = \frac{\partial \psi}{\partial x} = y$$

$$q = ui + vj = -xi + yj$$

$$\text{curl}\, q = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix}$$

$$= i(0) - j(0) + k(0) = 0$$

$\therefore$ Motion is irrotational.

(ii) $$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = -u\,dx - v\,dy$$

$$= x\,dx - y\,dy = M\,dx + N\,dy, \text{ say.}$$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

$\therefore$ $M\,dx + N\,dy$ is exact. Solution is

$$\int d\phi = \int x\,dx + \int -y\,dy = \frac{x^2 - y^2}{2} + c$$

or $$\phi = \frac{x^2 - y^2}{2} + c$$

This is the expression for velocity potential. **Ans.**

(iii) Stream lines are given by

ψ = const. But $\psi = xy$

$\therefore$ $xy = c$

gives stream lines. **Ans.**

Problem 19. *Show that velocity potential*

$$= \frac{1}{2} \log \left[\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \right]$$

gives a possible motion. Determine the form of stream lines and the curves of equal speed.

Solution. Given, $\phi = \frac{1}{2} \log [(x+a)^2 + y^2] - \frac{1}{2} \log [(x-a)^2 + y^2]$. ... (1)

$$\frac{\partial \phi}{\partial x} = \frac{x+a}{(x+a)^2 + y^2} - \frac{(x-a)}{(x-a)^2 + y^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{[(x+a)^2 + y^2] - 2(x+a)^2}{[(x+a)^2 + y^2]^2} - \frac{[(x-a)^2 + y^2] - 2(x-a)^2}{[(x-a)^2 + y^2]^2}$$

or $$\frac{\partial^2 \phi}{\partial x^2} = \frac{y^2 - (x+a)^2}{[(x+a)^2 + y^2]^2} - \frac{y^2 - (x-a)^2}{[(x-a)^2 + y^2]^2} \quad \text{... (2)}$$

By (1), $$\frac{\partial \phi}{\partial y} = \frac{y}{(x+a)^2 + y^2} - \frac{y}{(x-a)^2 + y^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{(x+a)^2 + y^2 - 2y^2}{[(x+a)^2 + y^2]^2} - \frac{(x-a)^2 + y^2 - 2y^2}{[(x-a)^2 + y^2]^2} \quad \text{... (3)}$$

Adding (2) and (3), $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ or $\nabla^2 \phi = 0$.

Thus the equation of continuity is satisfied and so (1) gives a possible liquid motion.

Second Part. To determine stream lines.

$$-\frac{\partial \phi}{\partial x} = u = -\frac{\partial \psi}{\partial y}, \; -\frac{\partial \phi}{\partial y} = v = \frac{\partial \psi}{\partial x}.$$

Hence $$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Now $$\frac{\partial \psi}{\partial y} = \frac{x+a}{(x+a)^2 + y^2} - \frac{x-a}{(x-a)^2 + y^2}$$

Integrating w.r.t. y,

$$\psi = \tan^{-1} \frac{y}{x+a} - \tan^{-1} \frac{y}{x-a} + F(x) \;\ldots \quad \text{... (4)}$$

where $F(x)$ is constant of integration. To determine $F(x)$.

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -\frac{y}{(x+a)^2 + y^2} + \frac{y}{(x-a)^2 + y^2} \quad \text{... (5)}$$

By (4), $$\frac{\partial \psi}{\partial x} = -\frac{y}{(x+a)^2 + y^2} + \frac{y}{(x-a)^2 + y^2} + F'(x) \quad \text{... (6)}$$

Equating (5) to (6), $F'(x) = 0$. Integrating this $F(x)$ = absolute const. and hence neglected. Since it has no effect on the fluid motion.

Now (4) becomes

$$\psi = \tan^{-1} \frac{y}{x+a} - \tan^{-1} \frac{y}{x-a} \quad \text{... (7)}$$

$$= \tan^{-1} \frac{-2ay}{x^2 - a^2 + y^2}$$

Stream lines are given by ψ = const., *i.e.*,

$$\tan^{-1} \left[\frac{-2ay}{x^2 - a^2 + y^2} \right] = \text{const. or } \frac{y}{x^2 - a^2 + y^2} = \text{const.}$$

If we take const. = 0, then we get $y = 0$, *i.e.*, x-axis.

If we take conts. = ∞, then we get circle $x^2 - a^2 + y^2 = 0$

i.e., $x^2 + y^2 = a^2$.

Thus stream lines include x axis and circle. Ans.

Third Part. To determine curves of equal speed.

By (1) and (7), we obtain

$$w = \phi + i\psi = \frac{1}{2}\log[(x+a)^2 + y^2] - \frac{1}{2}\log[(x-a)^2 + y^2]$$

$$+ i\tan^{-1}\frac{y}{x+a} - i\tan^{-1}\frac{y}{x-a}$$

$$= \log[(x+a) + iy] - \log[(x-a) + iy]$$

$$= \log(z+a) - \log(z-a)$$

$$\frac{dw}{dz} = \frac{1}{z+a} - \frac{1}{z-a} = \frac{-2a}{(z-a)(z+0)}$$

$$\left|\frac{dw}{dz}\right| = q = \frac{2a}{|z-a| \cdot |z+a|}$$

Write $|z-a| = r$, $|z+a| = r'$. Then speed $= \dfrac{2a}{rr'}$.

The curves of equal speed are given by

$\dfrac{2a}{rr'} =$ const., i.e., $rr' =$ const. which are Cassini ovals.

Problem 20. *Parallel line sources (perpendicular to the xyplane) of equal strength m are placed at the points $z = nia$, where $n = \ldots, -2, -1, 0, 1, 2, 3, \ldots$, prove that the complex potential is*

$$w = -m\log\sinh(\pi z/a).$$

Hence show that the complex potential for two dimensional doublets (line doublets), with their axes parallel to the x-axis, of strength μ at the same points, is given by

$$w = \mu\coth(\pi z/a).$$

(Agra 2000)

Solution. Sources of equal strength m are placed, at $z = \pm nia$ where $n = 0, 1, 2, 3, \ldots$ The complex potential due to this system at any point z is given by

$$w = -m\log(z-0) - \sum_{n=1}^{\infty} m\log(z - nia) - \sum_{n=1}^{\infty} m\log(z + nia)$$

$$= -m\log z - \sum_{n=1}^{\infty} m\log(z^2 + n^2a^2)$$

$$= -\sum_{n=1}^{\infty} m\log\left(1 + \frac{z^2}{n^2a^2}\right) \cdot n^2a^2 \cdot z$$

$$= -\sum_{n=1}^{\infty} m\log\left(1 + \frac{z^2}{n^2a^2}\right)\frac{\pi z}{a} - \sum_{n=1}^{\infty} m\log\left(n^2a^2\frac{a}{\pi}\right)$$

Neglecting constant, $w = -\sum_{n=1}^{\infty} m\log\frac{\pi z}{a}\left(1 + \frac{z^2}{n^2\pi^2}\right)$

Putting $\dfrac{\theta}{\pi} = \dfrac{z}{a}$, we get

$$w = -\sum_{n=1}^{\infty} m\log\theta\left(1 + \frac{\theta^2}{n^2\pi^2}\right)$$

or $$w = -m\log\theta \cdot \left(1 + \frac{\theta^2}{\pi^2}\right)\left(1 + \frac{\theta^2}{2^2\pi^2}\right)\left(1 + \frac{\theta^2}{3^2\pi^2}\right)$$

$$= -m\log\sinh\theta = -m\log\sinh\left(\frac{\pi z}{a}\right)$$

or $$w = -m\log\sinh(\pi z/a). \quad \ldots (1)$$

This proves the first required result.

Note that $w = -m\log(z-a)$ due to source $+m$ at $z = a$ and $w = m/(z-a)$ due to doublet $+m$ at $z = a$ with its axis along x-axis. i.e. $w = \dfrac{d}{dz}[m\log(z-a)]$ for a doublet $+m$ at $z = a$ with its axis along x-axis.

Therefore the complex potential for the doublets of strength m at these points is negative derivative of (1), so that

$$w = \frac{d}{dz}[m\log\sinh(\pi z/a)].$$

i.e., $$w = \frac{m\pi}{a}\coth\left(\frac{\pi z}{a}\right) = \mu\coth\left(\frac{\pi z}{a}\right)$$

This proves the second required result.

Miscellaneous Problems

Problem 21. *An area A is bounded by that part of the x-axis for which $x > a$ and by that branch of $x^2 - y^2 = a^2$ which is in the positive quadrant. There is a two dimensional unit sink at (a, 0) which sends out liquid uniformly in all directions. Show by means of the transformation $w = \log(z^2 - a^2)$ that in steady motion the stream lines of the liquid within the area A are portions of rectangular hyperbolas. Draw the stream lines corresponding to $\psi = 0, \pi/4$ and $\pi/2$. If ρ_1 and ρ_2 are the distances of a point P within the fluid from the points $(\pm a, 0)$, show that the velocity of the fluid at P is measured by $2PO/\rho_1\rho_2$, O being the origin.* (Garhwal 2001)

Solution. *Step I.* $w = \log(z^2 - a^2)$ is expressible as

$$\phi + i\psi = \log(x^2 - y^2 - a^2 + 2ixy)$$

This $$\Rightarrow \psi = \tan^{-1}\left(\frac{2xy}{x^2 - y^2 - a^2}\right)$$

Stream lines are given by $\psi =$ const. $= k$, say, then

$$\tan^{-1}\left(\frac{2xy}{x^2 - y^2 - a^2}\right) = k$$

or $$\tan k = 2xy/(x^2 - y^2 - a^2).$$

If $k = 0$, then (2) $\Rightarrow 2xy = 0 \Rightarrow x = 0, y = 0$.

If $k = \pi/2$, then (2) $\Rightarrow x^2 - y^2 - a^2 = 0 \Rightarrow x^2 - y^2 = a^2$.

Thus stream lines are parts of the curves $x^2 - y^2 = a^2$, $x = 0$ ... flows in an area A bounded by $x = 0$, $y = 0$, $x^2 - y^2 = a^2$ in the p...

Step II. $w = \log (z^2 - a^2)$ is expressible as

$$w = \log (z - a) + \log (z + a).$$

This proves that the liquid motion is generated by two sinks of strength unity at (a, 0) and $(-a, 0)$. Consequently, the image of sink -1 at $(a, 0)$ is an equal sink at $(-a, 0)$ relative to y-axis *i.e.*, relative to the area A.

Step III. To show that velocity $\quad q = 2.\ OP/\rho_1\rho_2$

We have $\quad w = \log (z^2 - a^2)$.

Hence $\quad \dfrac{dw}{dz} = \dfrac{2z}{z^2 - a^2}$.

This $\quad \Rightarrow q = \left|\dfrac{dw}{dz}\right| = \dfrac{2\,|z|}{|z - a|\,.\,|z + a|}$.

Let P be a point within the fluid. Then $|z| = |z - 0| = OP$,

$\rho_1 = |z - a| =$ distance between P and $(a, 0)$,

$\rho_2 = |z + a|$ distance between P and $(-a, 0)$

Thus $\quad q = \dfrac{2OP}{\rho_1 \rho_2}$.

Step IV. To determine stream lines corresponding to

$$\psi = 0, \frac{\pi}{4}, \frac{\pi}{2},$$

By (1), $\quad \tan \psi = \dfrac{2xy}{x^2 - y^2 - a^2}$

Putting $\quad \psi = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}$, we obtain

$$\frac{2xy}{x^2 - y^2 - a^2} = \tan 0 = 0, \quad \frac{2xy}{x^2 - y^2 - a^2} = \tan\frac{\pi}{4} = 1,$$

$$\frac{2xy}{x^2 - y^2 - a^2} = \infty$$

i.e. $\quad xy = 0;\ x^2 - y^2 - a^2 = 2xy\ ;\ x^2 - y^2 - a^2 = 0$

i.e. $\quad x = 0\ y = 0\ ;\ x^2 - y^2 - 2xy - a^2 = 0;\ x^2 - y^2 = a^2.$

Thus stream lines lie along

(i) x and y-axes relative to $\psi = 0$

(ii) the curve $x^2 - y^2 - 2xy - a^2 = 0$ relative to $\psi = \pi/4$

(iii) rectangular hyperbola $x^2 - y^2 = a^2$ relative to $\psi = \pi/2$.

Problem 22. *Show that the velocity vector* q *is everywhere tangent to the lines in xy-plane along which* $\psi (x, y) =$ const.

Solution. Given $\quad \psi (x, y) =$ const. $\quad \ldots (1)$

$$(1) \Rightarrow d\psi = 0 \Rightarrow \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0$$

But $\quad u = -\dfrac{\partial \psi}{\partial y},\ v = \dfrac{\partial \psi}{\partial x}$

Now the last gives $\quad v\,dx - u\,dy = 0,$

or $\quad \dfrac{dy}{dx} = \dfrac{v}{u}$

slope m_1 of the tangent to the curve (1) is $m_1 = v/u$.

But slope of direction of velocity q is $\dfrac{v}{u}$.

Consequently, direction of velocity is tangent to $\psi =$ const.

Problem 23. *A velocity field is given by* $\mathbf{q} = -x\mathbf{i} + (y + t)\,\mathbf{j}$. *Find the stream function and steam lines for this field at* $t = 2$.

Solution. $\quad \mathbf{q} = u\mathbf{i} + v\mathbf{j} = -x\mathbf{i} + \mathbf{j}\,(x + t)$

$$u = -x \qquad v = y + t$$

But $\quad u = -\dfrac{\partial \psi}{\partial y}, \quad v = \dfrac{\partial \psi}{\partial x}$

$$\frac{\partial \psi}{\partial y} = x, \quad \frac{\partial \psi}{\partial x} = y + t$$

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy$$

$$d\psi = (y + t)\,dx + x\,dy \qquad \ldots (1)$$

$$= M dx + N dy, \text{ say}$$

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

$\therefore\quad Mdx + Ndy$ is exact.

Solution of (1) is given by

$$\psi = \int (y + t)\,dx + \int 0\,dy = x\,(y + t) + c$$

or $\quad \psi = x\,(y + t) + c \qquad \ldots (2) \quad$ **Ans.**

This is the required expression for stream function. Stream lines at $t = 2$ are given by $(\psi)_{t=2} =$ const.

or $\quad x\,(y + 2) + c =$ const.

or $\quad x\,(y + 2) = a. \qquad$ **Ans.**

Problem 24. *Prove that in the two dimensional liquid motion due to any number of sources at different points on a circle, the circle is a stream line provided that there is no boundary and that the algebraic sum of strengths of the sources is zero.*

Show that the same is true if the region of flow is bounded by a circle in which cuts orthogonally the circle in question. **(Kanpur 1991)**

Solution. Suppose $A_1, A_2, A_3, \ldots$ are the positions of the sources of strengths $m_1, m_2, m_3, \ldots$

Take any point P on the circle and the diameter through it as the initial line.

Let $\angle A_1PA = \theta$, $\angle A_2PA_1 = \alpha_1$,

$\angle A_3PA_2 = \alpha_2$, ... etc.

Then stream line is given by

$$\psi = -m_1\theta - m_2(\theta + \alpha_1) - m_3(\theta + \alpha_1 + \alpha_2) \dots$$
$$= -(m_1 + m_2 + m_3 + \dots)\theta - [m_2\alpha_1 + m_3(\alpha_1 + \alpha_2) + \dots]$$

or $\psi = -\theta\Sigma m$ – const.

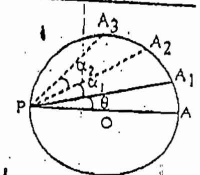


Fig. 3.29.

For whatever be the position of P, $\alpha_1, \alpha_2, \dots$ etc. do not change. Since the angle subtended at the circumference by an arc is always the same.

If the algebraic sum of the strengths is zero, *i.e.*, if $\Sigma m = 0$, then ψ = const. meaning thereby circle is a steam line. Hence the result.

Second Part. Let O' be the centre of a circle which cuts the given circle orthogonally. Join O' to A_1. If $O'A_1$ cuts the original at A', then A' is the inverse point of A_1 w.r.t. circle whose centre is O'.

Relative to the circle whose centre is O', the image of source $+m_1$ at A_1 is a source $+m_1$ at A' and sink $-m_1$ at O'. If the barriers are omitted, we are left with system $2\,\Sigma m$ on the original circle and $-\Sigma m$ at O' and as $\Sigma m = 0$, we again get the same result, *i.e.*, ϕ = const.

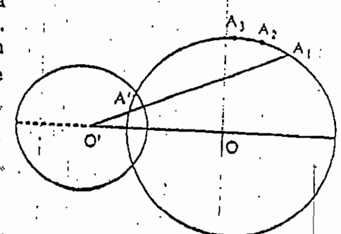


Fig. 3.30.

Problem 25. *Find the velocity potential when there is a source and an equal sink inside a circular cavity and show that one of the stream lines is an arc of the circle which passes through the source and sink and cuts orthogonally the boundary of the cavity.* (Kanpur 90)

Solution. Consider a source $+m$ at A and a sink $-m$ at B respectively inside an circular cavity whose centre is O and radius is a. Let $OA = b$, $OB = c$, $\angle BOA = \alpha$. Let A' and B', be respectively inverse points of A and B respectively. Then

$$OA \cdot OA' = a^2 = OB \cdot OB'$$

The image of source $+m$ at A is a sink $+m$ at A' and a sink $-m$ at O. The image of sink $-m$ at B is a sink $-m$ at B' and a source $+m$ at O. The source $+m$ and sink $-m$ both at O cancel each other. Thus w is given by

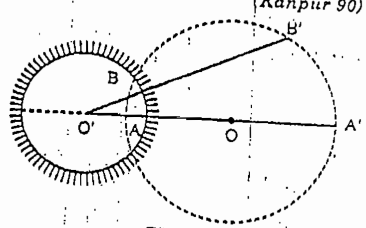


Fig. 3.31

$$w = -m\log(z - b) - m\log\left(z - \frac{a^2}{b}\right) + m\log(z - c\,e^{i\alpha}) + m\log\left(z - \frac{a^2}{c}e^{i\alpha}\right)$$

Equating real and imaginary parts from both sides, we can easily get velocity potential and stream function, respectively.

Since $OA \cdot OA' = OB \cdot OB' = a^2$.

Hence points A, A', B, B' are concyclic. Let the circle through these points meet the cavity in C and C'. Then $OA\,OA' = OC^2$. Hence OC is tangent at C to the circle through B, B', A, A'. It declares the fact that the two circles intersect orthogonally. Also the circle through A, A', B, B' passes through A' and B, *i.e.*, the same source and sink, hence it must be a stream line.

Problem 26. *Prove that for liquid circulating irrotationally in part of the plane between two non-intersecting circles the curves of constant velocity are Cassini's ovals.*

Solution. Suppose CC' is the line of centres. Take two points A and B s.t. they are inverse points w.r.t. both the circles. Consider a point P on one of the circles.

Write $PA = r$, $PC = r_1$.

Since $CA \cdot CB = CP^2$, hence ΔCPA and ΔCPB are similar so that

$$\frac{CP}{CB} = \frac{PA}{PB}\ \textit{i.e.,}\ \frac{CP}{CB} = \frac{r}{r_1}\ \text{ or }\ \frac{r}{r_1} = \text{const.}$$

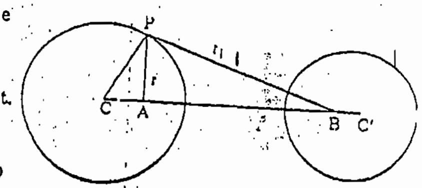


Fig. 3.32.

It means the equations of the two circles can be written as

$$\frac{r}{r_1} = k_1,\ \frac{r}{r_2} = k_2,\ \text{say.}$$

Also these two circles are stream lines, hence ψ must be of the form $\psi = f(r/r_1)$, but $f(r/r_1)$ is plane harmonic. Consequently $\psi = f(r/r_1) = A\log(r/r_1)$ as $\log r$ is the only function of r which is plane harmonic. Hence

$$\phi = -A(\theta - \theta_1)\ \text{as}\ \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$
$$w = \phi + i\psi = -A(\theta - \theta_1) + iA\log(r/r_1)$$
$$= iA\,[\log(r/r_1) + i(\theta - \theta_1)]$$
$$= iA\log\left[\frac{r}{r_1}e^{i(\theta - \theta_1)}\right] = iA\log\left[\frac{r\,e^{i\theta}}{r_1\,e^{i\theta_1}}\right]$$

Choosing A to be $(-a, 0)$ and B to be $(a, 0)$, then

$$w = iA\log\left(\frac{z + a}{z - a}\right).\ \text{This} \Rightarrow$$

$$q = \left|\frac{dw}{dz}\right| = |iA| \cdot \left|\frac{1}{z + a} - \frac{1}{z - a}\right| = \frac{2Aa}{|z + a| \cdot |z - a|}$$

or $$q = \frac{2Aa}{rr_1}.$$

Curves of constant velocity are given by

$$q = \text{const., i.e., } \frac{2Aa}{rr_1} = \text{const. i.e.,}$$

rr_1 = const. which are clearly Cassini's ovals.

Problem 27. *If a homogeneous liquid is acted on by a repulsive force from the origin, the magnitude of which at distance r from the origin is μr pre unit mass, show that it is possible for the liquid to move steadily, without being constrained by any boundaries, in the space between one branch of the hyperbola $x^2 - y^2 = a^2$ and the asymptotes and find the velocity potential.*

Solution. The liquid moves steadily between the space given by one branch of $x^2 - y^2 = a^2$ ...(1) and its asymptotes given by

$$x^2 - y^2 = 0. \qquad \dots(2)$$

(1) and (2) are clearly stream lines. For $x^2 - y^2$ is a harmonic function as it satisfies Laplace's equation. Thus

$$\psi = A(x^2 - y^2) = A r^2 (\cos^2\theta - \sin^2\theta) = Ar^2 \cos 2\theta$$

or $$\psi = Ar^2 \sin\left(\frac{\pi}{2} + 2\theta\right), \quad A \text{ being a constant.}$$

Using $\dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$, we get $\phi = Ar^2 \cos\left(\dfrac{\pi}{2} + 2\theta\right)$

$$w = \phi + i\psi = Ar^2\left[\cos\left(\frac{\pi}{2} + 2\theta\right) + i \sin\left(\frac{\pi}{2} + 2\theta\right)\right]$$

or $$w = Ar^2 e^{i\,[(\pi/2) + 2\theta]} = Ar^2 e^{i2\theta} = A(re^{i\theta})^2$$

For $$e^{i(\pi/2)} = 1,$$

Hence $w = Az^2$. Hence $q = \left|\dfrac{dw}{dz}\right| = 2A\,|z| = 2Ar$

In case of steady motion, the equation of motion is

$$\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega = \text{const.} \qquad \dots(1)$$

Given $$-\frac{\partial\Omega}{\partial r} = \mu r. \quad [F = -\nabla\Omega]$$

This $\Rightarrow$ $\Omega = -\dfrac{\mu}{2}r^2$, neglecting constant.

Putting the values in (1),

$$\frac{p}{\rho} + 2A^2r^2 - \frac{\mu}{2}r^2 = \text{const.}$$

Subjecting this to the condition p = const. on free surface, we get

$$2A^2r^2 - \frac{\mu}{2}r^2 = 0 \quad \text{or} \quad A = \sqrt{\mu/2}.$$

Hence, $$q = 2Ar = \frac{2\sqrt{\mu}}{2}\cdot r \quad \text{or} \quad q = r\sqrt{\mu}.$$

$$\text{velocity potential} = \phi = Ar^2 \cos\left(\frac{\pi}{2} + 2\theta\right) = -Ar^2 \sin 2\theta.$$

or $$\phi = -\frac{\sqrt{\mu}}{2}r^2 \sin 2\theta$$

Problem 28. *If the fluid fills the region of space on the positive side of x-axis, is a rigid boundary, and if there be a source + m at the point (0, a) and an equal sink at (0, b), and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is $\pi\rho m^2 (a - b)^2/ab\,(a + b)$, where ρ is the density of the fluid.*

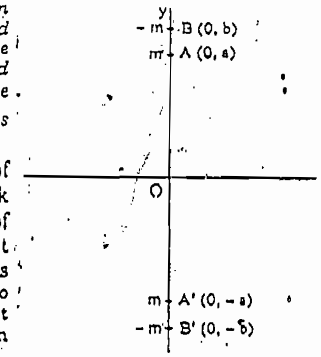


Fig. 3.33.

Solution. The object system consists of source + m at $A(0, a)$, *i.e.*, at $z = ia$ and sink $-m$ at $z = ib$. The image system consists of source $+m$ at $A'(z = -ia)$ and sink $-m$ at $B'(z = -ib)$ w.r.t. the positive line OX which is rigid boundary. The complex potential due to object system with rigid boundary is equivalent to the object system and its image system with no rigid boundary

$\therefore$ $$w = -m \log(z - ia) + m \log(z - ib) - m \log(z + ia) + m \log(z + ib)$$

or $$w = -m \log(z^2 + a^2) + m \log(z^2 + b^2)$$

$$\frac{dw}{dz} = -2mz\left[\frac{1}{z^2 + a^2} - \frac{1}{z^2 + b^2}\right] = \frac{2mz(a^2 - b^2)}{(z^2 + a^2)(z^2 + b^2)}$$

$\therefore$ $$q = \left|\frac{dw}{dz}\right| = \frac{2m(a^2 - b^2)\,|z|}{|z^2 + a^2|\,|z^2 + b^2|}.$$

For any point on x-axis, we have $z = x$ so that

$$q = \frac{2mx(a^2 - b^2)}{(x^2 + a^2)(x^2 + b^2)}$$

This is expression for velocity at any point on x-axis. Let p_0 be the pressure at $x = \infty$. By Bernoulli's equation for steady motion.

$\therefore$ $$\frac{p}{\rho} + \frac{1}{2}q^2 = C.$$

In view of $p = p_0$, $q = 0$ when $x = \infty$, we get $C = p_0/\rho$.

$$\frac{p_0 - p}{\rho} = \frac{1}{2}q^2.$$

Required pressure P on boundary is given by

$$P = \int_{-\infty}^{\infty} (p_0 - p)\,dx = \int_{-\infty}^{\infty} \frac{1}{2}\rho q^2\,dx$$

$$= \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{4m^2 x^2 (a^2-b^2)^2}{(x^2+a^2)^2 (x^2+b^2)^2}\, dx$$

$$= 4\rho m^2 (a^2-b^2)^2 \int_0^{\infty} \frac{x^2\, dx}{(x^2+a^2)^2 (x^2+b^2)^2}$$

$$= 4m^2\rho \int_0^{\infty} \left[\frac{a^2+b^2}{a^2-b^2} \left\{ \frac{1}{x^2+b^2} - \frac{1}{x^2+a^2} \right\} - \frac{a^2}{(x^2+a^2)^2} - \frac{b^2}{(x^2+b^2)^2} \right] dx$$

$$= 4m^2\rho \left[\frac{a^2+b^2}{a^2-b^2} \left\{ \frac{\pi}{2b} - \frac{\pi}{2a} \right\} - \frac{\pi}{4a} - \frac{\pi}{4b} \right]$$

$$= \frac{\pi\rho\, m^2 (a-b)^2}{ab\,(a+b)}. \qquad \text{Ans.}$$

For $\displaystyle\int_0^{\infty} \frac{dx}{x^2+a^2} = \left[\frac{1}{a} \tan^{-1} \frac{x}{a} \right]_0^{\infty} = \frac{\pi}{2a}$

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{a^3} \int_0^{\infty} \cos^2\theta\, d\theta, \; x = a\tan\theta$$

$$= \frac{1}{2a^3} \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta = \frac{\pi}{2} \cdot \frac{1}{2a^3} = \frac{\pi}{4a^3}.$$

Problem 29. *An infinite mass of liquid is moving irrotationally and steadily under the influence of a source of strength μ and an equal sink at a distance $2a$ from it. Prove that the kinetic energy of the liquid which passes in unit time across the plane which bisects at right angles in the line joining the source and sink is $\frac{8\pi}{7a^4}\pi\rho\mu^3$, ρ being the density of the liquid.*

Solution. Consider a source $+\mu$ at $A\,(a, 0, 0)$ and sink $-\mu$ at $B\,(-a, 0, 0)$ s.t. $AB = 2a$. Consider a point $P\,(0, y, 0)$ on Y-axis. Consider a circular strip bounded by the radius y and $y + \delta y$. Mass of the liquid passing through this strip is

$\rho\,(2\pi y \cdot \delta y)\, q = 2\pi\rho\, qy \cdot \delta y$ per unit time $= \delta m$, say.

Recall that $r^2 v =$ const. so that $v =$ const.$/r^2$ is the equation of continuity in case of spherical symmetry.

Hence velocity at P due to source at $A = \dfrac{\mu}{AP^2}$ along AP

velocity at P due to sink at $B = \dfrac{\mu}{BP^2}$ along PB,

$$= \frac{\mu}{AP^2} \text{ along } PB$$

$AP = PB$. Let $\angle PAO = \theta$.

Then resultant velocity at P along AB

$$= \frac{2\mu\cos\theta}{AP^2} = \frac{2\mu\cos\theta}{a^2+y^2} = \frac{2\mu a}{(a^2+y^2)^{3/2}} = q.$$

Required K.E. $= \dfrac{1}{2}\displaystyle\int_0^{\infty} \delta m . q^2 = \frac{1}{2}\int_0^{\infty} 2\pi\rho q y\, q^2\, dy$

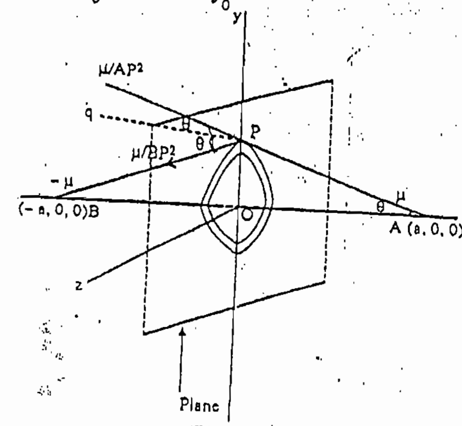


Fig. 3.34.

$$= \pi\rho \int_0^{\infty} \left[\frac{2\mu a}{(a^2+y^2)^{3/2}} \right]^3 y\, dy = 8\pi\rho\, \mu^3 a^3 \int_0^{\infty} \frac{y\, dy}{(a^2+y^2)^{9/2}}$$

$$= \frac{8\pi\rho\mu^3}{a^4} \int_0^{\pi/2} \sin t \cdot \cos^6 t\, dt, \text{ put } y = a\tan t$$

$$\frac{8\pi\rho\mu^3}{a^4} \left[-\frac{1}{7}\cos^7 t \right]_0^{\pi/2} = \frac{8}{7a^4}\pi\rho\mu^3.$$

Problem 30. *A single source is placed in an infinite perfectly elastic fluid, which is also a perfect conductor of heat; show that if the motion be steady, the velocity at distance r from the source satisfies the equation*

$$\left(V - \frac{k}{V} \right) \frac{\partial V}{\partial r} = \frac{2k}{r}$$

and hence that
$$r = \frac{1}{\sqrt{V}} e^{V^2/4k}.$$

Solution. Since the motion is steady and is due to a single source, hence the flow is purely radial. Equation of motion is

$$\frac{dq}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p.$$

Here we have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r},\; \nabla = \frac{\partial}{\partial r},\; \mathbf{F} = 0 \text{ as external forces are absent}$$

Hence $\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)u=-\frac{1}{\rho}\nabla p.$

Motion is steady, $\frac{\partial u}{\partial t}=0,\ p=k\rho$ (Boyle's law)

$$u\frac{\partial u}{\partial r}=-\frac{k}{\rho}\frac{\partial \rho}{\partial r}. \qquad \dots(1)$$

Motion has spherical symmetry and hence equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)=0.$$

But $\frac{\partial \rho}{\partial t}=0$ as the motion is steady.

Hence $\frac{\partial}{\partial r}(\rho u r^2)=0$ or $ur^2\frac{\partial \rho}{\partial r}+\rho r^2\frac{\partial u}{\partial r}+\rho u\,.\,2r=0$

or $$u\frac{\partial \rho}{\partial r}+\rho\frac{\partial u}{\partial r}+2\frac{\rho u}{r}=0. \qquad \dots(2)$$

Eliminating $\frac{\partial \rho}{\partial r}$ from (1) and (2), we get

$$u\left(-\frac{\rho u}{k}\frac{\partial u}{\partial r}\right)+\rho\frac{\partial u}{\partial r}+\frac{2\rho u}{r}=0$$

or $$\frac{2k}{r}=\left(u-\frac{k}{u}\right)\frac{\partial u}{\partial r} \qquad \dots(3)$$

This $\Rightarrow \frac{2k}{r}=\left(u-\frac{k}{u}\right)\frac{du}{dr}$, as $u=u(r)$

or $$\left(u-\frac{k}{u}\right)du=\frac{2k}{r}dr$$

Integrating, $\frac{u^2}{2}-k\log u=2k\log r+\log A$

or $$\frac{u^2}{2}=k\log(r^2A_1\,.\,u),\ \text{where } k\log A_1=\log A.$$

or $$r^2u\,A_1=e^{u^2/2k}.\ \text{ Take } A_1=1, \text{ we get}$$

$$r=\frac{1}{\sqrt{u}}e^{u^2/4k} \qquad \dots(4)$$

Replacing u by V in (3) and (4), we et the two required results.

Problem 31. *In two dimensional irrotational fluid motion, show that if the stream lines are confocal ellipses,*

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

$$\psi=A\log\left[\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}\right]+B$$

and the velocity at any point is inversely proportional to the square root of the rectangle under the focal radii of the point.

Solution. The conformal transformation $z=c\cos w$ $\dots(1)$
is known to yield the given type of confocal ellipses.

(1) $\Rightarrow x+iy=c\cos(\phi+i\psi)\Rightarrow x=c\cos\phi\cosh\psi,\ y=c\sin\phi\sinh\psi$

Eliminating ϕ, we get

$$\frac{x^2}{c^2\cosh^2\psi}+\frac{y^2}{c^2\sinh^2\psi}=1.$$

Stream lines are given by $\psi=$ const. By virtue of this, (2) declares that stream lines are confocal ellipses. Comparing (2) with the equation.

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,\ \text{ we get } c\cosh\psi=\sqrt{(a^2+\lambda)},\ c\sinh\psi=\sqrt{(b^2+\lambda)} \quad \dots(3)$$

This $\Rightarrow$ $c(\cosh\psi+\sinh\psi)=\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}$

or $$ce^{\psi}=\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}$$

or $$\psi=\log\left[\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}\right]-\log c \qquad \dots(4)$$

If $w=\phi+i\psi$ is the complex potential of some fluid motion, then so is Aw. Hence (4) gives

$$\psi=A\log\left[\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}\right]-B.$$

Velocity. (1) $\Rightarrow dz/dw=-c\sin w=-c\sqrt{[1-(z^2/c^2)]}$

$$q^{-1}=\frac{1}{q}=\left|-\frac{dz}{dw}\right|=\sqrt{|c^2-z^2|}=\sqrt{[|c-z|\,.\,|c+z|]} \qquad \dots(5)$$

By (3), $c^2(\cosh^2\psi-\sinh^2\psi)=(a^2+\lambda)-(b^2+\lambda)=a^2-b^2$

or $c^2=a^2-a^2(1-e^2)$. For $b^2=a^2(1-e^2)$

or $c=ae.$

Now (5) becomes $q^{-1}=\sqrt{[|z-ae|\,.\,|z+ae|]}$ $\dots(6)$

$(\pm ae, 0)$ are co-ordinates of foci, denoted by S and S'. P is a point z. Then $r_1=SP=|z-ae|$,

$$r_2=S'P=|z+ae|.$$

Now (6) is expressible as

$$q^{-1}=\sqrt{(r_1r_2)}\quad\text{or}\quad q=\frac{1}{\sqrt{(r_1r_2)}}$$

From this the reqired result follows.

Problem 32. *λ denoting a variable parameter, and f a given function, find the condition that $f(x,y,\lambda)=0$ should be a possible system of stream lines for steady irrotational motion in two dimensions.*

Solution. Suppose $f(x,y,\lambda)=0$ represents stream lines for different values of λ. Solving this equation, we get

$$\lambda=F(x,y). \qquad \dots(1)$$

We also know that $\psi=$ const. represents stream lines. So we can suppose that (1) and $\phi=c$ both represent the same stream lines. It means that

$$\psi=\psi(\lambda).\quad \text{Now } \frac{\partial\psi}{\partial x}=\frac{d\psi}{d\lambda}\frac{\partial\lambda}{\partial x}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{d^2 \psi}{d\lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2 + \frac{d\psi}{d\lambda} \cdot \frac{\partial^2 \lambda}{\partial x^2} \qquad \dots (3)$$

But the motion is irrotational and so, $\nabla^2 \psi = 0$.

i.e.,
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

In view of (3), this becomes

$$\left[\frac{d^2 \psi}{d\lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2 \lambda}{\partial x^2}\right] + \left[\frac{d^2 \psi}{d\lambda^2}\left(\frac{\partial \lambda}{\partial y}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2 \lambda}{\partial y^2}\right] = 0.$$

or,
$$\frac{d^2 \psi}{d\lambda^2}\left[\left(\frac{\partial \lambda}{\partial x}\right)^2 + \left(\frac{\partial \lambda}{\partial y}\right)^2\right] + \frac{d\psi}{d\lambda}\left[\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2}\right] = 0$$

This is the required condition. **Ans.**

* DYNAMICS *

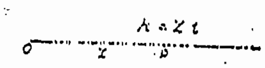
2

Rectilinear Motion. (S.H.M.)

§ 1. Introduction. When a point (or particle) moves along a straight line, its motion is said to be a rectilinear motion. Hence in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical.

§ 2. Velocity and acceleration.

Suppose a particle moves along a straight line OX where O is a fixed point on the line. Let P be the position of the particle at time t, where $OP=x$. If $\mathbf{r}$ denotes the position vector of P and $\mathbf{i}$ denotes the unit vector along OX, then $\mathbf{r}=\vec{OP}=x\,\mathbf{i}$.



Let $\mathbf{v}$ be the velocity vector of the particle at P. Then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(x\,\mathbf{i})=\frac{dx}{dt}\,\mathbf{i}+x\frac{d\mathbf{i}}{dt}=\frac{dx}{dt}\,\mathbf{i},$$

because $\mathbf{i}$ is a constant vector. Obviously the vector $\mathbf{v}$ is collinear with the vector $\mathbf{i}$. Thus for a particle moving along a straight line the direction of velocity is always along the line itself. If at P the particle be moving in the direction of x increasing (*i.e.*, in the direction OX) and if the magnitude of its velocity *i.e.*, its *speed* be v, we have

$$\mathbf{v}=v\,\mathbf{i}=\frac{dx}{dt}\,\mathbf{i}. \quad \text{Therefore } \frac{dx}{dt}=v.$$

On the other hand if at P the particle be moving in the direction of x decreasing (*i.e.*, in the direction XO) and if the magnitude of its velocity be v, we have

$$\mathbf{v}=-v\,\mathbf{i}=\frac{dx}{dt}\,\mathbf{i}. \quad \text{Therefore, } \frac{dx}{dt}=-v.$$

Remember. *In the case of a rectilinear motion the velocity of a particle at time t is dx/dt along the line itself and is taken with positive or negative sign according as the particle is moving in the direction of x increasing or x decreasing.*

Now let **a** be the acceleration vector of the particle at P. Then

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\frac{dx}{dt}\,\mathbf{i}\right)=\frac{d^2x}{dt^2}\,\mathbf{i}.$$

Thus the vector **a** is collinear with **i** *i.e.*, the direction of acceleration is always along the line itself. If at P the acceleration be acting in the direction of x increasing and if its magnitude be f, we have $\mathbf{a}=f\,\mathbf{i}=\frac{d^2x}{dt^2}\,\mathbf{i}$. Therefore $\frac{d^2x}{dt^2}=f$. On the other hand if at P the acceleration be acting in the direction of x decreasing and if its magnitude be f, we have

$$\mathbf{a}=-f\,\mathbf{i}=\frac{d^2x}{dt^2}\,\mathbf{i}\;;\text{ therefore }\frac{d^2x}{dt^2}=-f.$$

Remember. *In the case of a rectilinear motion the acceleration of a particle at time t is d^2x/dt^2 along the line itself and is taken with positive or negative sign according as it acts in the direction of x increasing or x decreasing.*

Since the acceleration is produced by the force, therefore while considering the sign of d^2x/dt^2 we must notice the direction of the acting force and not the direction in which the particle is moving. For example if the direction of the acting force is that of x increasing, then d^2x/dt^2 must be taken with positive sign whether the particle is moving in the direction of x increasing or in the direction of x decreasing.

Other Expressions for acceleration :

Let $v=\frac{dx}{dt}$. We can then write

$$\frac{d^2x}{dt^2}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\,\frac{dv}{dx}.$$

Thus $\frac{d^2x}{dt^2}$, $\frac{dv}{dt}$ and $v\,\frac{dv}{dx}$ are three expressions for representing the acceleration and any one of them may be used to suit the convenience in working out the problems.

Note. Often we denote dx/dt by $\dot{x}$ and d^2x/dt^2 by $\ddot{x}$.

Illustrative Examples :

Ex. 1. *If at time t the displacement x of a particle moving away from the origin is given by $x=a\sin t+b\cos t$, find the velocity and acceleration of the particle.* [Meerut 1977]

Sol. Given that $x=a\sin t+b\cos t$.

Differentiating w.r.t. 't', we have the velocity $v=dx/dt=a\cos t-b\sin t$.

Differentiating again, we have the acceleration $=dv/dt=-a\sin t-b\cos t=-x$.

Ex. 2. *A point moves in a straight line so that its distance s from a fixed point at any time t is proportional to t^n. If v be the velocity and f the acceleration at any time t, show that*

$$v^2=nfs/(n-1).$$ [Meerut 1981, 84 P, 85 S]

Sol. Here, distance $s\propto t^n$.

$\therefore$ let $s=k\,t^n$, ...(1)

where k is a constant of proportionality.

Differentiating (1), w.r.t. 't', we have the velocity $v=ds/dt=knt^{n-1}$. ...(2)

Again differentiating (2), the acceleration $f=dv/dt=kn\,(n-1)\,t^{n-2}$. ...(3)

$$\therefore\quad v^2=(knt^{n-1})^2=k^2n^2t^{2n-2}$$

$$=\frac{n.\{kn(n-1)\,t^{n-2}\}.kt^n}{(n-1)}$$

$$=\frac{nfs}{(n-1)},\text{ substituting from (1) and (3).}$$

Ex. 3. *A particle moves along a straight line such that its displacement x, from a point on the line at time t, is given by*

$$x=t^3-9t^2+24t+6.$$

Determine (i) the instant when the acceleration becomes zero, (ii) the position of the particle at that instant and (iii) the velocity of the particle, then. [Meerut 1971]

Sol. Here, $x=t^3-9t^2+24t+6$.

$\therefore$ the velocity $v=dx/dt=3t^2-18t+24$ and the acceleration $f=d^2x/dt^2=6t-18$.

(i) Now the acceleration $=0$, when $6t-18=0$ or $t=3$. Thus the acceleration is zero when $t=3$ seconds.

(ii) When $t=3$, position of the particle is given by $x=3^3-9.3^2+24.3+6=24$ units.

(iii) When $t=3$, the velocity $v=3\cdot3^2-18\cdot3+24=-3$ units. Thus when $t=3$, the velocity of the particle is 3 units in the direction of x decreasing.

Ex. 4. *A particle moves along a straight line and its distance from a fixed point on the line is given by $x=a\cos(\mu t+\epsilon)$. Show that its acceleration varies as the distance from the origin and is directed towards the origin.*

Sol. We have $x=a\cos(\mu t+\epsilon)$. ...(1)

Differentiating w.r.t. t, we get

$dx/dt = -a\mu \sin(\mu t + \epsilon)$,

and $d^2x/dt^2 = -a\mu^2 \cos(\mu t + \epsilon) = -\mu^2 x$. [from (1)]

Hence the acceleration varies as the distance x from the origin. The negative sign indicates that it is in the negative sense of x-axis i.e., towards the origin.

Ex. 5. *A particle moves along a straight line such that its distance x from a fixed point on it and the velocity v there are related by $v^2 = \mu(a^2 - x^2)$. Prove that the acceleration varies as the distance of the particle from the origin and is directed towards the origin.* [Agra 1975]

Sol. We have $v^2 = \mu(a^2 - x^2)$. ...(1)

Differentiating (1) w.r.t. x, we get

$$2v\frac{dv}{dx} = \mu(-2x). \quad \therefore \quad \frac{d^2x}{dt^2} = v\frac{dv}{dx} = -\mu x.$$

Hence the acceleration varies as the distance x from the origin. The negative sign indicates that it is in the direction of x decreasing i.e., towards the origin.

Ex. 6. *The velocity of a particle moving along a straight line, when at a distance x from the origin (centre of force) varies as $\sqrt{\{(a^2 - x^2)/x^2\}}$. Find the law of acceleration.* [Agra 1979]

Sol. Let v be the velocity of the particle when it is at a distance x from the origin. Then according to the question, we have $v = \mu\sqrt{\{(a^2 - x^2)/x^2\}}$, where μ is a constant.

$\therefore \quad v^2 = \mu^2 (a^2 - x^2)/x^2 = \mu^2 (a^2/x^2 - 1)$.

Differentiating w.r.t. x, we get

$$2v\frac{dv}{dx} = \mu^2\left(-\frac{2a^2}{x^3}\right). \quad \therefore \quad v\frac{dv}{dx} = \frac{d^2x}{dt^2} = -\frac{\mu^2 a^2}{x^3}.$$

Hence the acceleration varies inversely as the cube of the distance from the origin and is directed towards the centre of force.

Ex. 7. *The law of motion in a straight line being given by $s = \frac{1}{2} vt$, prove that the acceleration is constant.* [Meerut 1979]

Sol. We have $s = \frac{1}{2} vt = \frac{1}{2}\frac{ds}{dt} t$. $\left[\because \ v = \frac{ds}{dt}\right]$

Differentiating w.r.t. 't' we get

$$\frac{ds}{dt} = \frac{1}{2}\frac{d^2s}{dt^2} t + \frac{1}{2}\frac{ds}{dt} \quad \text{or} \quad \frac{1}{2}\frac{ds}{dt} = \frac{1}{2}\frac{d^2s}{dt^2} t$$

or $\quad \frac{ds}{dt} = \frac{d^2s}{dt^2} t$.

Differentiating again w.r.t. t, we get

$$\frac{d^2s}{dt^2} = \frac{d^2s}{dt^2} + \frac{d^3s}{dt^3} t \quad \text{or} \quad \frac{d^3s}{dt^3} t = 0 \quad \text{or} \quad \frac{d^3s}{dt^3} = 0$$

because $t \neq 0$.

Now $\frac{d^3s}{dt^3} = 0 \Rightarrow \frac{d}{dt}\left(\frac{d^2s}{dt^2}\right) = 0 \Rightarrow \frac{d^2s}{dt^2} = \text{constant}$.

Hence the acceleration is constant.

Ex. 8. *A point moves in a straight line so that its distance from a fixed point in that line is the square root of the quadratic function of the time; prove that its acceleration varies inversely as the cube of the distance from the fixed point.*

Sol. At any time t, let x be the distance of the particle from a fixed point on the line. Then according to the question, we have $x = \sqrt{(at^2 + 2bt + c)}$, where a, b, c are constants.

$\therefore \quad x^2 = at^2 + 2bt + c$. ...(1)

Differentiating w.r.t. t, we get

$$2x\frac{dx}{dt} = 2at + 2b$$

or $\quad \frac{dx}{dt} = \frac{at + b}{x}$. ...(2)

Differentiating again w.r.t. 't', we have

$$\frac{d^2x}{dt^2} = \frac{ax - (at+b)(dx/dt)}{x^2} = \frac{ax - (at+b)\{(at+b)/x\}}{x^2}, \text{ [from (2)]}$$

$$= \frac{ax^2 - (at+b)^2}{x^3} = \frac{a(at^2 + 2bt + c) - (a^2t^2 + 2abt + b^2)}{x^3}$$

$$= \frac{ac - b^2}{x^3} = (\text{some constant}) \cdot \frac{1}{x^3}.$$

Hence the acceleration varies inversely as the cube of the distance x from the fixed point.

Ex. 9. *If a point moves in a straight line in such a manner that its retardation is proportional to its speed, prove that the space described in any time is proportional to the speed destroyed in that time.*

Sol. Here it is given that the retardation $\propto$ speed.

$\therefore \quad -\frac{dv}{dt} = kv$, where k is a constant of proportionality

or $\quad -v\frac{dv}{dx} = kv$ or $dx = -\frac{1}{k} dv$.

Integrating, $x = -(v/k) + A$,

where A is constant of integration,

Suppose the particle starts from the origin with velocity u. Then $v=u$, $x=0$.

$$\therefore \quad 0=-\frac{u}{k}+A \quad \text{or} \quad A=\frac{u}{k}.$$

$$\therefore \quad x=-\frac{v}{k}+\frac{u}{k}=\frac{1}{k}(u-v)$$

or $$(u-v)=kx. \qquad \ldots(1)$$

Now the space described in time t is x and the speed destroyed in time $t=u-v$. Hence from (1), we conclude that the space described in any time is proportional to the speed destroyed in that time.

Ex. 10. *Prove that if a point moves with a velocity varying as any power (not less than unity) of its distance from a fixed point which it is approaching, it will never reach that point.*

Sol. If x is the distance of the particle from the fixed point O at any time t, then its speed v at that time is given by $v=kx^n$, where k is a constant and n is not less than 1.

Since the particle is moving towards the fixed point *i.e.*, in the direction of x decreasing, therefore

$$dx/dt=-v$$

or $$dx/dt=-kx^n. \qquad \ldots(1)$$

Case I. If $n=1$, then from (1), we have

$$dx/dt=-kx$$

or $$dt=-\frac{1}{k}\frac{dx}{x}.$$

Integrating, $t=-(1/k)\log x+A$, where A is a constant.

Putting $x=0$, the time t to reach the fixed point O is given by

$$t=-(1/k)\log 0+A=\infty$$

i.e., the particle will never reach the fixed point O.

Case II. If $n>1$, then from (1), we have

$$dt=-\frac{1}{k}x^{-n}\,dx.$$

Integrating, $$t=-\frac{1}{k}\frac{x^{-n+1}}{-n+1}+B, \text{ where } B \text{ is a constant}$$

or $$t=\frac{1}{k(n-1)x^{n-1}}+B.$$

Putting $x=0$, the time t to reach the fixed point O is given by

$$t=\infty+B=\infty$$

i.e., the particle will never reach the fixed point O.

Hence if $n\geq 1$, the particle will never reach the fixed point it is approaching.

Ex. 11. *The velocity of a particle moving along a straight line is given by the relation $v^2=ax^2+2bx+c$. Prove that the acceleration varies as the distance from a fixed point in the line.*

Sol. Here given that $v^2=ax^2+2bx+c$.

Differentiating w.r.t. 'x', we have

$$2v\frac{dv}{dx}=2ax+2b$$

or $$f=v\frac{dv}{dx}=ax+b=a\left(x+\frac{b}{a}\right).$$

Let P be the position of the particle at time t.

If $x=-(b/a)$ is the fixed point O', then the distance of the particle at time t from O'

$$=O'P=x-\left(-\frac{b}{a}\right)=x+\frac{b}{a}.$$

$$\therefore \quad f=a.O'P \quad \text{or} \quad f\propto O'P.$$

Hence the acceleration varies as the distance from a fixed point $x=-(b/a)$ in the line.

Ex. 12. *If t be regarded as a function of velocity v, prove that the rate of decrease of acceleration is given by $f^3\,(d^2t/dv^2)$, f being the acceleration.* [Meerut 80, 83]

Sol. Let f be the acceleration at time t. Then $f=dv/dt$.

Now the rate of decrease of acceleration $=-df/dt$

$$=-\frac{d}{dt}\left(\frac{dv}{dt}\right)=-\frac{d}{dt}\left(\frac{dt}{dv}\right)^{-1},$$

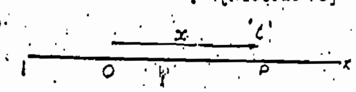
regarding t as a function of v

$$=-\left\{\frac{d}{dv}\left(\frac{dt}{dv}\right)^{-1}\right\}\cdot\frac{dv}{dt}=\left(\frac{dt}{dv}\right)^{-2}\frac{d^2t}{dv^2}\cdot\frac{dv}{dt}$$

$$=\left(\frac{dv}{dt}\right)^2\cdot\frac{dv}{dt}\cdot\frac{d^2t}{dv^2}=\left(\frac{dv}{dt}\right)^3\cdot\frac{d^2t}{dv^2}=f^3\frac{d^2t}{dv^2}.$$

§ 2. Motion under constant acceleration. *A particle moves in a straight line with a constant acceleration f, the initial velocity being u, to discuss the motion.* [Meerut 78]

Suppose a particle moves in a straight line OX starting from O with velocity u. Take O as origin. Let P be the position of the particle at any time t, where $OP=x$. The acceleration of P is constant and is f. Therefore the equation of motion of P is



$$\frac{d^2x}{dt^2}=f. \qquad \ldots(1)$$

If v is the velocity of the particle at any time t, then $v=dx/dt$. So integrating (1) w.r.t. t, we get

$v=dx/dt=ft+A$, where A is constant of integration.

But initially at O, $v=u$ and $t=0$; therefore $A=u$. Thus we have

$$v=\frac{dx}{dt}=u+ft. \qquad ...(2)$$

The equation (2) gives the velocity v of the particle at any time t.

Now integrating (2) w.r.t. 't', we get

$x=ut+\frac{1}{2}ft^2+B$, where B is a constant.

But at O, $t=0$ and $x=0$; therefore $B=0$. Thus we have

$$x=ut+\tfrac{1}{2}ft^2. \qquad ...(3)$$

The equation (3) gives the position of the particle at any time t.

The equation of motion (1) can also be written as

$$v\frac{dv}{dx}=f \quad \text{or} \quad 2v\frac{dv}{dx}=2f.$$

Integrating it w.r.t. x, we get

$v^2=2fx+C$. But at O, $x=0$ and $v=u$; therefore $C=u^2$. Hence we have

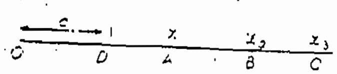
$$v^2=u^2+2fx. \qquad ...(4)$$

Thus in equations (2), (3) and (4) we have obtained the three well known formulae of rectiliner motion with constant acceleration.

Illustrative Examples

Ex. 13. *A particle moves in a straight line with constant acceleration and its distances from the origin O on the line (not necessarily the position at time $t=0$) at times t_1, t_2, t_3 are x_1, x_2, x_3 respectively. Show that if t_1, t_2, t_3 form an A. P. whose common difference is d and x_1, x_2, x_3 are in G. P., then the acceleration is*
$$(\sqrt{x_1}-\sqrt{x_3})^2/d^2.$$

Sol. Let O be the origin and D the point of start i.e., the position at $t=0$.



Let $OD=c$. Suppose u is the initial velocity and f the constant acceleration. Let A, B, C be the positions of the particle at times t_1, t_2, t_3 respectively and let $OA=x_1$, $OB=x_2$ and $OC=x_3$. Then

$$x_1-c=ut_1+\tfrac{1}{2}ft_1^2,\ x_2-c=ut_2+\tfrac{1}{2}ft_2^2,\ x_3-c=ut_3+\tfrac{1}{2}ft_3^2.$$

These equations give

$$x_1+x_3-2x_2=u.(t_1+t_3-2t_2)+\tfrac{1}{2}f(t_1^2+t_3^2-2t_2^2). \qquad ...(1)$$

But x_1, x_2, x_3 are in G.P., so that $x_2=\sqrt{(x_1x_3)}$. Also t_1, t_2, t_3 are in A.P. whose common difference is d. Therefore $t_1+t_3=2t_2$ and $t_3-t_1=2d$. Putting these values in (1), we get

$$x_1+x_3-2\sqrt{(x_1x_3)}=u.0+\tfrac{1}{2}f\left[t_1^2+t_3^2-2\left(\frac{t_1+t_3}{2}\right)^2\right].$$

$$\therefore \quad (\sqrt{x_1}-\sqrt{x_3})^2=\tfrac{1}{4}f[2t_1^2+2t_3^2-(t_1^2+t_3^2+2t_1t_3)]$$
$$=\tfrac{1}{4}f(t_3-t_1)^2=\tfrac{1}{4}f(2d)^2=fd^2.$$

Hence $\quad f=(\sqrt{x_1}-\sqrt{x_3})^2/d^2.$

Ex. 14. *Two cars start off to race with velocities u and u' and travel in a straight line with uniform accelerations f and f' respectively. If the race ends in a dead heat, prove that the length of the course is*

$$\{2(u-u')(uf'-u'f)\}/(f-f')^2.$$

Sol. Let s be the length of the course. By dead heat we mean that each car moves the distance s in the same time, say t. Then considering the motion of the first car we have $s=ut+\frac{1}{2}ft^2$; and considering the motion of the second car, we have $s=u't+\frac{1}{2}f't^2$. These equations can be written as

$$\tfrac{1}{2}ft^2+ut-s=0, \qquad ...(1)$$

and

$$\tfrac{1}{2}f't^2+u't-s=0. \qquad ...(2)$$

By the method of cross multiplication, we get from (1) and (2)

$$\frac{t^2}{\begin{vmatrix} u & -s \\ u' & -s \end{vmatrix}}=\frac{t}{\begin{vmatrix} -s & \frac{1}{2}f \\ -s & \frac{1}{2}f' \end{vmatrix}}=\frac{1}{\begin{vmatrix} \frac{1}{2}f & u \\ \frac{1}{2}f' & u' \end{vmatrix}}$$

or

$$\frac{t^2}{(u'-u)s}=\frac{t}{\frac{1}{2}s(f-f')}=\frac{1}{\frac{1}{2}(fu'-f'u)}.$$

Eliminating t, we have

$$\frac{(u'-u)s}{\frac{1}{2}(fu'-f'u)}=\left[\frac{\frac{1}{2}s(f-f')}{\frac{1}{2}(fu'-f'u)}\right]^2=\frac{s^2(f-f')^2}{(fu'-f'u)^2}.$$

Since $s\neq 0$, therefore $s=\{2(u'-u)(fu'-f'u)\}/(f-f')^2$
$=\{2(u-u')(uf'-u'f)\}/(f-f')^2.$

Ex. 15. *Two particles P and Q move in a straight line AB. The particle P starts from A in the direction AB with velocity u and constant acceleration f, and at the same time Q starts from B in the direction BA with velocity u_1 and constant acceleration f_1; if they pass one another at the middle point of AB and arrive at the other ends of AB with equal velocities, prove that*

$(u+u_1)(f-f_1)=8(fu_1-f_1u)$.

Sol. Let $AB=2s$. Let v be the velocity of either particle after moving the distance $AB=2s$. Then

$$v^2=u^2+2f(2s)=u_1^2+2f_1(2s).$$

$$\therefore \quad s=\frac{u^2-u_1^2}{4(f_1-f)}.$$

Now let t be the time taken by each particle to reach the middle point of AB. Then each particle moves distance s in time t. Therefore

$$s=ut+\tfrac{1}{2}ft^2=u_1t+\tfrac{1}{2}f_1t^2. \qquad \ldots(1)$$

Since $t\neq 0$, therefore from (1), we have $u+\frac{1}{2}ft=u_1+\frac{1}{2}f_1t$

or $$t=2(u-u_1)/(f_1-f).$$

Now considering the motion of the particle P to cover the first half of the journey AB and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$\frac{u^2-u_1^2}{4(f_1-f)}=u\frac{2(u-u_1)}{(f_1-f)}+\tfrac{1}{2}f\frac{4(u-u_1)^2}{(f_1-f)^2}$$

or $(u+u_1)(f_1-f)=8u(f_1-f)+8f(u-u_1)$ $\quad [\because \; u-u_1\neq 0]$

or $(u+u_1)(f_1-f)=8(uf_1-fu_1)$

or $(u+u_1)(f-f_1)=8(fu_1-f_1u)$.

Ex. 16. *A train travels a distance s in t seconds. It starts from rest and ends at rest. In the first part of journey it moves with constant acceleration f and in the second part with constant retardation f'. Show that if s is the distance between the two stations, then*

$$t=\sqrt{[2s(1/f+1/f')]}.$$

Sol. Let v be the velocity at the end of the first part of the motion, or say in the beginning of the second part of the motion and t_1 and t_2 be the times for the two motions respectively. Then $t=t_1+t_2$.

Let x be the distance described in the first part. Then the distance described in the second part is $s-x$. Considering the first part of the motion with constant acceleration f, we have

$$\left.\begin{aligned} v&=0+ft_1=ft_1,\\ \text{and} \quad v^2&=0+2fx=2fx.\end{aligned}\right\} \qquad \ldots(1)$$

Again considering the second part of the motion with constant retardation f', we have

$$\left.\begin{aligned} 0&=v-f't_2 \text{ i.e., } v=f't_2,\\ \text{and} \quad 0&=v^2-2f'(s-x) \text{ i.e., } v^2=2f'(s-x).\end{aligned}\right\} \qquad \ldots(2)$$

From (1) and (2), we have

$$(s-x)+x=\frac{v^2}{2f'}+\frac{v^2}{2f}, \quad \text{or} \quad s=\frac{v^2}{2}\left(\frac{1}{f}+\frac{1}{f'}\right). \qquad \ldots(3)$$

Also $t_1+t_2=v/f+v/f'=v(1/f+1/f')$. $\qquad \ldots(4)$

Substituting the value of v from (3) in (4), we get

$$t=t_1+t_2=\sqrt{\left\{\frac{2s}{(1/f+1/f')}\right\}}\cdot\left(\frac{1}{f}+\frac{1}{f'}\right)=\sqrt{\left[2s\left(\frac{1}{f}+\frac{1}{f'}\right)\right]}.$$

Ex. 17. *A point moving in a straight line with uniform acceleration describes distances a, b feet in successive intervals of t_1, t_2 seconds. Prove that the acceleration is $2(t_1b-t_2a)/[t_1t_2(t_1+t_2)]$.*

[Kanpur 1981; Meerut 69, 84S]

Sol. Let u be the initial velocity and f be the uniform acceleration of the particle. Then from $s=ut+\frac{1}{2}ft^2$, we have

$$a=ut_1+\tfrac{1}{2}ft_1^2 \qquad \ldots(1)$$

and $$a+b=u(t_1+t_2)+\tfrac{1}{2}f(t_1+t_2)^2. \qquad \ldots(2)$$

Subtracting (1) from (2), we have

$$b=ut_2+\tfrac{1}{2}f(t_2^2+2t_1t_2). \qquad \ldots(3)$$

Multiplying (3) by t_1 and (1) by t_2 and subtracting, we have

$$bt_1-at_2=\tfrac{1}{2}f(t_2^2+2t_1t_2)t_1-\tfrac{1}{2}ft_1^2t_2$$
$$=\tfrac{1}{2}f(t_2^2t_1+t_1^2t_2)=\tfrac{1}{2}ft_1t_2(t_2+t_1).$$

$$\therefore \quad f=\frac{2(bt_1-at_2)}{t_1t_2(t_1+t_2)}.$$

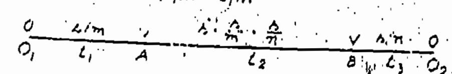
Ex. 18. *For 1/m of the distance between two stations a train is uniformly accelerated and for 1/n of the distance it is uniformly retarded; it starts from rest at one station and comes to rest at the other. Prove that the ratio of its greatest velocity to its average velocity is* $\left(1+\frac{1}{m}+\frac{1}{n}\right):1$.

[Meerut 1977]

Sol. Let O_1 and O_2 be two stations at a distance s apart and A and B two points between O_1 and O_2 such that

$$O_1A=s/m \quad \text{and} \quad BO_2=s/n.$$

$$\therefore \quad AB=s-s/m-s/n.$$



The train starts at rest from O_1 and moves with uniform acceleration f from O_1 to A. Let V be its velocity at the point A. It moves with constant velocity V from A to B and then moves with uniform retardation f' from B to O_2. The velocity at the station O_2 is zero.

Let t_1, t_2, t_3 be the times taken to travel the distances O_1A, AB and BO_2 respectively.

https://t.me/upsc_pdf

Now the greatest velocity of the train during its journey from O_1 to $O_2 = V$ and the average velocity of the train $= s/(t_1+t_2+t_3)$.

$\therefore$ the required ratio $= \dfrac{\text{greatest velocity}}{\text{average velocity}} = \dfrac{V}{s/(t_1+t_2+t_3)}$

$$= \frac{V(t_1+t_2+t_3)}{s} \qquad \ldots(1)$$

For motion from O_1 to A, using the formula $v = u + ft$, we have

$V = 0 + ft_1$. $\quad \therefore \quad f = \dfrac{V}{t_1}$.

Now using the formula $s = ut + \frac{1}{2} ft^2$ for the same motion, we have

$$\frac{s}{m} = 0 + \frac{1}{2}\,\frac{V}{t_1}\,.\,t_1^2$$

or $$t_1 = \frac{2s}{Vm}. \qquad \ldots(2)$$

For motion from A to B, $AB = V.t_2$.

$$\therefore \quad t_2 = \frac{AB}{V} = \frac{s - s/m - s/n}{V}. \qquad \ldots(3)$$

For motion from B to O_2, using the formula $v = u + ft$, we have

$0 = V - f't_3$, $\quad \therefore \quad f' = V/t_3$.

Using the formula $s = ut + \frac{1}{2} ft^2$ for the same motion, we have

$$\frac{s}{n} = Vt_3 - \frac{1}{2}\,.\,\frac{V}{t_3}\,.\,t_3^2 = \frac{Vt_3}{2}$$

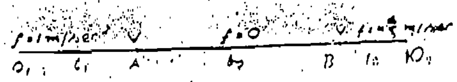
or $$t_3 = \frac{2s}{Vn}. \qquad \ldots(4)$$

Substituting from (2), (3) and (4) in (1), the required ratio

$$= \frac{V\left\{\dfrac{2s}{Vm} + \dfrac{1}{V}\left(s - \dfrac{s}{m} - \dfrac{s}{n}\right) + \dfrac{2s}{Vn}\right\}}{s} = \frac{\dfrac{1}{m} + \dfrac{1}{n} + 1}{1}.$$

Ex. 19. *The greatest possible acceleration of a train is 1 m/sec² and the greatest possible retardation is $\frac{4}{3}$ m/sec². Find the least time taken to run between two stations 12 km. appart if the maximum speed is 22 m/sec.* [Meerut 1972, 76, 88S]

Sol. Let a train start from the station O_1 and move with uniform acceleration 1 m/sec² upto A for time t_1 seconds.



Let the velocity of the train at A be $V = 22$ m/sec. Then the train moves with constant velocity V from A to B for time t_2 seconds. In the last the train moves from B to the second station O_2 under constant retardation $\frac{4}{3}$ m/sec.² for time t_3 seconds. Thus the least time to travel between the two stations O_1 and O_2 is $(t_1+t_2+t_3)$ seconds.

Also $O_1O_2 = 12$ km. $= 12000$ meters.

Now using the formula $v = u + ft$ for the parts O_1A and BO_2 of the journey, we have

$V = 22 = 0 + 1\,.\,t_1$ so that $t_1 = 22$,

and $0 = 22 - \frac{4}{3}\,t_3$ so that $t_3 = \frac{33}{2}$.

Now $O_1A =$ (Average velocity from O_1 to A) $\times t_1$

$$= \frac{0+22}{2} \times 22 = 242 \text{ meters},$$

and $BO_2 = \dfrac{22+0}{2} \times \dfrac{33}{2} = \dfrac{363}{2}$ meters.

$$\therefore \quad AB = O_1O_2 - O_1A - BO_2 = 12000 - 242 - \frac{363}{2}$$

$$= \frac{23153}{2} \text{ meters.}$$

$$\therefore \quad t_2 = \frac{AB}{V} = \frac{23153}{2 \times 22} = \frac{23153}{44} \text{ seconds.}$$

$\therefore$ the required time $= (t_1+t_2+t_3)$ seconds

$$= \left(22 + \frac{33}{2} + \frac{23153}{44}\right) \text{ seconds} = \frac{24847}{44} \text{ seconds}$$

$= 9$ minutes 25 seconds approximately.

Ex. 20. *Two points move in the same straight line starting at the same moment from the same point in the same direction. The first moves with constant velocity u and the second with constant acceleration f (its initial velocity being zero). Show that the greatest distance between the points before the second catches first is $u^2/2f$ at the end of the time u/f from the first.*

Sol. If s_1 and s_2 are the distances moved by the two particles in time t, then

$$s_1 = ut \text{ and } s_2 = 0 + \tfrac{1}{2} ft^2.$$

$\therefore$ the distance s between the two particles at time t is given by $\quad s = s_1 - s_2 = ut - \frac{1}{2} ft^2 = \dfrac{f}{2}\left(\dfrac{2u}{f}\,t - t^2\right)$

or $$s=\frac{f}{2}\left[\frac{u^2}{f^2}-\left(t-\frac{u}{f}\right)^2\right]. \qquad \ldots(1)$$

Now s is greatest if $(t-u/f)^2=0$ i.e., if $t=u/f$.

Also the greatest value of $s=\frac{f}{2}\cdot\frac{u^2}{f^2}=\frac{u^2}{2f}$.

Ex. 21. *The speed of a train increases at constant rate α from zero to v, then remains constant for an interval and finally decreases to zero at a constant rate β. If l be the total distance described, prove that the total time occupied is $(l/v)+(v/2)(1/\alpha+1/\beta)$. Also find the least value of time when $\alpha=\beta$.* [Allahabad 1975]

Sol. Let t_1, t_2, t_3 be the times taken to cover the distances x, y, z of the first, second and last phase of the journey. Whole distance $l=x+y+z$.

Equations for the first and last part of the journey are

$$\left.\begin{array}{l} v^2=2\alpha x, \\ v=\alpha t_1 \end{array}\right\} \ldots(1); \quad \text{and} \quad \left.\begin{array}{l} v^2=2\beta z, \\ v=\beta t_3 \end{array}\right\} \qquad (2)$$

From (1), on eliminating α, we have $x=\frac{1}{2}vt_1$; and from (2), on eliminating β, we have $z=\frac{1}{2}vt_3$.

Also considering the motion for the middle part of the journey, we have $y=vt_2$.

Thus $x+y+z=v\left(\frac{1}{2}t_1+t_2+\frac{1}{2}t_3\right)$

i.e., $$l=v\left[(t_1+t_2+t_3)-\tfrac{1}{2}(t_1+t_3)\right]$$

or $$\frac{l}{v}=(t_1+t_2+t_3)-\tfrac{1}{2}(t_1+t_3).$$

$\therefore$ the total time occupied i.e., $t_1+t_2+t_3=(l/v)+\frac{1}{2}(t_1+t_3)$

$$=\frac{l}{v}+\frac{1}{2}\left(\frac{v}{\alpha}+\frac{v}{\beta}\right), \qquad \text{[from (1) and (2)]}$$

$$=\frac{l}{v}+\frac{1}{2}v\left(\frac{1}{\alpha}+\frac{1}{\beta}\right). \qquad \ldots(3)$$

Let t denote the total time occupied when $\alpha=\beta$.

Then putting $\alpha=\beta$ in the above result (3), we have

$t=\frac{l}{v}+\frac{v}{\alpha}$. Therefore $\frac{dt}{dv}=-\frac{l}{v^2}+\frac{1}{\alpha}$.

For least value of t, we have $dt/dv=0$, i.e., $-\frac{l}{v^2}+\frac{1}{\alpha}=0$

$$\text{i.e., } \frac{l}{v}=\frac{v}{\alpha} \quad \text{i.e., } \quad v=\sqrt{(l\alpha)}.$$

Also then the time $=2\left(\frac{l}{v}\right)=\frac{2l}{\sqrt{(l\alpha)}}=2\sqrt{(l/\alpha)}$. This time is least because $d^2t/dv^2=2l/v^3$ which is positive for $v=\sqrt{(l\alpha)}$.

Ex. 22. *A lift ascends with constant acceleration f, then with constant velocity and finally stops under constant retardation f. If the total distance ascended is s and the total time occupied is t, show that the time during which the lift is ascending with constant velocity is $\sqrt{(t^2-(4s/f))}$.*

Sol. Let v be the maximum velocity produced during the ascent. Since this velocity is produced under a constant acceleration f during the first part of the ascent and destroyed under the same retardation f during the last part of the ascent, therefore, the distances as well as the times for these two ascents are equal. Let x be the distance and t_1 the time for each of these two parts. We have then

$$\left.\begin{array}{l} v^2=2fx, \\ v=ft_1 \end{array}\right\} \qquad \ldots(1)$$

and

for the first and last part of the motion.

Also considering the middle part of the motion, we have

$$v(t-2t_1)=s-2x.$$

From (1) and (2), on eliminating v and x, we have

$$ft_1(t-2t_1)=s-\frac{v^2}{f}=s-\frac{f^2t_1^2}{f}=s-ft_1^2.$$

$$\therefore\ ft_1^2-ftt_1+s=0.$$

Solving this as a quadratic in t_1, we get

$$t_1=\frac{ft\pm\sqrt{(f^2t^2-4fs)}}{2f}$$

or $$2t_1=t\pm\sqrt{\left(t^2-\frac{4s}{f}\right)} \quad \text{or} \quad t-2t_1=\sqrt{\left(t^2-\frac{4s}{f}\right)}.$$

This gives the time of ascent with constant velocity.

Ex. 23. *Prove that the shortest time from rest to rest in which a steady load of P tons can lift a weight of W tons through a vertical distance h feet is $\sqrt{\{(2h/g) \cdot P/(P-W)\}}$ seconds.*

Sol. The time will be shortest if the load acts continuously during the first part of the ascent. Let f be the acceleration during the first part of the ascent. Then by Newton's second law of motion, f is given by

$$P-W=(W/g)f. \qquad \ldots(1)$$

During the second part of the ascent, P ceases to act and W then moves only under gravity. Therefore the retardation is g.

Let x and y be the distances and t_1, t_2 the corresponding times for the two parts in the ascent.

If v be the velocity at the end of the first part of the ascent or at the beginning of the second part of the ascent, we have then

$$\left.\begin{aligned} v^2&=2fx,\\ v&=ft_1\end{aligned}\right\} \quad \ldots(2)$$

[Equations for the first part of the ascent]

and
$$\left.\begin{aligned} v^2&=2gy\\ v&=gt_2\end{aligned}\right\} \quad \ldots(3)$$

[Equations for the second part of the ascent]

Also $x+y=h$ (given).

From (2) and (3), we get

$$\frac{v^2}{2f}+\frac{v^2}{2g}=x+y$$

i.e.,
$$\frac{v^2}{2}\left(\frac{1}{f}+\frac{1}{g}\right)=h. \quad \ldots(4)$$

Also
$$\frac{v}{f}+\frac{v}{g}=t_1+t_2. \quad \ldots(5)$$

Now the total time of ascent

$$=t_1+t_2=\left(\frac{1}{f}+\frac{1}{g}\right)v \qquad \text{[from (5)]}$$

$$=\left(\frac{1}{f}+\frac{1}{g}\right)\sqrt{\left[2h\Big/\left(\frac{1}{f}+\frac{1}{g}\right)\right]} \qquad \text{[from (4)]}$$

$$=\sqrt{\left[2h\left(\frac{1}{f}+\frac{1}{g}\right)\right]}=\sqrt{\left[\frac{2h}{g}\left(\frac{g}{f}+1\right)\right]}$$

$$=\sqrt{\left[\frac{2h}{g}\left(\frac{W}{P-W}+1\right)\right]} \qquad \text{[from (1)]}$$

$$=\sqrt{\left[\frac{2h}{g}\cdot\frac{P}{P-W}\right]}.$$

Ex. 24. *Prove that the mean kinetic energy of a particle of mass m moving under a constant force, in any interval of time is $\frac{1}{6}m(u_1^2+u_1u_2+u_2^2)$, where u_1 and u_2 are the initial and final velocities.*

Sol. Let the interval of time during which the particle moves be T. If the particle moves under a constant acceleration f and v be its velocity at any time t, we have $v=u_1+ft$.

Now the mean kinetic energy of the particle during the time T

$$=\frac{1}{T}\int_0^T \tfrac{1}{2}mv^2\,dt=\frac{m}{2T}\int_0^T (u_1+ft)^2\,dt=\frac{m}{2fT}\cdot\frac{1}{3}\left[(u_1+ft)^3\right]_0^T$$

$$=\frac{m}{6fT}\left[(u_1+fT)^3-u_1^3\right]=\frac{m}{6(u_2-u_1)}(u_2^3-u_1^3)$$

$[\because u_2=u_1+fT$ and so $u_2-u_1=fT]$

$$=\tfrac{1}{6}m(u_1^2+u_1u_2+u_2^2).$$

Ex. 25. *A bullet fired into a target loses half its velocity after penetrating 3 cm. How much further will it penetrate?*

[Meerut 1972, 76, 79S, 83, 86P, 88]

Sol. If u cm./sec. is the initial velocity of the bullet then its velocity after penetrating 3 cm. will be $\frac{1}{2}u$ cm./sec. Let f cm./sec². be the retardation of the bullet.

Then from $v^2=u^2+2fs$, we have

$$(u/2)^2=u^2-2.f.3 \text{ giving } f=u^2/8.$$

If the bullet penetrates further by a cm, then from $v^2=u^2+2fs$, we have

$$0=(u/2)^2-2.(u^2/8).a.$$

$$\therefore\quad a=1 \text{ cm}.$$

Ex. 26. *A load W is to be raised by a rope from rest to rest, through a height h; the greatest tension which the rope can safely bear is nW. Show that the least time in which the ascent can be made is $[2nh/(n-1)g]^{1/2}$.* [Meerut 1986]

Sol. Obviously the time for ascent is least when the acceleration of the load is greatest. If m is the mass of the load, then $W=mg$ or $m=W/g$. Let f be the greatest acceleration of the load in the upward direction. Since the rope can bear the greatest tension nW, therefore when f is the greatest acceleration of the load, then the tension T in the rope is nW.

$\therefore$ by Newton's second law of motion $P=mf$, we have

$$T-W=nW-W=mf \text{ or } f=(n-1)\cdot(W/m)=(n-1)g. \quad \ldots(1)$$

Let the load W move upwards upto the height h_1 under the acceleration f. After that the tension in the rope ceases to act and therefore above the height h_1 the load will move under gravity which acts vertically downwards. If the load comes to rest after moving through a subsequent height h_2 above the height h_1, then according to the question

$$h_1+h_2=h. \quad \ldots(2)$$

If V is the maximum velocity of the load acquired at the end of the first part and t_1, t_2 are the times taken for describing the heights h_1 and h_2 respectively, then from $v=u+ft$, we have

$$V=0+ft_1 \quad\text{and}\quad 0=V-gt_2.$$

$$\therefore\quad t_1=V/f \quad\text{and}\quad t_2=V/g.$$

Also from $v^2=u^2+2fs$, we have

$$V^2=0+2fh_1 \quad\text{and}\quad 0=V^2-2gh_2.$$

$$\therefore\quad h_1=\frac{V^2}{2f} \quad\text{and}\quad h_2=\frac{V^2}{2g}.$$

Now from $h_1+h_2=h$, we have

$$\frac{V^2}{2f}+\frac{V^2}{2g}=h \text{ or } \frac{V^2}{2}\left(\frac{1}{f}+\frac{1}{g}\right)=h.$$

$$\therefore \quad V=\sqrt{(2h/(1/f+1/g))}. \qquad \ldots(3)$$

$\therefore$ the least time of ascent

$$=t_1+t_2=\frac{V}{f}+\frac{V}{g}=V\left(\frac{1}{f}+\frac{1}{g}\right)$$

$$=\sqrt{\left\{\frac{2h}{(1/f+1/g)}\right\}}\cdot\left(\frac{1}{f}+\frac{1}{g}\right) \quad \text{[substituting for } V \text{ from (3)]}$$

$$=\sqrt{\left[2h\left(\frac{1}{f}+\frac{1}{g}\right)\right]}$$

$$=\sqrt{\left[2h\left\{\frac{1}{(n-1)\, g}+\frac{1}{g}\right\}\right]} \quad \text{[substituting for } f \text{ from (1)]}$$

$$=\left[\frac{2nh}{(n-1)g}\right]^{1/2}$$

§ 3. Newton's Laws of Motion. [Allahabad 1979; Meerut 81]

The Newton's laws of motion are as follows:

Law 1. *Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled by some external force or forces to change its state.*

Law 2. *The rate of change of momentum of a body is proportional to the impressed force, and takes place in the direction in which the force acts.*

Law 3. *To every action there is an equal and opposite reaction.*

§ 4. Equation of motion *of a particle moving in a straight line as deduced from the Newton's second law of motion.*

Let v be the velocity at time t of a particle of mass m moving in a straight line under the action of the impressed force P. Since from Newton's second law of motion the rate of change of momentum is proportional to the impressed force, therefore

$$P \propto \frac{d}{dt}(mv), \quad [\because \text{ by def. momentum}=\text{mass}\times\text{velocity}]$$

or $\quad P=k\dfrac{d}{dt}(mv)$, where k is some constant

or $\quad P=km\dfrac{dv}{dt}$ provided m is constant

or $\quad P=kmf. \qquad \ldots(1)$

[$\because f=\text{acceleration}=dv/dt$].

Let us suppose that a unit force is that which produces a unit acceleration in a particle of unit mass. Then

$$P=1, \text{ when } m=1 \text{ and } f=1.$$

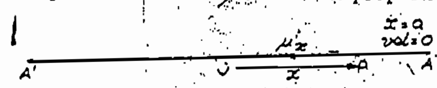
$\therefore$ from (1), we have $k=1$.

Hence we have, $P=mf$, which is the required equation of motion of the particle.

§ 5. Simple Harmonic Motion (S.H.M.) Definition. *The kind of motion, in which a particle moves in a straight line in such a way that its acceleration is always directed towards a fixed point on the line (called the centre of force) and varies as the distance of the particle from the fixed point, is called simple harmonic motion.*

[Meerut 1976, 78, 79, 81, 82, 85, 86; Kanpur 76, 77; Lucknow 80; Agra 77]

Let O be the centre of force taken as origin. Suppose the particle starts from rest from the point A where $OA=a$. It begins to move towards the centre of attraction O. Let P be the position of the particle after time t, where $OP=x$. By the definition of S.H.M. the magnitude of acceleration at P is proportional to x.



Let it be μx, where μ is a constant called, the intensity of force. Also on account of a centre of attraction at O, the acceleration of P is towards O i.e., in the direction of x decreasing. Therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=-\mu x, \qquad \ldots(1)$$

where the negative sign has been taken because the force acting on P is towards O i.e., in the direction of x decreasing. The equation (1) gives the acceleration of the particle at any position.

Multiplying both sides of (1) by $2dx/dt$, we get

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}=-2\mu x\frac{dx}{dt}.$$

Integrating with respect to t, we get

$$v^2=\left(\frac{dx}{dt}\right)^2=-\mu x^2+C,$$

where C is a constant of integration and v is the velocity at P.

Initially at the point A, $x=a$ and $v=0$; therefore $C=\mu a^2$. Thus, we have

$$v^2=\left(\frac{dx}{dt}\right)^2=-\mu x^2+\mu a^2.$$

or

$$v^2=\mu(a^2-x^2). \qquad \ldots(2)$$

The equation (2) gives the velocity at any point P. From (2) we observe that v^2 is maximum when $x^2=0$ or $x=0$. Thus in a S.H.M. the velocity is maximum at the centre of force O. Let this maximum velocity be v_1. Then at O, $x=0$, $v=v_1$. So from (2) we get $v_1^2=\mu a^2$ or $v_1=a\sqrt{\mu}$.

Also from (2) we observe that $v=0$ when $x^2=a^2$ i.e., $x=\pm a$. Thus in a S.H.M. the velocity is zero at points equidistant from the centre of force.

Now from (2), on taking square root, we get $dx/dt=-\sqrt{\mu}\sqrt{(a^2-x^2)}$, where the $-$ive sign has been taken because at P the particle is moving in the direction of x decreasing.

Separating the variables, we get

$$-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2-x^2)}}=dt. \qquad \ldots(3)$$

Integrating both sides, we get

$$\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}=t+D, \text{ where } D \text{ is a constant.}$$

But initially at A, $x=a$ and $t=0$; therefore $D=0$.

Thus we have

$$\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}=t \quad \text{or} \quad x=a\cos(\sqrt{\mu}t). \qquad \ldots(4)$$

[Lucknow 1978]

The equation (4) gives a relation between x and t, where t is the time measured from A. If t_1 be the time from A to O, then at O, we have $t=t_1$ and $x=0$. So from (4), we get $t_1=\frac{1}{\sqrt{\mu}}\cos^{-1}0$

$=\frac{1}{\sqrt{\mu}}\frac{\pi}{2}=\frac{\pi}{2\sqrt{\mu}}$, which is independent of the initial displacement a of the particle. *Thus in a S.H.M. the time of descent to the centre of force is independent of the initial displacement of the particle.*

[Meerut 1984, 85]

Note. The time of descent t_1 from A to O can also be found from (3) with the help of the definite integrals $-\frac{1}{\sqrt{\mu}}\int_a^0\frac{dx}{\sqrt{(a^2-x^2)}}=\int_0^{t_1}dt.$ For fixing the limits of integration, we observe that at A, $x=a$ and $t=0$ while at O, $x=0$ and $t=t_1$.

Nature of Motion. The particle starts from rest at A where its acceleration is maximum and is μa towards O. It begins to move towards the centre of attraction O and as it approaches the centre of force O, its velocity goes on increasing. When the particle reaches O its acceleration is zero and its velocity is maximum and is $a\sqrt{\mu}$ in the direction OA'. Due to this velocity gained at O the particle moves towards the left of O. But on account of the centre

of attraction at O a force begins to act upon the particle against its direction of motion. So its velocity goes on decreasing and it comes to instantaneous rest at A' where $OA'=OA$. The rest at A' is only instantaneous. The particle at once begins to move towards the centre of attraction O and retracing its path it again comes to instantaneous rest at A. Thus the motion of the particle is oscillatory and it continues to oscillate between A and A'. To start from A and to come back to A is called one *complete oscillation.*

Few Important Definitions :

1. Amplitude. *In a S.H.M. the distance from the centre of force of the position of maximum displacement is called the amplitude of the motion.* Thus the amplitude is the distance of a position of instantaneous rest from the centre of force. In the formulae (2) and (4) of this article the amplitude is a.

2. Time period. [Kanpur 1977]. *In a S.H.M. the time taken to make one complete oscillation is called time period or periodic time.* Thus if T is the time period of the S.H.M., then

$T=4$. (time from A to O) $=4.\frac{\pi}{2\sqrt{\mu}}=\frac{2\pi}{\sqrt{\mu}}$, which is independent of the amplitude a.

[Meerut 1989]

3. Frequency. *The number of complete oscillations in one second is called the frequency of the motion.* Since the time taken to make one complete oscillation is $\frac{2\pi}{\sqrt{\mu}}$ seconds, therefore if n is the frequency, then $n.\frac{2\pi}{\sqrt{\mu}}=1$ or $n=\frac{\sqrt{\mu}}{2\pi}$.

Thus the frequency is the reciprocal of the periodic time.

Important Remark 1. In a S.H.M. if the centre of force is not at origin but is at the point $x=b$, then the equation of motion is $d^2x/dt^2=-\mu(x-b)$. Similarly $d^2x/dt^2=-\mu(x+b)$ is the equation of a S.H.M. in which the centre of force is at the point $x=-b$.

Important Remark 2. In the above article when after instantaneous rest at A' the particle begins to move towards A, we have from (2)

$$\frac{dx}{dt}=+\sqrt{\mu}\sqrt{(a^2-x^2)},$$

where the $+$ive sign has been taken because the particle is moving in the direction of x increasing.

Separating the variables, we have $\frac{dx}{\sqrt{(a^2-x^2)}}=\sqrt{\mu}\,dt.$

Integrating, we get $-\cos^{-1}(x/a)=\sqrt{\mu}t+B$. Now the time from A to A' is $\pi/\sqrt{\mu}$. Therefore at A', we have $t=\pi/\sqrt{\mu}$ and

$x=-a$. These give $-\cos^{-1}(-a/a)=\sqrt{\mu}.(\pi/\sqrt{\mu})+B$
or $-\cos^{-1}(-1)=\pi+B$ or $-\pi=\pi+B$ or $B=-2\pi$. Thus we have
$-\cos^{-1}(x/a)=\sqrt{\mu}\,t-2\pi$ or $\cos^{-1}(x/a)=2\pi-\sqrt{\mu}\,t$
or $x=a\cos(2\pi-\sqrt{\mu}\,t)$ or $x=a\cos\sqrt{\mu}\,t$. Thus in S.H.M. the equation $x=a\cos\sqrt{\mu}\,t$ is valid throughout the entire motion from A to A' and back from A' to A.

4. Phase and Epoch. From equation (1), we have

$$\frac{d^2x}{dt^2}+\mu x=0,$$

which is a linear differential equation with constant coefficients and its general solution is given by

$$x=a\cos(\sqrt{\mu}\,t+\epsilon). \qquad \ldots(5)$$

The constant ϵ is called the starting phase or the epoch of the motion and the quantity $\sqrt{\mu}\,t+\epsilon$ is called the argument of the motion.

The phase at any time t of a S.H.M. is the time that has elapsed since the particle passed through its extreme position in the positive direction.

From (5), x is maximum when $\cos(\sqrt{\mu}\,t+\epsilon)$ is maximum *i.e.*, when $\cos(\sqrt{\mu}\,t+\epsilon)=1$.

Therefore if t_1 is the time of reaching the extreme position in the positive direction, then

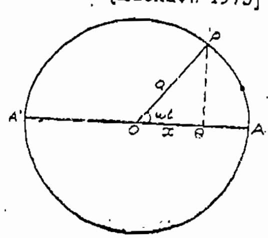
$$\cos(\sqrt{\mu}\,t_1+\epsilon)=1.$$

or, $$\sqrt{\mu}\,t_1+\epsilon=0 \quad \text{or} \quad t_1=-\frac{\epsilon}{\sqrt{\mu}}.$$

$\therefore$ the phase at time $t=t-t_1=t+\dfrac{\epsilon}{\sqrt{\mu}}$.

5. Periodic Motion. *A point is said to have a periodic motion when it moves in such a manner that after a certain fixed interval of time called periodic time it acquires the same position and moves with the same velocity in the same direction. Thus S.H.M. is a periodic motion.*

§ 6. Geometrical representation of S.H.M. [Lucknow 1975]



Let a particle move with a uniform angular velocity ω round the circumference of a circle of radius a. Suppose AA' is a fixed diameter of the circle. If the particle starts from A and P is its position at time t, then $\angle AOP=\omega t$.

Draw PQ perpendicular to the diameter AA'.

If $OQ=x$, then

$$x=a\cos\omega t. \qquad \ldots(1)$$

As the particle P moves round the circumference, the foot Q of the perpendicular on the diameter AA' oscillates on AA' from A to A' and from A' to A back. Thus the motion of the point Q is periodic.

From (1), we have

$$\frac{dx}{dt}=-a\omega\sin\omega t \qquad \ldots(2)$$

and $$\frac{d^2x}{dt^2}=-a\omega^2\cos\omega t=-\omega^2 x. \qquad \ldots(3)$$

The equations (2) and (3) give the velocity and acceleration of Q at any time t.

The equation (3) shows that Q executes a simple harmonic motion with centre at the origin O. From equation (1), we see that the amplitude of this S.H.M. is a because the maximum value of x is a.

The periodic time of Q = The time required by P to turn through an angle 2π with a uniform angular velocity ω

$$=\frac{2\pi}{\omega}.$$

Thus if a particle describes a circle with constant angular velocity, the foot of the perpendicular from it on any diameter executes a S.H.M.

§ 7. Important results about S.H.M.

We summarize the important relations of a S.H.M. as follows : (Remember them).

(i) Referred to the centre as origin the equation of S.H.M. is $\ddot{x}=-\mu x$, or the equation $\ddot{x}=-\mu x$ represents a S.H.M. with centre at the origin.

(ii) The velocity v at a distance x from the centre and the distance x from the centre at time t are respectively given by

$$v^2=\mu(a^2-x^2) \text{ and } x=a\cos\sqrt{\mu}\,t,$$

where a is the amplitude and the time t has been measured from the extreme position in the positive direction.

(iii) Maximum acceleration $=\mu a$, (at extreme points)

(iv) Maximum velocity $=\sqrt{\mu}\,a$, (at the centre)

(v) Periodic time $T=\dfrac{2\pi}{\sqrt{\mu}}$.

(vi) *Frequency* $n=\frac{1}{T}=\frac{\sqrt{\mu}}{2\pi}$.

Illustrative Examples.

Ex. 27. *The maximum velocity of a body moving with S.H.M. is 2 ft./sec. and its period is $\frac{1}{5}$ sec. What is its amplitude?*

Sol. Let the amplitude be a ft. Then the maximum velocity $=a\sqrt{\mu}$ ft./sec. $=2$ ft./sec. (given).

$$\therefore \quad a\sqrt{\mu}=2. \quad \ldots(1)$$

Also the time period $T=2\pi/\sqrt{\mu}$ seconds $=\frac{1}{5}$ seconds (given)

$$\therefore \quad \frac{2\pi}{\sqrt{\mu}}=\frac{1}{5}. \quad \ldots(2)$$

Multiplying (1) and (2) to eliminate μ, we have

$$2\pi a=\frac{2}{5}, \quad \therefore \quad a=\frac{1}{5\pi}.$$

$\therefore$ the required amplitude $=\frac{1}{5\pi}$ ft. $=\cdot 064$ ft. nearly.

Ex. 28. *At what distance from the centre the velocity in a S.H.M. will be half of the maximum?*

Sol. Take the centre of the motion as origin. Let a be the amplitude. In a S.H.M., the velocity v of the particle at a distance x from the centre is given by

$$v^2=\mu\,(a^2-x^2). \quad \ldots(1)$$

From (1), v is max. when $x=0$. Therefore max velocity $=\sqrt{\mu}a$.

Let x_1 be the distance from the centre of the point where the velocity is half of the maximum. *i.e.*, where the velocity is $\frac{1}{2}a\sqrt{\mu}$. Then putting $x=x_1$ and $v=\frac{1}{2}a\sqrt{\mu}$ in (1), we get

$$\tfrac{1}{4}a^2\mu=\mu\,(a^2-x_1^2), \text{ or } \tfrac{1}{4}a^2=a^2-x_1^2$$

or $$x_1^2=\frac{3a^2}{4} \text{ or } x_1=\pm a\sqrt{3}/2.$$

Thus there are two points, each at a distance $a\sqrt{3}/2$ from the centre, where the velocity is half of the maximum.

Ex. 29. *A particle moves in a straight line and its velocity at a distance x from the origin is $k\sqrt{(a^2-x^2)}$, where a and k are constants. Prove that the motion is simple harmonic and find the amplitude and the periodic time of the motion.*

Sol. We know that in a rectilinear motion the expression for velocity at a distance x from the origin is dx/dt. So according to the question, we have

$$\left(\frac{dx}{dt}\right)^2=k^2\,(a^2-x^2). \quad \ldots(1)$$

Differentiating (1) w.r.t. t, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}=k^2\left(-2x\frac{dx}{dt}\right).$$

$\therefore$ $\frac{d^2x}{dt^2}=-k^2x$, which is the equation of a S. H. M. with centre at the origin and $\mu=k^2$. Hence the given motion is simple harmonic.

The time period $T=2\pi/\sqrt{\mu}=2\pi/\sqrt{k^2}=2\pi/k$.

Now to find the amplitude we are to find the distance from the centre of a point where the velocity is zero. So putting $dx/dt=0$ in (1), we get $0=k^2\,(a^2-x^2)$ or $x=\pm a$. Since here the centre is at origin, therefore the amplitude $=a$.

Ex. 30. *Show that if the displacement of a particle in a straight line is expressed by the equation $x=a\cos nt+b\sin nt$, it describes a simple harmonic motion whose amplitude is $\sqrt{(a^2+b^2)}$ and period is $2\pi/n$.* [Meerut 1977]

Sol. Given $x=a\cos nt+b\sin nt$, ...(1)

$\therefore$ $dx/dt=-an\sin nt+bn\cos nt$, ...(2)

and $d^2x/dt^2=-an^2\cos nt-bn^2\sin nt=-n^2\,(a\cos nt+b\sin nt)$
$=-n^2x$, from (1).

Now $d^2x/dt^2=-n^2x$ is the equation of a S. H. M. with centre at the origin and $\mu=n^2$. Hence the given motion is simple harmonic.

The time period $T=2\pi/\sqrt{\mu}=2\pi/\sqrt{n^2}=2\pi/n$. Also the amplitude is the distance from the centre of a point where the velocity is zero. Since here the centre is at origin, therefore the amplitude is the value of x when $dx/dt=0$. Putting $dx/dt=0$ in (2), we get

$0=-an\sin nt+bn\cos nt$ or $\tan nt=b/a$.

$\therefore$ $\sin nt=b/\sqrt{(a^2+b^2)}$ and $\cos nt=a/\sqrt{(a^2+b^2)}$.

Substituting these in (1), we have

$$\text{the amplitude}=a\cdot\frac{a}{\sqrt{(a^2+b^2)}}+b\cdot\frac{b}{\sqrt{(a^2+b^2)}}=\frac{a^2+b^2}{\sqrt{(a^2+b^2)}}=\sqrt{(a^2+b^2)}.$$

Ex. 31. *The speed v of a particle moving along the axis of x is given by the relation $v^2=n^2\,(8bx-x^2-12b^2)$. Show that the motion is simple harmonic with its centre at $x=4b$, and amplitude $=2b$.*

Sol. Given $v^2=(dx/dt)^2=n^2\,(8bx-x^2-12b^2)$. ...(1)

Differentiating (1) w.r.t. t, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}=n^2(8b-2x)\frac{dx}{dt}.$$

$\therefore \quad \dfrac{d^2x}{dt^2}=n^2(4b-x)=-n^2(x-4b)$, which is the equation of a S.H.M. with centre at the point $x-4b=0$ i.e., at the point $x=4b$. [Note that centre is the point where the acceleration d^2x/dt^2 is zero.]

Now $v=0$ where $8bx-x^2-12b^2=0$ i.e., $x^2-8bx+12b^2=0$ i.e., $(x-6b)(x-2b)=0$ i.e., $x=6b$, or $2b$. Thus the positions of instantaneous rest are given by $x=2b$ and $x=6b$. The distance of any of these two positions from the centre $x=4b$ is the amplitude.

Hence the amplitude is the distance of the point $x=6b$ from the point $x=4b$. Thus the amplitude $=6b-4b=2b$.

Ex. 32. *The speed v of the point P which moves in a line is given by the relation $v^2=a+2bx-cx^2$, where x is the distance of the point P from a fixed point on the path, and a, b, c are constants. Show that the motion is simple harmonic if c is positive; determine the period and the amplitude of the motion.* [Kanpur 1979]

Sol. Here given that, $v^2=a+2bx-cx^2$. ...(1)

Differentiating both sides of (1) w.r.t. x, we have

$$2v\frac{dv}{dx}=2b-2cx.$$

$$\therefore \quad \frac{d^2x}{dt^2}=v\frac{dv}{dx}=-c\left(x-\frac{b}{c}\right). \quad ...(2)$$

Since c is positive, therefore the equation (2) represents a S. H. M. with the centre of force at the point $x=b/c$.

Hence the relation (1) represents a S. H. M. of period

$$T=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{c}}, \text{ because in the equation (2), } \mu=c.$$

To determine the amplitude, putting $v=0$ in (1), we have

$$a+2bx-cx^2=0$$

or

$$cx^2-2bx-a=0.$$

$$\therefore \quad x=\frac{b\pm\sqrt{(b^2+ac)}}{c}.$$

$\therefore$ the distances of the two positions of instantaneous rest A and A' from the fixed point O are given by

$$OA=\frac{b+\sqrt{(b^2+ac)}}{c} \quad \text{and} \quad OA'=\frac{b-\sqrt{(b^2+ac)}}{c}.$$

The distance of any of these two positions from the centre $x=(b/c)$ is the amplitude of the motion.

$$\therefore \quad \text{the amplitude}=\frac{b+\sqrt{(b^2+ac)}}{c}-\frac{b}{c}=\frac{\sqrt{(b^2+ac)}}{c}.$$

Ex. 33. *In a S. H. M. of period $2\pi/\omega$ if the initial displacement be x_0 and the initial velocity u_0, prove that*

(i) amplitude $=\sqrt{\left(x_0^2+\dfrac{u_0^2}{\omega^2}\right)}$

(ii) position at time $t=\sqrt{\left(x_0^2+\dfrac{u_0^2}{\omega^2}\right)}\cdot\cos\left\{\omega t-\tan^{-1}\left(\dfrac{u_0}{\omega x_0}\right)\right\}$,

and (iii) time to the position of rest $=\dfrac{1}{\omega}\tan^{-1}\left(\dfrac{u_0}{\omega x_0}\right)$.

Sol. We know that in a S. H. M. the time period $=2\pi/\sqrt{(\mu)}$.

Since here the time period is $2\pi/\omega$, therefore $2\pi/\sqrt{(\mu)}=2\pi/\omega$ i.e., $\mu=\omega^2$.

Now taking the centre of the motion as origin, the equation of the given S. H. M. is

$$\frac{d^2x}{dt^2}=-\omega^2x. \quad ...(1)$$

Multiplying (1) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\omega^2x^2+A, \text{ where } A \text{ is a constant.}$$

But initially at $x=x_0$, the velocity $\dfrac{dx}{dt}=u_0$.

Therefore $u_0^2=-\omega^2x_0^2+A$ or $A=u_0^2+\omega^2x_0^2$.

Thus we have

$$\left(\frac{dx}{dt}\right)^2=-\omega^2x^2+u_0^2+\omega^2x_0^2=\omega^2\left(x_0^2+\frac{u_0^2}{\omega^2}-x^2\right) \quad ...(2)$$

(i) Now the amplitude is the distance from the centre of a point where the velocity is zero. Since here the centre is origin, therefore the amplitude is the value of x when velocity is zero.

Putting $\dfrac{dx}{dt}=0$ in (2), we get $x=\pm\sqrt{\left(x_0^2+\dfrac{u_0^2}{\omega^2}\right)}$.

Here the required amplitude is $\sqrt{\left(x_0^2+\dfrac{u_0^2}{\omega^2}\right)}$.

(ii) Assuming that the particle is moving in the direction of x increasing, we have from (2)

$$\frac{dx}{dt}=\omega\sqrt{\left\{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)-x^2\right\}}$$

or

$$dt=\frac{1}{\omega}\frac{dx}{\sqrt{\{(x_0^2+u_0^2/\omega^2)-x^2\}}}.$$

Integrating, $t=-\dfrac{1}{\omega}\cos^{-1}\left\{\dfrac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}+B$,

where B is a constant.

But initially, when $t=0$, $x=x_0$.

$$\therefore \quad B=\frac{1}{\omega}\cos^{-1}\left\{\frac{x_0}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right).$$

$$\therefore \quad t=-\frac{1}{\omega}\cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}+\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$$

or $$\cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}=-\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$$

or $$\frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}=\cos\left[-\left\{\omega t-\tan^{-1}\frac{u_0}{\omega x_0}\right\}\right]$$

$$=\cos\left(\omega t-\tan^{-1}\frac{u_0}{\omega x_0}\right)$$

or $$x=\sqrt{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)}\cos\left(\omega t-\tan^{-1}\frac{u_0}{\omega x_0}\right), \quad \ldots(3)$$

which gives the position of the particle at time t.

(iii) Substituting the value of x from (3) in (2), we get

$$\left(\frac{dx}{dt}\right)^2=\omega^2\left(x_0^2+\frac{u_0^2}{\omega^2}\right)\sin^2\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}.$$

Putting $\frac{dx}{dt}=0$, we get

$$0=\omega^2\left(x_0^2+\frac{u_0^2}{\omega^2}\right)\sin^2\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$$

or $$\sin\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}=0$$

or $$\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)=0 \quad \text{or} \quad t=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right).$$

Hence the time of the position of rest $=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$.

Ex. 34. *Show that in a simple harmonic motion of amplitude a and period 'T', the velocity v at a distance x from the centre is given by the relation $v^2T^2=4\pi^2(a^2-x^2)$.*

Find the new amplitude if the velocity were doubled when the particle is at a distance $\frac{1}{2}a$ from the centre; the period remaining unaltered.

Sol. Let the equation of S. H. M. with centre as origin be $d^2x/dt^2=-\mu x$.

The time period $T=2\pi/\sqrt{\mu}$. $\quad \ldots(1)$

$\therefore$ Let a be the amplitude. Then the velocity v at a distance x from the centre is given by

$$v^2=\mu(a^2-x^2). \quad \ldots(2)$$

From (1), $\mu=4\pi^2/T^2$. Putting this value of μ in (2), we have

$$v^2=\frac{4\pi^2}{T^2}(a^2-x^2) \quad \text{or} \quad v^2T^2=4\pi^2(a^2-x^2). \quad \ldots(3)$$

Let v_1 be the velocity at a distance $\frac{1}{2}a$ from the centre. Then putting $x=\frac{1}{2}a$ and $v=v_1$ in (3), we get

$$v_1^2T^2=4\pi^2(a^2-\tfrac{1}{4}a^2)=3\pi^2a^2. \quad \ldots(4)$$

Let a_1 be the new amplitude when the velocity at the point $x=\frac{1}{2}a$ is doubled *i.e.*, when the velocity at the point $x=\frac{1}{2}a$ is any how made $2v_1$. Since the period remains unchanged, therefore, putting $v=2v_1$, $a=a_1$ and $x=\frac{1}{2}a$ in (3), we get

$$4v_1^2T^2=4\pi^2(a_1^2-\tfrac{1}{4}a^2)$$

or $4\times3\pi^2a^2=4\pi^2(a_1^2-\frac{1}{4}a^2)$ $\quad$ [$\because$ from (4), $v_1^2T^2=3\pi^2a^2$]

or $a_1^2=3a^2+\frac{1}{4}a^2=13a^2/4$. Hence the new amplitude $a_1=(a\sqrt{13})/2$.

Ex. 35. *Show that the particle executing S.H.M. requires one sixth of its period to move from the position of maximum displacement to one in which the displacement is half the amplitude.* (Kanpur 1973)

Sol. Let the equation of S.H.M. with centre as origin be $d^2x/dt^2=-\mu x$.

The time period $T=2\pi/\sqrt{\mu}$.

Let a be the amplitude of the motion. Then

$$(dx/dt)^2=\mu(a^2-x^2).$$

Suppose the particle is moving from the position of maximum displacement $x=a$ in the direction of x decreasing. Then

$$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{(a^2-x^2)} \quad \text{or} \quad dt=-\frac{1}{\sqrt{\mu}\sqrt{(a^2-x^2)}}. \quad \ldots(1)$$

Let t_1 be the time from the maximum displacement $x=a$ to the point $x=\frac{1}{2}a$. Then integrating (1), we get

$$\int_0^{t_1}dt=-\frac{1}{\sqrt{\mu}}\int_a^{a/2}\frac{dx}{\sqrt{(a^2-x^2)}}.$$

$$\therefore \quad t_1=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^{a/2}=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{1}{2}-\cos^{-1}1\right]$$

$$=\frac{1}{\sqrt{\mu}}\left[\frac{\pi}{3}-0\right]=\frac{1}{\sqrt{\mu}}\cdot\frac{\pi}{3}=\frac{1}{6}\left(\frac{2\pi}{\sqrt{\mu}}\right)=\frac{1}{6}\cdot(\text{time period } T).$$

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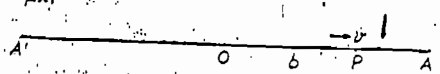
Ex. 36. *A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where $OP=b$ with velocity v in the direction OP; prove that the time which elapses before it returns to P is*

$$\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

[Lucknow 1979; Meerut 72, 83, 87, 90; Kanpur 74]

Sol. Let the equation of the S.H.M. with centre O as origin be $d^2x/dt^2=-\mu x$.



The time period $T=2\pi/\sqrt{\mu}$.

Let the amplitude be a. Then $(dx/dt)^2=\mu(a^2-x^2)$. ...(1)

When the particle passes through P its velocity is given to be v in the direction OP. Also $OP=b$. So putting $x=b$ and $dx/dt=v$ in (1), we get

$$v^2=\mu(a^2-b^2). \quad ...(2)$$

Let A be an extremity of the motion. From P the particle comes to instantaneous rest at A and then returns back to P. In S.H.M. the time from P to A is equal to the time from A to P.

$\therefore$ the required time $=2$ (time from A to P).

Now for the motion from A to P, we have

$$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{(a^2-x^2)} \quad \text{or} \quad dt=-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2-x^2)}}.$$

Let t_1 be the time from A to P. Then at A, $t=0$, $x=a$ and at P, $t=t_1$ and $x=b$. Therefore integrating (3), we get

$$\int_0^{t_1} dt=\frac{1}{\sqrt{\mu}}\int_a^b \frac{-dx}{\sqrt{(a^2-x^2)}}; \quad \text{or} \quad t_1=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

$$=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{b}{a}-\cos^{-1}1\right]=\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}.$$

Hence the required time $=2t_1=\frac{2}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}$

$$=\frac{2}{\sqrt{\mu}}\tan^{-1}\left\{\frac{\sqrt{(a^2-b^2)}}{b}\right\}=\frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right)$$

$$\left[\because \text{ from (2), } \sqrt{(a^2-b^2)}=\frac{v}{\sqrt{\mu}}\right]$$

$$=\frac{2}{2\pi/T}\tan^{-1}\left\{\frac{v}{b(2\pi/T)}\right\} \quad [\because T=2\pi/\sqrt{\mu} \text{ so that } \sqrt{\mu}=2\pi/T]$$

$$=\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

Ex. 37. *A point moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of motion is*

$$2\pi\sqrt{\left(\frac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}.$$

[Meerut 1977; Kanpur 84]

Sol. Let the equation of the S.H.M. with centre O as origin be $d^2x/dt^2=-\mu x$. Then the time period $T=2\pi/\sqrt{\mu}$.

If a be the amplitude of the motion, we have

$$v^2=\mu(a^2-x^2),$$

where v is the velocity at a distance x from the centre.

But when $x=x_1$, $v=v_1$ and when $x=x_2$, $v=v_2$.

Therefore from (1), we have

$$v_1^2=\mu(a^2-x_1^2) \text{ and } v_2^2=\mu(a^2-x_2^2).$$

These give $v_2^2-v_1^2=\mu\{(a^2-x_2^2)-(a^2-x_1^2)\}=\mu(x_1^2-x_2^2)$

i.e., $\mu=(v_2^2-v_1^2)/(x_1^2-x_2^2)$.

Hence the time period $T=2\pi/\sqrt{\mu}=2\pi\sqrt{\left(\frac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}$.

Ex. 38. *A particle is moving with S.H.M. and while making an excursion from one position of rest to the other, its distances from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3; prove that the time of a complete oscillation is*

$$2\pi\Big/\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right).$$

Sol. Take the middle point of the path as origin. Let the equation of the S.H.M. be $d^2x/dt^2=-\mu x$. Then the time period

$$T=2\pi/\sqrt{\mu}.$$

Let a be the amplitude of the motion. If the time t be measured from the position of instantaneous rest $x=a$, we have

$$x=a\cos\sqrt{\mu}t,$$

where x is the distance of the particle from the centre at time t.

Let x_1, x_2, x_3 be the distances of the particle from the centre at the ends of t_1^{th}, $(t_1+1)^{th}$ and $(t_1+2)^{th}$ seconds. Then from (1),

$$x_1=a\cos\sqrt{\mu}t_1, \quad ...(2)$$

$$x_2=a\cos\sqrt{\mu}(t_1+1), \quad ...(3)$$

and $$x_3=a\cos\sqrt{\mu}(t_1+2). \quad ...(4)$$

$\therefore$ $x_1+x_3=a[\cos\sqrt{\mu}t_1+\cos\sqrt{\mu}(t_1+2)]$

$=2a\cos\sqrt{\mu}(t_1+1)\cos\sqrt{\mu}=2x_2\cos\sqrt{\mu}$, [from (3)]

$\therefore$ $\cos\sqrt{\mu}=(x_1+x_3)/2x_2$ or $\sqrt{\mu}=\cos^{-1}\{(x_1+x_3)/2x_2\}$.

Hence the time period $T=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\cos^{-1}\{(x_1+x_3)/2x_2\}}$.

Ex. 39 (a). *At the ends of three successive seconds the distances of a point moving with S.H.M. from the mean position measured in the same direction are 1, 5 and 5. Show that the period of a complete oscillation is $2\pi/\theta$ where $\cos\theta = 3/5$.*

[Meerut 1969, 72, 90P]

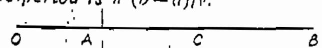
Sol. Proceed as in Ex. 38.

Ex. 39 (b). *At the end of three successive seconds, the distances of a point moving with simple harmonic motion from its mean position measured in the same direction are 1, 3 and 4. Show that the period of complete oscillation is*

$$\frac{2\pi}{\cos^{-1}(5/6)}.$$

[Meerut 1987P, 88P]

Ex. 40. *A body moving in a straight line OAB with S.H.M. has zero velocity when at the points A and B whose distances from O are a and b respectively, and has velocity v when half way between them. Show that the complete period is $\pi(b-a)/v$.*



Sol. In the figure, A and B are the positions of instantaneous rest in a S.H.M. Let C be the middle point of AB. Then C is the centre of the motion. Also it is given that $OA = a$, $OB = b$.

$\therefore$ The amplitude of the motion $= \frac{1}{2}AB = \frac{1}{2}(OB - OA) = \frac{1}{2}(b-a)$.

Now in a S.H.M. the velocity at the centre $= (\sqrt{\mu}) \times$ amplitude. Since in this case the velocity at the centre is given to be v, therefore $v = \frac{1}{2}(b-a)\cdot\sqrt{\mu}$ or $\sqrt{\mu} = 2v/(b-a)$.

Hence time period $T = 2\pi/\sqrt{\mu} = 2\pi \cdot \{(b-a)/2v\} = \pi(b-a)/v$.

Ex. 41. *A point executes S.H.M. such that in two of its positions velocities are u, v and the two corresponding accelerations are α, β; show that the distance between the two positions is $(v^2-u^2)/(\alpha+\beta)$ and the amplitude of the motion is*

$$\frac{\{(v^2-u^2)(\alpha^2 v^2 - \beta^2 u^2)\}^{1/2}}{\alpha^2-\beta^2}.$$

[Meerut 1990S; Allahabad 77]

Sol. Let the equation of the S.H.M. with centre as origin be $d^2x/dt^2 = -\mu x$.

If a be the amplitude of the motion, we have

$$(dx/dt)^2 = \mu(a^2 - x^2),$$

where dx/dt is the velocity at a distance x from the centre.

Let x_1 and x_2 be the distances from the centre of the two positions where u and v are the velocities and α and β are the accelerations respectively. Then

$$\alpha = \mu x_1, \quad \ldots(1)$$
$$\beta = \mu x_2, \quad \ldots(2)$$
$$u^2 = \mu(a^2 - x_1^2), \quad \ldots(3)$$
and
$$v^2 = \mu(a^2 - x_2^2). \quad \ldots(4)$$

Adding (1) and (2), we get $\alpha + \beta = \mu(x_1 + x_2)$. ...(5)

Also subtracting (3) from (4), we get

$$v^2 - u^2 = \mu(x_1^2 - x_2^2) = \mu(x_1 - x_2)(x_1 + x_2) = (\alpha+\beta)(x_1 - x_2). \quad \text{[from (5)]}$$

$\therefore$ $(x_1 - x_2) = (v^2 - u^2)/(\alpha+\beta)$. This gives the distance between the two positions.

Now to get the amplitude a it is obvious that we have to eliminate x_1, x_2 and μ from the equations (1), (2), (3) and (4). Substituting for x_1 and x_2 from (1) and (2) in (3) and (4), we have

$$u^2 = \mu\left(a^2 - \frac{\alpha^2}{\mu^2}\right) \quad i.e., \quad a^2\mu^2 - u^2\mu - \alpha^2 = 0 \quad \ldots(6)$$

and
$$v^2 = \mu\left(a^2 - \frac{\beta^2}{\mu^2}\right) \quad i.e., \quad a^2\mu^2 - v^2\mu - \beta^2 = 0. \quad \ldots(7)$$

By the method of cross multiplication, we have from (6) and (7),

$$\frac{\mu^2}{u^2\beta^2 - v^2\alpha^2} = \frac{\mu}{-a^2\alpha^2 + a^2\beta^2} = \frac{1}{a^2u^2 - a^2v^2}.$$

Equating the two values of μ^2 found from the above equations, we get

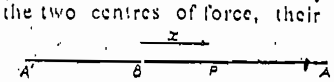
$$\frac{\alpha^2 v^2 - u^2\beta^2}{a^2(v^2-u^2)} = \left[\frac{a^2(\alpha^2-\beta^2)}{a^2(v^2-u^2)}\right]^2, \text{ or } \frac{\alpha^2v^2 - u^2\beta^2}{a^2(v^2-u^2)} = \frac{(\alpha^2-\beta^2)^2}{(v^2-u^2)^2}.$$

$$\therefore \quad a^2 = \frac{(\alpha^2v^2 - \beta^2u^2)(v^2-u^2)}{(\alpha^2-\beta^2)^2} \text{ or } a = \frac{\{(v^2-u^2)(\alpha^2v^2-\beta^2u^2)\}^{1/2}}{(\alpha^2-\beta^2)}.$$

IFoS 2009 Ex. 42. *A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'; the particle is displaced slightly towards one of them, show that the time of a small oscillation is $2\pi/\sqrt{(\mu+\mu')}$.*

[Agra 1980, 86; Rohilkhand 88]

Sol. Suppose A and A' are the two centres of force, their intensities being μ and μ' respectively. Let a particle of mass m be in equilibrium at B under the attraction of these two centres. If $AB = a$ and $A'B = a'$, the forces of attraction at B due to the centres A and A' are $m\mu a$ and $m\mu' a'$ respectively in opposite directions. As these two forces balance, we have



$$m\mu a = m\mu' a'. \quad \ldots(1)$$

Now suppose the particle is slightly displaced towards A and then let go. Let P be the position of the particle after time t, where $BP=x$.

The attraction at P due to the centre A is $m\mu.AP$ or $m\mu(a-x)$ in the direction PA i.e., in the direction of x increasing. Also the attraction at P due to the centre A' is $m\mu'.A'P$ or $m\mu'(a'+x)$ in the direction PA' i.e., in the direction of x decreasing. Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m(d^2x/dt^2)=m\mu(a-x)-m\mu'(a'+x), \qquad \ldots(2)$$

where the force in the direction of x increasing has been taken with +ive sign and the force in the direction of x decreasing has been taken with −ive sign.

Simplifying the equation (2), we get

$$m(d^2x/dt^2)=m(\mu a-\mu x-\mu' a'-\mu' x)$$

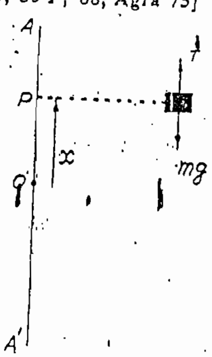
or $$d^2x/dt^2=-(\mu+\mu')x. \qquad [\because \text{ by (1), } m\mu a=m\mu' a']$$

This is the equation of a S. H. M. with centre at the origin. Hence the motion of the particle is simple harmonic with centre at B and its time period is $2\pi/\sqrt{(\mu+\mu')}$.

Ex. 43. *A body is attached to one end of an inelastic string, and the other end moves in a vertical line with S.H.M. of amplitude a, making n oscillations per second. Show that the string will not remain tight during the motion unless $n^2<g/(4\pi^2a)$.*

[Meerut 1970, 80, 86 P, 88; Agra 75]

Sol. Suppose the string remains tight during the motion so that the body also moves in an identical S.H.M. Let m be the mass of the body.



Let the body move in S.H.M. between A and A' and suppose O is the centre of the motion, where $OA=a$.

Since the body makes n oscillations per second, therefore its time period $\dfrac{2\pi}{\sqrt{\mu}}=\dfrac{1}{n}$.

This gives $\mu=4\pi^2n^2$.

At time t, let the body be in a position P, where $OP=x$. The impressed force acting on the body is $T-mg$ along OP. Here T is the tension of the string. By Newton's law, the equation of motion of the body is

$$m(d^2x/dt^2)=T-mg.$$

$$\therefore \quad T=mg+m(d^2x/dt^2).$$

Obviously T is least when d^2x/dt^2 is least. But the least value of d^2x/dt^2 is $-\mu a$. Hence least $T=mg-m\mu a$.

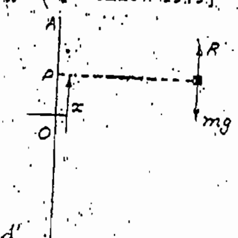
The string will remain tight if this least tension is positive i.e., if $m\mu a<mg$

i.e., if $m4\pi^2n^2a<mg$ $\qquad [\because \mu=4\pi^2n^2]$

i.e., if $n^2<g/(4\pi^2a)$. Hence the result.

Ex. 44. *A horizontal shelf is moved up and down with S. H. M. of period $\frac{1}{2}$ sec. What is the amplitude admissible in order that a weight placed on the shelf may not be jerked off?* [Lucknow 1979]

Sol. Let m be the mass of the body placed on the shelf. Suppose along with the shelf, the body moves in an identical S. H. M. between A and A'. Let O be the centre of the motion so that $OA=a$ is the amplitude.



The time period $2\pi/\sqrt{\mu}=\frac{1}{2}$; (given)

$$\therefore \quad \mu=16\pi^2.$$

Let P be the position of the body at time t, where $OP=x$. The impressed force acting on the body is $R-mg$ along OP. Here R is the reaction of the shelf. By Newton's law the equation of motion of the body is

$$m(d^2x/dt^2)=R-mg.$$

$$R=mg+m(d^2x/dt^2).$$

Obviously R is least when d^2x/dt^2 is least and the least value of d^2x/dt^2 is $-\mu a$. Hence least $R=mg-m\mu a$.

The body will not be jerked off if this least value of R remains non-negative i.e., if $m\mu a\leq mg$

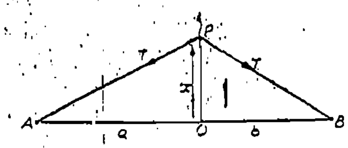
i.e., if $m16\pi^2a\leq mg$ $\qquad [\because \mu=16\pi^2]$

i.e., if $a\leq g/(16\pi^2)$. Hence the greatest admissible value of the amplitude $a=g/(16\pi^2)$.

Ex. 45. *A particle of mass m is attached to a light wire which is stretched tightly between two fixed points with a tension T. If a, b be the distance of the particle from the two ends, prove that the period of small transverse oscillation of mass m is*

$$\frac{2\pi}{\sqrt{\left\{\dfrac{T(a+b)}{mab}\right\}}}.$$

Sol. Let a light wire be stretched tightly between the fixed points A and B with a tension T. Let a particle of mass m be attached at the point O of the wire where $AO=a$ and $OB=b$.



Let the particle be displaced slightly perpendicular to AB (*i.e.*, in the transverse direction) and then let go. Let P be the position of the particle at any time t, where $OP=x$. Since the displacement is small, therefore the tension in the string in any displaced position can be taken as T which is the tension in the string in the original position. The equation of motion of the particle is

$$m\frac{d^2x}{dt^2}=-(T\cos\angle OPA+T\cos\angle OPB)$$

$$=-T\left(\frac{OP}{AP}+\frac{OP}{BP}\right)=-T\left(\frac{x}{\sqrt{(a^2+x^2)}}+\frac{x}{\sqrt{(b^2+x^2)}}\right)$$

$$=-T\left\{\frac{x}{a}\left(1+\frac{x^2}{a^2}\right)^{-1/2}+\frac{x}{b}\left(1+\frac{x^2}{b^2}\right)^{-1/2}\right\}$$

$$=-T\left[\frac{x}{a}\left(1-\tfrac{1}{2}\cdot\frac{x^2}{a^2}+\ldots\right)+\frac{x}{b}\left(1-\tfrac{1}{2}\cdot\frac{x^2}{b^2}+\ldots\right)\right]$$

$$=-T\left(\frac{x}{a}+\frac{x}{b}\right),$$ neglecting higher powers of x/a and x/b which are very small

$$=-T\left(\frac{a+b}{ab}\right)x.$$

$\therefore$ $$\frac{d^2x}{dt^2}=-\frac{T(a+b)}{mab}x=-\mu x,\ \text{where } \mu=\frac{T(a+b)}{mab}.$$

This is the standard equation of a S. H. M. with centre at the origin. The time period

$$T=\frac{2\pi}{\sqrt{\mu}}=2\pi\bigg/\sqrt{\left\{\frac{T(a+b)}{mab}\right\}}=2\pi\sqrt{\left\{\frac{mab}{T(a+b)}\right\}}.$$

Ex. 46. *If in a S. H. M. u, v, w be the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of force, show that the period T is given by the equation*

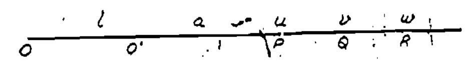
$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a)=\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1\end{vmatrix}.$$

[Kanpur 1980, 85, 88; Meerut 84]

Sol. Let O and O' be the centre of force and the fixed point respectively on the line of motion and let

$OO'=l.$ Let u, v, w be the velocities of the particle at P, Q, R respectively where

$$O'P=a,\ O'Q=b,\ O'R=c.$$



For a S.H.M. of amplitude A, the velocity V at a distance x from the centre of force is given by

$$V^2=\mu(A^2-x^2). \quad \ldots(1)$$

At P, $x=OP=l+a$, $V=u$
at Q, $x=OQ=l+b$; $V=v$
and at R, $x=OR=l+c$, $V=w$.

$\therefore$ from (1), we have

$$u^2=\mu\{A^2-(l+a)^2\}$$

or $$\frac{u^2}{\mu}=A^2-l^2-a^2-2al$$

or $$\left(\frac{u^2}{\mu}+a^2\right)+2l.a+(l^2-A^2)=0. \quad \ldots(2)$$

Similarly,

$$\left(\frac{v^2}{\mu}+b^2\right)+2l.b+(l^2-A^2)=0. \quad \ldots(3)$$

and $$\left(\frac{w^2}{\mu}+c^2\right)+2l.c+(l^2-A^2)=0. \quad \ldots(4)$$

Eliminating $2l$ and (l^2-A^2) from (2), (3) and (4), we have

$$\begin{vmatrix} \frac{u^2}{\mu}+a^2 & a & 1 \\ \frac{v^2}{\mu}+b^2 & b & 1 \\ \frac{w^2}{\mu}+c^2 & c & 1\end{vmatrix}=0$$

or $$\begin{vmatrix} \frac{u^2}{\mu} & a & 1 \\ \frac{v^2}{\mu} & b & 1 \\ \frac{w^2}{\mu} & c & 1\end{vmatrix}+\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1\end{vmatrix}=0$$

$$\text{or} \quad -\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \frac{1}{\mu} \begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}$$

$$\text{or} \quad \mu \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$\text{or} \quad \mu\,(a-b)\,(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \qquad \ldots(5)$$

But the time period $T=\dfrac{2\pi}{\sqrt{\mu}}$, so that $\mu=\dfrac{4\pi^2}{T^2}$.

Hence from (5), we have

$$\frac{4\pi^2}{T^2}\,(a-b)\,(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

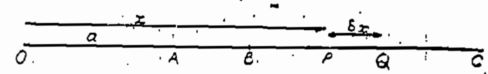
§ 8. Hooke's Law :

Statement. *The tension of an elastic string is proportional to the extension of the string beyond its natural length.*

If x is the stretched length of a string of natural length l, then by Hooke's law the tension T in the string is given by $T=\lambda\,\dfrac{x-l}{l}$, where λ is called the modulus of elasticity of the string. Remember that the direction of the tension is always opposite to the extension.

Theorem. *Prove that the work done against the tension in stretching a light elastic string, is equal to the product of its extension and the mean of its final and initial tensions.* [Kanpur 1977]

Proof. Let $OA=a$ be the natural length of a string whose one end is fixed at O. Let the string be stretched beyond its natural



length. Let B and C be the two positions of the free end A of the string during its any extension and let $OB=b$ and $OC=c$.

Then by Hooke's law,

$$\text{the tension at } B=T_B=\lambda\,\frac{b-a}{a}, \qquad \ldots(1)$$

$$\text{and} \qquad \text{the tension at } C=T_C=\lambda\,\frac{c-a}{a}, \qquad \ldots(2)$$

where λ is the modulus of elasticity of the string.

Now we find the work done against the tension in stretching the string from B to C.

Let P be any position of the free end of the string during its extension from B to C and let $OP=x$.

Then the tension at $P=T_P=\lambda\,\dfrac{x-a}{a}$.

Now suppose the free end of the string is slightly stsetched from P to Q, where $PQ=\delta x$. Then the work done against the tension in stretching the string from P to Q

$$=T_P\,\delta x=\lambda\,\frac{(x-a)}{a}\,\delta x.$$

$\therefore$ the work done against the tension in stretching the string from B to C

$$=\int_b^c \frac{\lambda}{a}\,(x-a)\,dx=\frac{\lambda}{2a}\Big[(x-a)^2\Big]_b^c$$

$$=\frac{\lambda}{2a}\,[(c-a)^2-(b-a)^2]=\frac{\lambda}{2a}\,[\{(c-a)-(b-a)\}\,\{(c-a)+(b-a)\}]$$

$$=(c-b)\cdot\frac{1}{2}\left[\frac{\lambda}{a}\,(c-a)+\frac{\lambda}{a}\,(b-a)\right]$$

$$=(c-b)\cdot\tfrac{1}{2}\,[T_C+T_B], \qquad \text{[from (1) and (2)]}$$

$$=BC\times(\text{mean of the tension at } B \text{ and } C).$$

Hence, the work done against the tension in stretching the string is equal to the product of the extension and the mean of the initial and final tensions.

Now we shall discuss a few simple and interesting cases of simle harmonic motion.

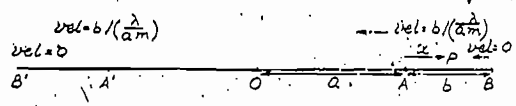
§ 9. Particle attached to one end of a horizontal elastic string.

A particle of mass m is attached to one end of a horizontal elastic string whose other end is fixed to a point on a smooth hori-

zontal table. The particle is pulled to any distance in the direction of the string and then let go; to discuss the motion.

[Lucknow 1977; Allahabad 76]

Let a string OA of natural length a lie on a smooth horizontal table. The end O of the string is attached to a fixed point of the table and a particle of mass m is attached to the other end A. The mass m is pulled upto B, where $AB=b$; and then let go.



Let P be the position of the particle after time t, where $AP=x$. The table being smooth, the only horizontal force acting on the particle at P is the tension T in the string OP. Since the direction of tension is always opposite to the extension, therefore, the force T acts in the direction PA i.e., in the direction of x decreasing. Also by Hooke's law $T=\lambda(x/a)$. Hence the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=-\lambda\frac{x}{a} \text{ or } \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x. \qquad \ldots(1)$$

The equation (1) shows that the motion of the particle is simple harmonic with centre at the origin A. The equation of motion (1) holds good so long as the string is stretched. Since the string becomes slack just as the particle reaches A, therefore the equation (1) holds good for the motion of the particle from B to A.

Multiplying (1) by $2(dx/dt)$ and integrating, we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{\lambda}{am}x^2+C, \text{ where } C \text{ is a constant.}$$

At the point B, $x=b$ and $dx/dt=0$; $\therefore$ $C=(\lambda/am)\,b^2$.

Thus we have $\left(\dfrac{dx}{dt}\right)^2=\dfrac{\lambda}{am}(b^2-x^2)$. $\qquad \ldots(2)$

This equation gives velocity in any position from B to A. Putting $x=0$ in (2), we have the velocity at $A=\sqrt{(\lambda/am)}\,b$, in the direction AO.

The time from B to A is $\frac{1}{4}$ of the complete time period of a S.H.M. whose equation is (1).

Character of the motion. The motion from B to A is simple harmonic. When the particle reaches A, the string becomes slack and the simple harmonic motion ceases. But due to the velocity

gained at A the particle continues to move to the left of A. So long as the string is loose there is no force on the particle to change its velocity because the only force here is that of tension and the tension is zero so long as the string is loose. Thus the particle moves from A to A' with uniform velocity $\sqrt{(\lambda/am)}\cdot b$ gained by it at A. Here A' is a point on the other side of O such that $OA'=OA$. When the particle passes A' the string again becomes tight and begins to extend. The tension again comes into picture and the particle begins to move in S. H. M. But now the force of tension acts against the direction of motion of the particle. So the velocity of the particle starts decreasing and the particle comes to instantaneous rest at B', where $A'B'=AB$. The time from A' to B' is the same as that from B to A. At B' the particle at once begins to move towards A' because of the tension which attracts it towards A'. Retracing its path the particle again comes to instantaneous rest at B and thus it continues to oscillate between B and B'.

During one complete oscillation the particle covers the distance between A and B and also that between A' and B' twice while moving in S. H. M. Also it covers the distance between A and A' twice with uniform velocity $\sqrt{(\lambda/am)}\,b$. Hence the total time for one complete oscillation

$=$ the complete time period of a S.H.M. whose equation is (1) $+$ the time taken to cover the distance $4a$ with uniform velocity $\sqrt{(\lambda/am)}\,b$

$$=\frac{2\pi}{\sqrt{(\lambda/am)}}+\frac{4a}{\sqrt{(\lambda/am)}\,b}=2\pi\sqrt{\left(\frac{am}{\lambda}\right)}+\frac{4a}{b}\sqrt{\left(\frac{am}{\lambda}\right)}$$

$$=2\left(\pi+\frac{2a}{b}\right)\sqrt{\left(\frac{am}{\lambda}\right)}.$$

Illustrative Examples :

Ex. 47. *One end of an elastic string (modulus of elasticity λ) whose natural length is a, is fixed to a point on a smooth horizontal table and the other is tied to a particle of mass m, which is lying on the table. The particle is pulled to a distance from the point of attachment of the string equal to twice its natural length and then let go. Show that the time of a complete oscillation is*

$$2(\pi+2)\sqrt{\left(\frac{am}{\lambda}\right)}.$$

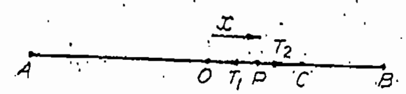
[Lucknow 1981]

Sol. Proceed exactly in the same way as in § 9. Here, the particle is pulled to a distance from the point of attachment of the string equal to twice its natural length. Therefore initially the increase b in the length of the string is equal to $2a-a$ i.e., a.

Now proceed as in § 9, taking $b=a$.

Ex. 48. *A light elastic string whose modulus of elasticity is λ is stretched to double its length and is tied to two fixed points distant $2a$ apart. A particle of mass m tied to its middle point is displaced in the line of the string through a distance equal to half its distance from the fixed points and released. Find the time of a complete oscillation and the maximum velocity acquired in the subsequent motion.*

Sol. Let an elastic string of natural length a be stretched between two fixed points A and B distant $2a$ apart, O, being the middle point of AB. We have, $OA=OB=a$.



Natural length of the portions OA and OB each is $a/2$ (since the string is stretched to double its length). A particle of mass m attached to the middle point O is displaced towards B upto a point C, where $OC=a/2$ and then let go. Let P be the position of the particle after any time t, where $OP=x$. [Note that we have taken O as origin. The direction OP is that of x increasing and the direction PO is that of x decreasing]. At P there are two horizontal forces acting on the particle :

(i) The tension T_1 in the string AP acting in the direction PA i.e., in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB i.e., in the direction of x increasing.

[Note that the string AP is extended in the direction AP and so the tension T_1 in it acts in the opposite direction PA].

By Hooke's law, $T_1=\lambda \dfrac{a+x-\frac{1}{2}a}{a/2}$ and $T_2=\lambda \dfrac{a-x-\frac{1}{2}a}{a/2}$.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=T_2-T_1=\lambda\frac{a-x-a/2}{a/2}-\lambda\frac{a+x-a/2}{a/2}=-\frac{4\lambda x}{a}.$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{4\lambda}{am}x. \quad \ldots(1)$$

Thus the motion is S.H.M. with centre at the origin O. Since we have displaced the particle towards B only upto the point C so that the portion BC of the string is just in its natural length, therefore during the entire motion of the particle both the portions of

the string remain taut and so the entire motion of the particle is governed by the above equation. Thus the particle makes oscillations in S.H.M. about O and the time period of one complete oscillation

$=$ the time period of S.H.M. whose equation is (1)

$$=2\pi\Big/\sqrt{\left(\frac{4\lambda}{am}\right)}=\pi\sqrt{(am/\lambda)}.$$

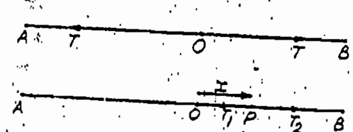
The amplitude (*i.e.*, the maximum displacement from the centre) of this S.H.M. is $a/2$.

$\therefore$ the maximum velocity $=(\sqrt{\mu})\times$ amplitude

$$=\sqrt{(4\lambda/am)}\cdot(a/2)=\sqrt{(a\lambda/m)}.$$

Ex. 49. *A particle of mass m executes simple harmonic motion in the line joining the points A and B on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T; show that the time of an oscillation is $2\pi\{ml l'/T(l+l')\}^{1/2}$, where l, l' are the extensions of the strings beyond their natural legths.*

Sol. A particle of mass m rests at O being pulled by two horizontal strings AO and BO whose other ends are connected to two fixed points A and B. Let a, a' be the natural lengths of the strings AO and BO whose extensions beyond their natural lengths are l and l' respectively. Let λ and λ' be the respective moduli of elasticity of the two strings AO and BO. At O the particle is in equilibrium under the tensions of the two strings. Therefore



$$\frac{\lambda l}{a}=\frac{\lambda' l'}{a'}=T \text{ (given)}. \quad \ldots(1)$$

From (1), we have $\dfrac{T}{l}=\dfrac{\lambda}{a}$ and $\dfrac{T}{l'}=\dfrac{\lambda'}{a'}$ $\quad \ldots(2)$

Now suppose the particle is slightly pulled towards B and then let go. It beings to move towards O. Let P be the position of the particle after any time t, where $OP=x$. [Note that we have taken O as origin. The direction OP is that of x increasing and the direction PO is that of x decreasing.]

At P there are two horizontal forces acting on the particle :

(i) The tension T_1 in the string AP acting in the direction PA, *i.e.*, in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB, *i.e.*, in the direction of x increasing. [Note that the string AP is extended in the direction AP and so the tension T_1 in it acts in the opposite direction PA.]

By Hooke's law, $T_1=\lambda\,\dfrac{(l-x)}{a}$ and $T_2=\lambda'\,\dfrac{(l'-x)}{a'}$.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\,\frac{d^2x}{dt^2}=T_2-T_1=\frac{\lambda'(l'-x)}{a'}-\frac{\lambda(l+x)}{a}$$

$$=-\frac{\lambda' x}{a'}-\frac{\lambda x}{a}, \qquad \left[\because \text{ by (1) } \frac{\lambda' l'}{a'}=\frac{\lambda l}{a}\right]$$

$$=-x\left(\frac{\lambda'}{a'}+\frac{\lambda}{a}\right).$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{x}{m}\left(\frac{\lambda'}{a'}+\frac{\lambda}{a}\right)=-\frac{x}{m}\left(\frac{T}{l'}+\frac{T}{l}\right), \qquad \text{from (2)}$$

$$=-\frac{T(l+l')}{mll'}\,x, \qquad \ldots(3)$$

showing that the motion of the particle is simple harmonic with centre at the origin O.

Since we have given only a slight displacement to the particle towards B, therefore during the entire motion of the particle both the strings remain taut and so the entire motion of the particle is governed by the equation (3). Thus the particle makes small oscillations in S.H.M. about O and the time period of one complete oscillation

$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{\{T(l+l')/mll'\}}}=2\pi\left[\frac{mll'}{T(l+l')}\right]^{1/2}.$$

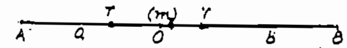
Remark. In order that the entire motion of the particle should remain simple harmonic with centre at O, the particle must be pulled towards B only upto that distance which does not allow the string OB to become slack.

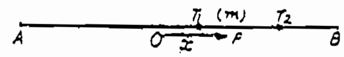
Ex. 50. *Two light elastic strings are fastened to a particle of mass m and their other ends to fixed points so that the strings are taut. The modulus of each is λ, the tension T, and length a and b. Show that the period of an oscillation along the line of the strings is*

$$2\pi\left[\frac{mab}{(T+\lambda)(a+b)}\right]^{1/2}.$$

[Meerut 1981, 84, 85]

Sol. Let the two light elastic strings be fastened to a particle of mass m at O and their other ends be attached to two fixed points A and B so that the strings are taut and $OA=a$, $OB=b$. If l and l' are the natural lengths of the strings OA and OB respectively, then in the position of equilibrium of the particle at O,





tension in the string OA = tension in the string $OB=T$, (as given).

Applying Hooke's law, we have

$$T=\lambda\,\frac{a-l}{l}=\lambda\,\frac{b-l'}{l'}, \qquad \ldots(1)$$

From $T=\lambda\,\dfrac{a-l}{l}$, we have $Tl=\lambda a-\lambda l$

i.e., $\qquad l(T+\lambda)=\lambda a$

i.e., $\qquad \dfrac{\lambda}{l}=\dfrac{T+\lambda}{a}. \qquad \ldots(2)$

Similarly $\qquad \dfrac{\lambda}{l'}=\dfrac{T+\lambda}{b}. \qquad \ldots(3)$

Now suppose the particle is slightly pulled towards B and then let go. It begins to move towards O. Let P be the position of the particle after any time t, where $OP=x$. The direction OP is that of x increasing and the direction PO is that of x decreasing.

At P there are two horizontal forces acting on the particle.

(i) The tension T_1 in the string AP acting in the direction PA *i.e.*, in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB *i.e.*, in the direction of x increasing.

By Hooke's law, $T_1=\lambda\,\dfrac{a+x-l}{l}$, $\quad T_2=\lambda\,\dfrac{b-x-l'}{l'}$.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\,\frac{d^2x}{dt^2}=T_2-T_1=\frac{\lambda(b-x-l')}{l'}-\frac{\lambda(a+x-l)}{l}$$

$$=-\frac{\lambda}{l'}\,x-\frac{\lambda}{l}\,x, \qquad \left[\because \text{ from (1), } \frac{\lambda(b-l')}{l'}=\frac{\lambda(a-l)}{l}\right]$$

$$=-\left[\frac{T+\lambda}{b}+\frac{T+\lambda}{a}\right]x, \qquad \text{[from (2) and (3)]}$$

$$= -\frac{(T+\lambda)(a+b)}{ab}\,x,$$

$$\therefore\quad \frac{d^2x}{dt^2} = -\frac{(T+\lambda)(a+b)}{mab}\,x, \qquad \ldots(4)$$

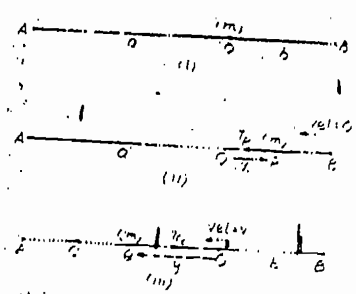
showing that the motion of the particle is simple harmonic with centre at the origin O.

Since we have given only a slight displacement to the particle towards B, therefore during the entire motion of the particle both the strings remain taut and the entire motion of the particle is governed by the equation (4). Thus the particle makes small oscillations in S. H. M. about O and the time period of one complete oscillation

$$= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\{(T+\lambda)(a+b)/mab\}}} = 2\pi\left[\frac{mab}{(T+\lambda)(a+b)}\right]^{1/2}.$$

Ex. 51. *An elastic string of natural length $(a+b)$ where $a > b$ and modulus of elasticity λ has a particle of mass m attached to it at a distance a from one end, which is fixed to a point A of a smooth horizontal plane. The other end of the string is fixed to a point B so that the string is just unstretched. If the particle be held at B and then released, show that it will oscillate to and fro through a distance $\dfrac{b(\sqrt{a}+\sqrt{b})}{\sqrt{a}}$ in a periodic time $\pi(\sqrt{a}+\sqrt{b})\sqrt{(m/\lambda)}$.*

Sol. Let AB be an elastic string of natural length $a+b$ attached to two fixed points A and B distant $a+b$ apart. Let a particle of mass m be attached to the point O of the string such that $OA=a$, $OB=b$ and $a > b$.



When the particle is held at B, the portion AO of the string is stretched while the portion OB is slack and so when the particle is released from B, it moves towards O starting from rest at B.

If P is the position of the particle between O and B, [see fig. (ii)], at any time t after its release from B and $OP=x$, then the tension in the string AP is $T_P=\lambda\dfrac{x}{a}$ acting towards O and the tension in the string PB is zero because it is slack.

$\therefore$ the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -T_P = -\frac{\lambda}{a}\,x$$

or

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{am}\,x, \qquad \ldots(1)$$

which represents a S. H. M. with centre at O and amplitude OB.

If t_1 be the time from B to O, then

$$t_1 = \tfrac{1}{4}\times\text{time period of the S. H. M. represented by (1)}$$

$$= \tfrac{1}{4}\cdot\frac{2\pi}{\sqrt{(\lambda/am)}} = \frac{\pi}{2}\sqrt{\left(\frac{am}{\lambda}\right)}. \qquad \ldots(2)$$

Now multiplying both sides of (1) by $2(dx/dt)$ and then integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{\lambda}{am}\,x^2 + k, \text{ where } k \text{ is a constant.}$$

But at the point B, $x=OB$ and $dx/dt=0$;

$$\therefore\quad 0 = -\frac{\lambda}{am}\,b^2 + k \quad\text{or}\quad k = \frac{\lambda b^2}{am}.$$

$$\therefore\quad \left(\frac{dx}{dt}\right)^2 = \frac{\lambda}{am}(b^2 - x^2). \qquad \ldots(3)$$

If V is the velocity of the particle at O, where $x=0$, then from (3), we have

$$V^2 = \frac{\lambda}{am}\cdot b^2 \quad\text{or}\quad V = \sqrt{\left(\frac{\lambda}{am}\right)}\cdot b. \qquad \ldots(4)$$

At the point O, the tension in either of the two portions of the string is zero and the velocity of the particle is V to the left of O, due to which the particle moves towards the left of O. As the particle moves to the left of O, the string OA becomes slack and the string OB is stretched.

If Q is the position of the particle between O and A, [see fig. (iii)], at any time t, since it starts moving from O to the left of it and $OQ=y$, then the tension in the string QB is $T_Q=\lambda\dfrac{y}{b}$ acting towards O and the tension in the string $QA=0$ because it is slack.

$\therefore$ The equation of motion of the particle at Q is

$$m\frac{d^2y}{dt^2} = -T_Q = -\frac{\lambda y}{b}$$

or

$$\frac{d^2y}{dt^2} = -\frac{\lambda}{bm}\,y. \qquad \ldots(4)$$

Multiplying both sides of (4) by $2(dy/dt)$ and then integrating, we have

$$\left(\frac{dy}{dt}\right)^2=-\frac{\lambda}{bm}y^2+D, \text{ where } D \text{ is a constant.}$$

But at O, $y=0$ and $\left(\frac{dy}{dt}\right)^2=V^2=\frac{\lambda}{am}b^2$.

$$\therefore \frac{\lambda}{am}b^2=-\frac{\lambda}{bm}.0+D \quad \text{or} \quad D=\frac{\lambda}{am}b^2.$$

$$\therefore \left(\frac{dy}{dt}\right)^2=\frac{\lambda}{m}\left(\frac{b^2}{a}-\frac{1}{b}y^2\right)$$

or
$$\left(\frac{dy}{dt}\right)^2=\frac{\lambda}{bm}\left(\frac{b^3}{a}-y^2\right). \quad \ldots(5)$$

If the particle comes to instantaneous rest at the point C between O and A such that $OC=c$, then at C, $y=c$ and $dy/dt=0$.

$\therefore$ from (5), we have

$$0=\frac{\lambda}{bm}\left(\frac{b^3}{a}-c^2\right), \quad \text{or} \quad c=b\sqrt{\left(\frac{b}{a}\right)}.$$

From C the particle retraces its path and comes to instantaneous rest at B.

The particle thus oscillates to and fro through a distance

$$BC=BO+OC=b+c=b+b\sqrt{\left(\frac{b}{a}\right)}=\frac{b(\sqrt{a}+\sqrt{b})}{\sqrt{a}}.$$

The equation (4) represents a S. H. M. with centre at O, amplitude OC and time period $T'=2\pi\Big/\sqrt{\left(\frac{\lambda}{bm}\right)}=2\pi\sqrt{\left(\frac{bm}{\lambda}\right)}$.

If t_2 be the time from O to C, we have

$$t_2=\tfrac{1}{4}.(T')=\frac{\pi}{2}\sqrt{\left(\frac{bm}{\lambda}\right)}.$$

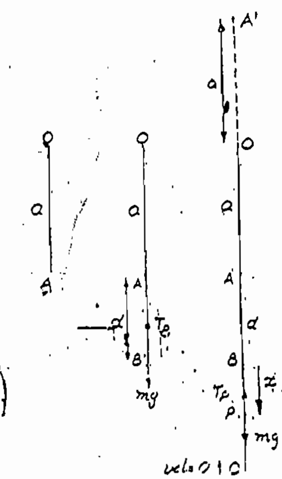
Hence the required periodic time for making a complete oscillation between B and C

$$=2.(\text{time from } B \text{ to } C)=2\,(t_1+t_2)$$

$$=2\left[\frac{\pi}{2}\sqrt{\left(\frac{am}{\lambda}\right)}+\frac{\pi}{2}\sqrt{\left(\frac{bm}{\lambda}\right)}\right]=\pi(\sqrt{a}+\sqrt{b})\sqrt{\left(\frac{m}{\lambda}\right)}.$$

§ 10. Particle suspended by an elastic string. *A particle of mass m is suspended from a fixed point by a light elastic string of natural length a and modulus of elasticity λ. The particle is pulled down a little in the line of the string and released; to discuss the motion.* (Meerut 1988 S)

Let one end of the string OA of natural length a be attached to the fixed point O and a particle of mass m be attached to the other end A. Due to the weight mg of the particle the string OA is stretched and if B is the position of equilibrium of the particle such that $AB=d$, then the tension T_B in the string will balance the weight of the particle



i.e.,
$$mg=T_B$$

or
$$mg=\lambda\frac{AB}{OA}=\lambda\frac{d}{a}. \quad \ldots(1)$$

The particle is pulled down to a point C such that $BC=c$ and then released. At the point C, the tension in the string is greater than the weight of the particle and so the particle starts moving vertically upwards with velocity zero at C. Let P be the position of the particle at any time t, where $BP=x$. The tension in the string when the particle is at P is $T_P=\lambda\frac{d+x}{a}$, acting vertically upwards.

The resultant force acting on the particle at P in the vertically upwards direction $=T_P-mg=\lambda\left(\frac{d+x}{a}\right)-mg=\frac{\lambda d}{a}+\frac{\lambda x}{a}-mg$

$$=\frac{\lambda x}{a}, \quad \left[\because \frac{\lambda d}{a}=mg, \text{ from (1)}\right].$$

Also the acceleration of the particle at P is d^2x/dt^2 in the direction of x increasing, i.e., in the vertically downwards direction.

$\therefore$ by Newton's law, the equation of motion of P is given by

$$m\frac{d^2x}{dt^2}=-\frac{\lambda x}{a} \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x. \quad \ldots(2)$$

This equation holds good so long as the tension operates i.e., when the string is extended beyond its natural length.

Equation (2) is the standard equation of a S.H.M. with centre at the origin B and the amplitude of the motion is $BC=c$.

The periodic time T of the S.H.M. represented by the equation (2) is given by

$$T = 2\pi \Big/ \sqrt{\left(\frac{\lambda}{am}\right)} = 2\pi \sqrt{\left(\frac{am}{\lambda}\right)}. \quad ...(3)$$

The motion of the particle remains simple harmonic as long as there is tension in the string *i.e.*, as long the particle remains in the region from C to A.

In case the string becomes slack during the motion of the particle, the particle will begin to move freely under gravity.

Now there are two cases.

Case I. If $BC \leq AB$ *i.e.*, $c \leq d$. In this case the particle will not rise above A and it will come to instantaneous rest before or just reaching A. The whole motion will be S.H.M. with centre at B, amplitude BC and period T given by (3).

Case II. If $BC > AB$ *i.e.*, $c > d$. In this case the particle will rise above A, and the motion will be simple harmonic upto A and above A the particle will move freely under gravity.

Multiplying both sides of (2) by $2\,(dx/dt)$ and then integrating, we have $\left(\frac{dx}{dt}\right)^2 = -\frac{\lambda}{am} x^2 + k$, where k is a constant.

But at C, $x = BC = c$ and $dx/dt = 0$.

$$\therefore \quad 0 = -\frac{\lambda}{am} c^2 + k \quad \text{or} \quad k = \frac{\lambda}{am} c^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{\lambda}{am} (c^2 - x^2). \quad ...(4)$$

Now if V is the velocity of the particle at A, where $x = -BA = -d$, then, from (4), we have

$$V^2 = \frac{\lambda}{am} (c^2 - d^2) \quad \text{or} \quad V = \sqrt{\left[\frac{\lambda}{am} (c^2 - d^2)\right]}, \quad ...(5)$$

the direction of V being vertically upwards.

If h is the height to which the particle rises above A, then

$$h = \frac{V^2}{2g} = \frac{\lambda (c^2 - d^2)}{2amg}, \quad ...(6)$$

provided $h \leq 2a$.

Also in this case the maximum height attained by the particle during its entire motion

$$= CB + BA + h,$$
$$= c + d + h. \quad ...(7)$$

If $h \leq 2a$ *i.e.*, if $h \leq AA'$, then the particle, after coming to instantaneous rest, will retrace its path *i.e.*, it will fall freely under gravity upto A and below A it will move in S.H.M. till it comes to instantaneous rest at C.

If $h = 2a = AA'$, then the particle will just come to rest at A' and will then move downwards, retracing its path.

In this case the maximum height attained by the particle

$$= c + d + 2a. \quad ...(8)$$

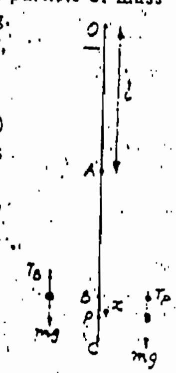
If $h > 2a$ *i.e.*, if $h > AA'$, then the particle will rise above A' also and so the string will again become stretched and the particle will again begin to move in simple harmonic motion. After coming to instantaneous rest the particle will retrace its path.

Illustrative Examples

Ex. 52 (a). *An elastic string without weight, of which the unstretched length is l and modulus of elasticity is the weight of n oz. is suspended by one end and a mass m oz. is attached to the other end. Show that the time of a small vertical oscillation is*

$$2\pi \sqrt{(ml/ng)}. \quad \text{[Meerut 1971, 76, 78, 79]}$$

Sol. $OA = l$ is the natural length of a string whose one end is fixed at O. B is the position of equilibrium of a particle of mass m oz. attached to the other end of the string. Considering the equilibrium of the particle at B, we have mg = the tension T_B in the string OB.



$$\therefore \quad mg = ng \frac{AB}{l}. \quad ...(1)$$

because modulus of elasticity of the string is given to be ng.

Now suppose the particle is pulled slightly upto C (so that $BC < AB$), and then let go. It starts moving vertically upwards with velocity zero at C. Let P be its position at any point t, where $BP = x$. The direction BP is that of x increasing and the direction PB is that of x decreasing. At P there are two forces acting on the particle :

(i) The weight mg acting vertically downwards *i.e.*, in the direction of x increasing,

and (ii) the tension $T_P = ng \frac{AB + x}{l}$ in the string OP, acting vertically upwards *i.e.*, in the direction of x decreasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-ng\frac{AB+x}{l}=mg-ng\frac{AB}{l}-ng\frac{x}{l}$$

$$=-ng\frac{x}{l}, \qquad \left[\because \text{ from (1), } mg=ng\frac{AB}{l}\right]$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{ng}{lm}x, \qquad \ldots(2)$$

which is the equation of a simple harmonic motion with centre at the origin B and amplitude BC.

Since $BC<AB$, therefore during the entire motion of the particle the string will not become slack.

Thus the entire motion of the particle is governed by the equation (2) and the particle will make oscillations in simple harmonic motion about the centre B.

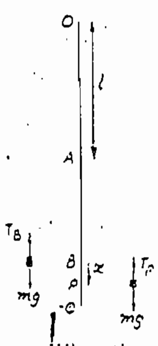
The time of one oscillation

$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{(ng/lm)}}=2\pi\sqrt{\left(\frac{lm}{ng}\right)}.$$

Ex. 52 (b). *A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical extensions corresponding to these two weights, prove that*

$$g(t_1^2-t_2^2)=4\pi^2(c_1-c_2).$$ [Rohilkhand 1985]

Sol. One end of a string OA of natural length l is attached to a fixed point O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string. Then AB is the statical extension in the string corresponding to this particle of mass m. Let $AB=d$.



In the equilibrium position of the particle of mass m at B, the tension $T_B=\lambda\,(d/l)$ in the string OB balances the weight mg of the particle.

$$\therefore \frac{\lambda d}{l}=mg \text{ or } \frac{\lambda}{lm}=\frac{g}{d}. \qquad \ldots(1)$$

Now suppose the particle at B is slightly pulled down upto C and then let go. Let P be the position of the particle at any time t where $BP=x$. When the particle is at P, the tension T_P in the string P is $\lambda\dfrac{d+x}{l}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=-\frac{\lambda(d+x)}{l}+mg,$$

[Note that the weight mg of the particle has been taken with the +ive sign because it is acting vertically downwards i.e., in the direction of x increasing.]

or
$$m\frac{d^2x}{dt^2}=-\frac{\lambda d}{l}-\frac{\lambda x}{l}+mg$$

$$=-\frac{\lambda x}{l}, \qquad \left[\because \frac{\lambda d}{l}=mg\right]$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{\lambda}{lm}x=-\frac{g}{d}x, \quad \text{[from (1)]}.$$

Hence the motion of the particle is simple harmonic about the centre B and its period is $\dfrac{2\pi}{\sqrt{(g/d)}}$ i.e., $2\pi\sqrt{\left(\dfrac{d}{g}\right)}$.

But according to the question, the period is t_1 when $d=c_1$ and the period is t_2 when $d=c_2$.

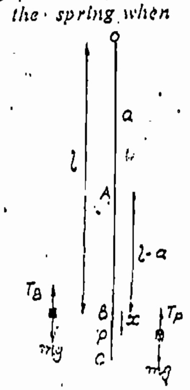
$$\therefore t_1=2\pi\sqrt{(c_1/g)} \text{ and } t_2=2\pi\sqrt{(c_2/g)},$$

so that
$$t_1^2-t_2^2=(4\pi^2/g)(c_1-c_2)$$

or
$$g(t_1^2-t_2^2)=4\pi^2(c_1-c_2).$$

Ex. 53. *A mass m hangs from a light spring and is given a small vertical displacement. If l is the length of the spring when the system is in equilibrium and n the number of oscillations per second, show that the natural length of the spring is $l-(g/4\pi^2n^2)$.*

Sol. Let $OA=a$ be the natural length of the spring which extends to a length $OB=l$ when a particle of mass m hangs in equilibrium. In the position of equilibrium of the particle at B, the tension T_B in the spring is $\lambda\{(l-a)/a\}$ and it balances the weight mg of the particle.



$$\therefore \lambda\{(l-a)/a\}=mg. \qquad \ldots(1)$$

Now suppose the particle at B is slightly pulled down upto C and then let go. It moves towards B starting at rest from C. Let P be

the position of the particle after any time t, where $BP=x$. When the particle is at P, the tension T_P in the spring OP, is $\lambda \frac{l+x-a}{a}$, acting vertically upwards *i.e.*, in the direction of x decreasing.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{l+x-a}{a}=mg-\lambda\frac{l-a}{a}-\frac{\lambda x}{a}$$

$$=-\frac{\lambda x}{a}, \text{ [from (1)].}$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x=-\frac{g}{l-a}x, \left[\because \text{ from (1), } \frac{\lambda}{am}=\frac{g}{l-a}\right].$$

Hence the motion of the particle is simple harmonic with centre at the origin B and the time period T (*i.e.*, the time for one complete oscillation) $=2\pi\sqrt{\left(\frac{l-a}{g}\right)}$ seconds.

Since n is given to be the number of oscillations per second, therefore $n.T=1$ or $n^2T^2=1$

or $$n^2\cdot\frac{4\pi^2(l-a)}{g}=1 \quad\text{or}\quad l-a=\frac{g}{4\pi^2n^2}$$

or $$a=l-\frac{g}{4\pi^2n^2}.$$

This gives the natural length a of the spring.

Ex. 54. *A heavy particle attached to a fixed point by an elastic string hangs freely, stretching the string by a quantity e. It is drawn down by an additional distance f and then let go; determine the height to which it will arise if $f^2-e^2=4ae$, a being the unstretched length of the string.*

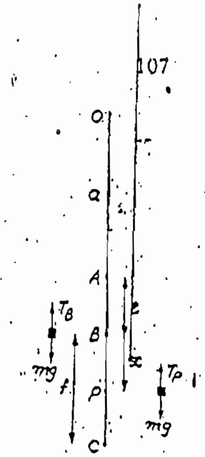
Sol. Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string. It is given that $AB=e$. In the position of equilibrium of the particle at B, the tension T_B in the string OB is $\lambda(e/a)$ and it balances the weight mg of the particle.

$$\therefore \quad mg=\lambda(e/a). \quad \ldots(1)$$

Now suppose the particle is pulled down to a point C, such that $BC=f$, and then let go. It moves towards B starting with

velocity zero at C. Let P be the position of the particle after any time t, where $BP=x$. Note that we have taken B as the origin. When the particle is at P, there are two forces acting upon it :

(i) the tension $T_P=\lambda\frac{OP-OA}{OA}=\lambda\frac{e+x}{a}$



in the string OP, acting vertically upwards *i.e.*, in the direction of x decreasing, and (ii) the weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{e+x}{a}=mg-\frac{\lambda e}{a}-\frac{\lambda x}{a}$$

$$=-\frac{\lambda x}{a}, \quad \left[\because \text{ from (1), } mg=\frac{\lambda e}{a}\right]$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x=-\frac{g}{e}x, \quad \left[\because \text{ from (1), } \frac{\lambda}{am}=\frac{g}{e}\right].$$

Thus the equation of motion of the particle is

$$\frac{d^2x}{dt^2}=-\frac{g}{e}x, \quad \ldots(2)$$

which is the equation of a simple harmonic motion with centre at the origin B and amplitude BC. The equation (2) governs the motion of the particle so long as the string does not become slack.

Since $f^2-e^2=4ae=+$ive, therefore $f>e$ *i.e.*, $BC>AB$. So when the particle, while moving in simple harmonic motion, reaches the point A, its velocity is not zero. But at A the string becomes slack and so above A the particle will move freely under gravity.

Let us first find the velocity at A for the S.H.M. given by (2). Multiplying both sides of (2) by $2\,(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{e}x^2+k, \text{ where } k \text{ is a constant.}$$

But at C, $x=BC=f$ and $\left(\frac{dx}{dt}\right)=0$. Therefore $0=-\left(\frac{g}{e}\right)f^2+k$

or $$k=\left(\frac{g}{e}\right)f^2.$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = -\frac{g}{e}x^2 + \frac{g}{e}f^2 = \frac{g}{e}(f^2 - x^2). \qquad \ldots(3)$$

The equation (3) gives the velocity of the particle at any point from C to A. Let v_1 be the velocity of the particle at A. Then at A, $x = -e$ and $\left(\frac{dx}{dt}\right)^2 = v_1^2$. Therefore, from (3), we have

$$v_1^2 = \frac{g}{e}(f^2 - e^2) = \frac{g}{e}\, 4ae \qquad [\because f^2 - e^2 = 4ae]$$

$= 4ag$, the direction of v_1 being vertically upwards.

Above A the motion of the particle is freely under gravity. If the particle rises to a height h above A, we have

$0 = v_1^2 - 2gh$, [using the formula $v^2 = u^2 + 2fs$]

$= 4ag - 2gh$, $[\because v_1^2 = 4ag]$.

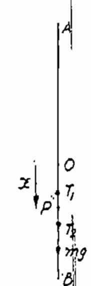
$\therefore$ $2gh = 4ag$ or $h = 2a$.

Hence the total height to which the particle rises above C

$= CB + BA + h = f + e + 2a$.

Ex. 55. *A heavy particle is attached to one point of a uniform elastic string. The ends of the string are attached to two points in a vertical line. Show that the period of a vertical oscillation in which the string remains taut is $2\pi\sqrt{(mh/2\lambda)}$, where λ is the coefficient of elasticity of the string and h the harmonic mean of the unstretched lengths of the two parts of the string.*

Sol. Let a particle of mass m be attached to a point O of a string whose ends have been fastened to two fixed points A and B in a vertical line. The string is taut and the particle is in equilibrium at O. Let $OA = a$ and $OB = b$. Also let a_1 and b_1 be the natural lengths of the stretched portions OA and OB of the string.



Considering the equilibrium of the particle at O, we have the resultant upward force = the resultant downward force

i.e., the tension in OA = the tension in OB + the weight of the particle

i.e., $$\lambda\frac{(a - a_1)}{a_1} = \lambda\frac{(b - b_1)}{b_1} + mg. \qquad \ldots(1)$$

Now suppose the particle is slightly displaced towards B and then let go. During this slight displacement of the particle both the portions of the string remain taut. Let P be the position of the particle after any time t, where $OP = x$.

When the particle is at P, there are three forces acting upon it :

(i) The tension $T_1 = \lambda\,\dfrac{a + x - a_1}{a_1}$ in the string AP acting in the direction PA i.e., in the direction of x decreasing.

(ii) The tension $T_2 = \lambda\,\dfrac{b - x - b_1}{b_1}$ in the string BP acting in the direction PB i.e., in the direction of x increasing.

(iii) The weight mg of the particle acting vertically downwards i.e., in the direction of x increasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -\lambda\frac{a + x - a_1}{a_1} + \lambda\frac{b - x - b_1}{b_1} + mg$$

$$= -\lambda\frac{a - a_1}{a_1} + \lambda\frac{b - b_1}{b_1} - mg - \frac{\lambda x}{a_1} - \frac{\lambda x}{b_1}$$

$$= -\lambda\left(\frac{1}{a_1} + \frac{1}{b_1}\right)x \qquad [\text{by (1)}]$$

$$= -\lambda\left(\frac{a_1 + b_1}{a_1 b_1}\right)x.$$

$\therefore$ $\dfrac{d^2x}{dt^2} = -\dfrac{\lambda}{m}\dfrac{(a_1 + b_1)}{a_1 b_1}x$, which is the equation of motion of a S.H.M. with centre at the origin O. This equation of motion holds good so long as both the portions of the string remain taut. But the initial displacement given to the particle below O being small, both the portions of the string must remain taut for ever. Hence this equation governs the entire motion of the particle. Thus the entire motion of the particle is simple harmonic about the centre O and the time period of one complete oscillation

$$= 2\pi\bigg/\sqrt{\left\{\frac{\lambda(a_1 + b_1)}{m\, a_1 b_1}\right\}} = \pi\sqrt{\left\{\frac{m\,(2a_1 b_1)}{2\lambda\,(a_1 + b_1)}\right\}} = 2\pi\sqrt{\left(\frac{mh}{2\lambda}\right)},$$

where $h = \dfrac{2a_1 b_1}{a_1 + b_1}$ is the harmonic mean between a_1 and b_1.

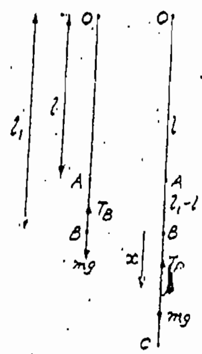
Ex. 56. *A light elastic string of natural length l has one extremity fixed at a point O and the other attached to a stone, the weight of which in equilibrium would extend the string to a length l_1. Show that if the stone be dropped from rest at O, it will come to instantaneous rest at a depth $\sqrt{(l_1^2 - l^2)}$ below the equilibrium position.* [Kanpur 1978; Meerut 80, 84 P, 88 P, Allahabad 75]

Sol. $OA = l$ is the natural length of a string whose one end is fixed at O. B is the position of equilibrium of a stone of mass m

attached to the other end of the string and $OB=l_1$. When the stone rests at B, the tension T_0 of the string balances the weight of the stone. Therefore

$$T_0=\lambda\frac{(l_1-l)}{l}=mg, \qquad \ldots(1)$$

where λ is the modulus of elasticity of the string.



Now the stone is dropped from O. It falls the distance OA $(=l)$ freely under gravity. If v_1 be the velocity gained by the stone at A, we have $v_1=\sqrt{(2gl)}$ downwards. When the stone falls below A, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from A to B, the force of tension acting vertically upwards remains less than the weight of the stone acting vertically downwards. Therefore during the fall from A to B the velocity of the stone goes on increasing. When the stone begins to fall below B, its velocity goes on decreasing because now the force of tension exceeds the weight of the stone. Let the stone come to instantaneous rest at C, where $BC=a$.

During the motion of the stone below A, let P be its position after any time t, where $BP=x$. [Note that we have taken the position of equilibrium B of the stone as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing].

When the stone is at P, there are two forces acting upon it:

(i) The tension $T_P=\lambda\dfrac{(l_1+x)-l}{l}$ in the string OP acting in the direction OP i.e., in the direction of x decreasing.

(ii) The weight mg of the stone acting vertically, downwards i.e., in the direction of x increasing.

Hence by Newton's second law of motion $(P=mf)$, the equation of motion of the stone at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{(l_1+x)-l}{l}=mg-\lambda\frac{(l_1-l)}{l}-\frac{\lambda x}{l}$$

$$=-\frac{\lambda x}{l}, \qquad \text{[from (1)]}.$$

[Note that the force acting in the direction of x increasing has been taken with + ive sign and that in the direction of x decreasing with − ive sign].

Thus $\dfrac{d^2x}{dt^2}=-\dfrac{\lambda}{lm}x$, $\qquad \ldots(2)$

which is the equation of a S.H.M. with centre at the origin B. The equation (2) holds good so long as the string is stretched i.e., for the motion of the stone between A and C.

Multiplying (2) by $2\,(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{\lambda}{lm}x^2+D, \text{ where } D \text{ is a constant.}$$

At A, $x=-(l_1-l)$ and $dx/dt=\sqrt{(2gl)}$;

$$\therefore\quad 2gl=-\frac{\lambda}{lm}(l_1-l)^2+D \quad\text{or}\quad D=2gl+\frac{\lambda}{lm}(l_1-l)^2.$$

Thus, we have $\left(\dfrac{dx}{dt}\right)^2=-\dfrac{\lambda}{lm}x^2+2gl+\dfrac{\lambda}{lm}(l_1-l)^2$. $\qquad \ldots(3)$

The equation (3) gives velocity of the stone at any point between A and C. At C, $x=a$, $dx/dt=0$. Therefore (3) gives

$$0=-\frac{\lambda}{lm}a^2+2gl+\frac{\lambda}{lm}(l_1-l)^2$$

or $\quad -\dfrac{g}{(l_1-l)}a^2+2gl+\dfrac{g}{(l_1-l)}(l_1-l)^2=0$

$$\left[\because \text{ from (1)}, \frac{\lambda}{lm}=\frac{g}{l_1-l}\right]$$

or $\quad \dfrac{a^2}{l_1-l}=2l+l_1-l=l_1+l$

or $\quad a^2=(l_1-l)(l_1+l)=l_1^2-l^2$.

$$\therefore\quad a=\sqrt{(l_1^2-l^2)}.$$

Ex. 57. *A light elastic string whose natural length is a has one end fixed to a point O, and to the other end is attached a weight which in equilibrium would produce an extension e. Show that if the weight be let fall from rest at O, it will come to stay instantaneously at a point distant $\sqrt{(2ae+e^2)}$ below the position of equilibrium.*

Sol. Proceed as in the preceding example 56. Take $l=a$, $l_1-l=e$ or $l_1=e+a$. Then the required distance $=\sqrt{(l_1^2-l^2)}$ $=\sqrt{[(e+a)^2-a^2]}=\sqrt{(2ae+e^2)}$.

Ex. 58. *A light elastic string of natural length a has one extremity fixed at a point O and the other attached to a body of mass m. The equilibrium length of the string with the body attached*

is a $\sec\theta$. *Show that if the body be dropped from rest at O it will come to instantaneous rest at a depth* $a\tan\theta$ *below the position of equilibrium.*

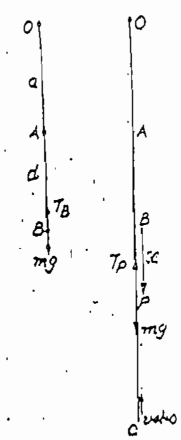
Sol. Proceed as in Example 56. Take $l=a$ and $l_1=a\sec\theta$. We have then, the required depth below the equilibrium position

$=\sqrt{(a^2\sec^2\theta-a^2)}=a\sqrt{(\sec^2\theta-1)}=a\tan\theta$.

Ex. 59. *A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time*

$$\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3}+2\sqrt{3}\right],$$ *where a is the natural length of the string.*

[Lucknow 1976; Kanpur 83; Agra 80; Meerut 88]



Sol. Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and $AB=d$. If T_B is the tension in the string OB, then by Hooke's law,

$$T_B=\lambda\frac{OB-OA}{OA}=\lambda\frac{d}{a},$$

where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B, we have

$$mg=T_B=\lambda\frac{d}{a}=mg\frac{d}{a}. \quad [\because \lambda=mg, \text{ as given}]$$

$\therefore \quad d=a.$

Now the particle is pulled down to a point C such that $OC=4a$ and then let go. It starts moving towards B with velocity zero at C. Let P be the position of the particle at time t, where $BP=x$. [Note that we have taken the position of equilibrium B as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing.]

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P=\lambda\dfrac{a+x}{a}=\dfrac{mg}{a}(a+x)$ in the string OP acting in the direction PO i.e., in the direction of x decreasing.

(ii) The weight mg of the particle acting vertically downwards i.e., in the direction of x increasing.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\frac{mg}{a}(a+x)=-\frac{mgx}{a},$$

Thus $$\frac{d^2x}{dt^2}=-\frac{g}{a}x, \quad \ldots(1)$$

which is the equation of a S.H.M. with centre at the origin B and the amplitude $BC=2a$ which is greater than $AB=a$.

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{a}x^2+k, \text{ where } k \text{ is a constant.}$$

At the point C, $x=BC=2a$, and the velocity $dx/dt=0$;—

$$\therefore \quad k=\frac{g}{a}\cdot 4a^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{g}{a}(4a^2-x^2). \quad \ldots(2)$$

Taking square root of (2), we have

$$\frac{dx}{dt}=-\sqrt{\left(\frac{g}{a}\right)}\sqrt{(4a^2-x^2)},$$

the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt=-\sqrt{\left(\frac{a}{g}\right)}\frac{dx}{\sqrt{(4a^2-x^2)}} \quad \ldots(3)$$

If t_1 be the time from C to A, then integrating (3) from C to A, we get

$$\int_0^{t_1}dt=-\sqrt{\left(\frac{a}{g}\right)}\int_{2a}^{-a}\frac{dx}{\sqrt{(4a^2-x^2)}}$$

or $$t_1=\sqrt{\left(\frac{a}{g}\right)}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a}$$

$$=\sqrt{\left(\frac{a}{g}\right)}\left[\cos^{-1}(-\tfrac{1}{2})-\cos^{-1}(1)\right]=\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3}.$$

Let v_1 be the velocity of the particle at A. Then at A, $x=-a$ and $(dx/dt)^2=v_1^2$.

So from (2), we have $v_1^2=(g/a)(4a^2-a^2)$

or $v_1=\sqrt{(3ag)}$, the direction of v_1 being vertically upwards.

Thus the velocity at A is $\sqrt{(3ag)}$ and is in the upwards direction so that the particle rises above A. Since the tension of the string vanishes at A, therefore at A the simple harmonic motion ceases and the particle when rising above A moves freely under

gravity. Thus the particle rising from A with velocity $\sqrt{(3ag)}$ moves upwards till this velocity is destroyed. The time t_2 for this motion is given by

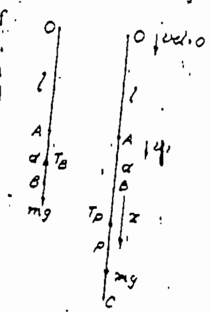
$$0=\sqrt{(3ag)}-gt_2, \text{ so that } t_2=\sqrt{\left(\frac{3a}{g}\right)}.$$

Conditions being the same, the equal time t_2 is taken by the particle in falling freely back to A. From A to C the particle will take the same time t_1 as it takes from C to A. Thus the whole time taken by the particle to return to $C=2(t_1+t_2)$

$$=2\left[\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3}+\sqrt{\left(\frac{3a}{g}\right)}\right]=\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3}+2\sqrt{3}\right].$$

Ex. 60. *A heavy particle of mass m is attached to one end of an elastic string of natural length l, whose other end is fixed at O. The particle is then let fall from rest at O. Show that, part of the motion is simple harmonic, and that, if the greatest depth of the particle below O is $l\cot^2(\theta/2)$, the modulus of elasticity of the string is $\frac{1}{2}mg\tan^2\theta$.* [Meerut 1988]

Sol. Let $OA=l$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and let $AB=d$. In the equilibrium position at B, the tension T_B in the string OB balances the weight mg of the particle. Therefore,



$$T_B=\lambda\frac{d}{l}=mg, \quad \text{...(1)}$$

where λ is the modulus of elasticity of the string. Now the particle is dropped at rest from O. It falls the distance OA freely under gravity. If v_1 be the velocity gained by it at A, we have $v_1=\sqrt{(2gl)}$ in the downward direction. When the particle falls below A, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from A to B the force of tension acting vertically upwards remains less than the weight of the particle acting vertically downwards. Therefore during the fall from A to B the velocity of the particle goes on increasing. When the particle begins to fall below B, its velocity goes on decreasing because now the force of tension exceeds the weight of the particle. Let the particle come to instantaneous rest at C, where $OC=l\cot^2\frac{1}{2}\theta$, as given.

During the motion of the particle below A, let P be its position after any time t, where $BP=x$. [Note that we have taken the position of equilibrium B of the particle as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing.]

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P=\lambda\frac{d+x}{l}$ in the string OP, acting in the direction PO i.e., in the direction of x decreasing.

(ii) The weight mg of the particle acting vertically downwards i.e., in the direction of x increasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{d+x}{l}$$

$$=mg-\frac{\lambda d}{l}-\frac{\lambda x}{l}=-\frac{\lambda x}{l}, \text{ by (1)}.$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{\lambda}{lm}x=-\frac{g}{d}x. \quad \text{...(2)}$$

$$\left[\because \text{ from (1), } \frac{\lambda}{lm}=\frac{g}{d}\right]$$

The equation (2) represents a S. H. M. with centre at the point B and amplitude BC. Hence the motion of the particle below A is simple harmonic.

Multiplying (2) by $2(dx/dt)$ and integrating w.r.t. 't', we get $\left(\frac{dx}{dt}\right)^2=-\frac{g}{d}x^2-D$, where D is a constant.

At the point A, $x=-d$ and the velocity $=dx/dt=\sqrt{(2gl)}$.

$$\therefore D=2gl+gd.$$

$$\therefore \text{ we have, (velocity)}^2=\left(\frac{dx}{dt}\right)^2=-\frac{g}{d}x^2+2gl+gd. \quad \text{...(3)}$$

The above equation (3) gives the velocity of the particle at any point between A and C. At C, $x=BC=OC-OB=l\cot^2\frac{1}{2}\theta-(l+d)$, and $dx/dt=0$. Therefore (3) gives

$$0=-\frac{g}{d}[(l\cot^2\tfrac{1}{2}\theta-l)-d]^2+2gl+gd$$

$$=-\frac{g}{d}[(l\cot^2\tfrac{1}{2}\theta-l)^2+d^2-2ld(\cot^2\tfrac{1}{2}\theta-1)]+2gl+gd$$

$$=-\left[\frac{g}{d}(l\cot^2\tfrac{1}{2}\theta-l)^2-2gl\cot^2\tfrac{1}{2}\theta\right]$$

$$=-\left[\frac{\lambda}{ml}(l\cot^2\tfrac{1}{2}\theta-l)^2-2gl\cot^2\tfrac{1}{2}\theta\right], \left[\because \frac{g}{d}=\frac{\lambda}{ml} \text{ by (1)}\right].$$

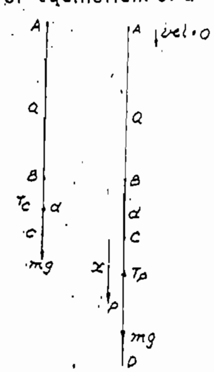
$$\lambda=\frac{2mgl^2\cot^2\frac{1}{2}\theta}{(l\cot^2\frac{1}{2}\theta-l)^2}=\frac{2mg\cot^2\frac{1}{2}\theta}{(\cot^2\frac{1}{2}\theta-1)^2}$$

$$=\frac{2mg\cot^2\frac{1}{2}\theta}{(\cos^2\frac{1}{2}\theta-\sin^2\frac{1}{2}\theta)^2}\cdot\sin^4\tfrac{1}{2}\theta$$

$$=\tfrac{1}{2}\cdot\frac{mg.4\cos^2\frac{1}{2}\theta\sin^2\frac{1}{2}\theta}{\cos^2\theta}=\tfrac{1}{2}mg\cdot\frac{\sin^2\theta}{\cos^2\theta}=\tfrac{1}{2}mg\tan^2\theta.$$

Ex. 61. *One end of a light elastic string of natural length a and modulus of elasticity $2mg$ is attached to a fixed point A and the other end to a particle of mass m. The particle initially held at rest at A, is let fall. Show that the greatest extension of the string is $\frac{1}{2}a\ (1+\sqrt{5})$ during the motion and show that the particle will reach back A again after a time $(\pi+2\tan^{-1} 2)\ \sqrt{(2a/g)}$.*

Sol. $AB=a$ is the natural length of an elastic string whose one end is fixed at A. Let C be the position of equilibrium of a particle of mass m attached to the other end of the string and let $BC=d$. In the position of equilibrium of the particle at C, the tension $T_C=\lambda\,\frac{d}{a}=2mg\,\frac{d}{a}$ in the string AC balances the weight mg of the particle.



$\therefore\quad mg=2mg\ (d/a)$ or $d=a/2$. $\quad\ldots(1)$

Now the particle is dropped at rest from A. It falls the distance AB freely under gravity. If v_1 be the velocity gained at B, we have $v_1=\sqrt{(2ga)}$ in the downward direction. When the particle falls below B, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from B to C the velocity of the particle goes on increasing as the tension remains less than the weight of the particle and when the particle begins to fall below C, its velocity goes on decreasing because now the force of tension exceeds the weight of the particle. Let the particle come to instantaneous rest at D.

During the motion of the particle below B, let P be its position after any time t, where $CP=x$. If T_P be the tension in the string AP, we have $T_P=\lambda\,\frac{d+x}{a}=2mg\,\frac{\frac{1}{2}a+x}{a}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\,\frac{d^2x}{dt^2}=mg-T_P=mg-2mg\,\frac{\frac{1}{2}a+x}{a}=-\frac{2mg}{a}\,x.$$

$$\therefore\quad\frac{d^2x}{dt^2}=-\frac{2g}{a}\,x,\qquad\ldots(2)$$

which is the equation of a S. H. M. with centre at the point C and amplitude CD.

Multiplying (2) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}\,x^2+k,\ \text{where } k \text{ is a constant.}$$

At the point B, the velocity

$$=dx/dt=\sqrt{(2ga)}\ \text{and}\ x=-d=-\frac{a}{2}.$$

$$\therefore\quad k=2ga+\frac{2g}{a}\cdot\frac{a^2}{4}=2ga+\frac{2ga}{4}=\frac{5ag}{2}.$$

$$\therefore\quad\text{We have}\left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}\,x^2+\frac{5ag}{2}.\qquad\ldots(3)$$

The equation (3) gives the velocity of the particle at any point between B and D. At D, $x=CD$ and $dx/dt=0$. So putting $dx/dt=0$ in (3), we have

$$0=-\frac{2g}{a}\,x^2+\frac{5ag}{2}\quad\text{or}\quad x^2=\frac{5a^2}{4}$$

or $\quad x=\frac{a}{2}\sqrt{5}=CD.$

$\therefore\quad$ the greatest extension of the string

$$=BC+CD=\tfrac{1}{2}a+\tfrac{1}{2}a\sqrt{5}=\tfrac{1}{2}a\ (1+\sqrt{5}).$$

Now from (3), we have $\left(\frac{dx}{dt}\right)^2=\frac{2g}{a}\left[\frac{5}{4}a^2-x^2\right].$

$$\therefore\quad\frac{dx}{dt}=\sqrt{\left(\frac{2g}{a}\right)}\sqrt{\left[\frac{5}{4}a^2-x^2\right]},$$ the $+$ive sign has been taken, because the particle is moving in the direction of x increasing.

Separating the variables, we have $dt=\sqrt{\left(\frac{a}{2g}\right)}\,\frac{dx}{\sqrt{[\frac{5}{4}a^2-x^2]}}$

If t_1 is the time from B to D, then

$$\int_0^{t_1}dt=\sqrt{\left(\frac{a}{2g}\right)}\int_{-a/2}^{(a\sqrt{5})/2}\frac{dx}{\sqrt{[\frac{5}{4}a^2-x^2]}}$$

or

$$t_1=\sqrt{\left(\frac{a}{2g}\right)}\left[\sin^{-1}\left\{\frac{x}{(a\sqrt{5})/2}\right\}\right]_{-a/2}^{(a\sqrt{5})/2}$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\left[\sin^{-1}1+\sin^{-1}\frac{1}{\sqrt{5}}\right]=\sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2}+\tan^{-1}\tfrac{1}{2}\right)$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2}+\cot^{-1} 2\right)=\sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2}+\frac{\pi}{2}-\tan^{-1} 2\right)$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\ (\pi-\tan^{-1} 2).$$

And if t_2 is the time from A to B, (while falling freely under gravity), then

$$a=0.t_2+\frac{1}{2}\ gt_2^2 \quad \text{or} \quad t_2=\sqrt{\left(\frac{2a}{g}\right)}.$$

$\therefore$ the total time to return back to $A=2$ (time from A to D)

$$=2\ (t_2+t_1)=2\left[\sqrt{\left(\frac{a}{2g}\right)}\ (\pi-\tan^{-1} 2)+\sqrt{\left(\frac{2a}{g}\right)}\right]$$

$$=\sqrt{\left(\frac{2a}{g}\right)}\ [\pi-\tan^{-1} 2+2].$$

This proves the required result.

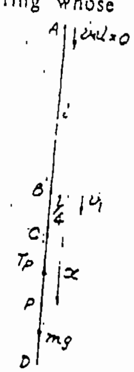
Ex. 62. *A light elastic string AB of length l is fixed at A and is such that if a weight w be attached to B, the string will be stretched to a length 2l. If a weight ¼w be attached to B and let fall from the level of A prove that (i) the amplitude of the S.H.M. that ensues is 3l/4; (ii) the distance through which it falls is 2l; and (iii) the period of oscillation is*

$$\sqrt{\left(\frac{l}{4g}\right)}\ (4\sqrt{3}+\pi+2\ \sin^{-1} \tfrac{1}{3}).$$

[Meerut 1981S, 88S]

Sol. $AB=l$ is the natural length of an elastic string whose one end is fixed at A. Let λ be the modulus of elasticity of the string. If a weight w be attached to the other end of the string, it extends the string to a length $2l$ while hanging in equilibrium. Therefore

$$w=\lambda\ \frac{2l-l}{l}=\lambda. \qquad \ldots(1)$$



Now in the actual problem a particle of weight $\frac{1}{4}w$ or mass $\frac{1}{4}(w/g)$ is attached to the free end of the string. Let C be the position of equilibrium of this weight $\frac{1}{4}w$. Then considering the equilibrium of this weight at C, we have

$$\tfrac{1}{4}w=\lambda\ \frac{BC}{l}=w\ \frac{BC}{l} \qquad [\because \text{ by } (1),\ \lambda=w]$$

$\therefore\ BC=\frac{1}{4}l.$

Now the weight $\frac{1}{4}w$ is dropped from A. It falls the distance $AB\ (=l)$ freely under gravity. If v_1 be the velocity gained by this weight at B, we have $v_1=\sqrt{(2gl)}$ in the downward direction. When this weight falls below B, the string begins to extend beyond its natural length and the tension begins to operate. The velocity of the weight continues increasing upto C, after which it starts decreasing. Suppose the weight comes to instantaneous rest at D, where $CD=a$.

During the motion of the weight below B, let P be its position after any time t, where $CP=x$. [Note that we have taken C as origin and CP is the direction of x increasing]. If T_P be the tension in the string AP, we have $T_P=w\ \dfrac{\frac{1}{4}l+x}{l}$ acting vertically upwards.

The equation of motion of this weight $w/4$ at P is

$$\frac{1}{4}\frac{w}{g}\ \frac{d^2x}{dt^2}=\frac{1}{4}\ w-w\ \frac{\frac{1}{4}l+x}{l}=\frac{1}{4}\ w-\frac{1}{4}\ w-w\ \frac{x}{l}$$

or
$$\frac{1}{4}\frac{w}{g}\ \frac{d^2x}{dt^2}=-w\ \frac{x}{l} \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{4g}{l}\ x, \qquad \ldots(2)$$

which is the equation of a S.H.M. with centre at the origin C, and amplitude $CD\ (=a)$. The equation (2) holds good so long as the string is stretched *i.e.*, for the motion of the weight from B to D.

Multiplying (2) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{4g}{l}\ x^2+k, \text{ where } k \text{ is a constant.}$$

At B, $x=-\frac{1}{4}l$ and $dx/dt=\sqrt{(2gl)}$;

$$\therefore\quad 2gl=-\frac{4g}{l}\cdot\frac{1}{16}\ l^2+k \quad \text{or} \quad k=\frac{9}{4}\ gl.$$

Thus, we have
$$\left(\frac{dx}{dt}\right)^2=-\frac{4g}{l}\ x^2+\frac{9}{4}\ gl=\frac{4g}{l}\left(\frac{9}{16}\ l^2-x^2\right) \qquad \ldots(3)$$

The equation (3) gives velocity at any point between B and D. At D, $x=a$, $dx/dt=0$. Therefore (3) gives

$$0=\frac{4g}{l}\left(\frac{9}{16}\ l^2-a^2\right) \quad \text{or} \quad a=\tfrac{3}{4}l.$$

Hence the amplitude a of the S.H.M. that ensues is $\frac{3}{4}l$.

Also the total distance through which the weight falls $=AB+BC+CD=l+\frac{1}{4}l+\frac{3}{4}l=2l.$

Now let t_1 be the time taken by the weight to fall freely under gravity from A to B.

Then using the formula $v=u+ft$, we get

$$\sqrt{(2gl)}=0+gt_1 \quad \text{or} \quad t_1=\sqrt{(2l/g)}.$$

Again let t_2 be the time taken by the weight to fall from B to D while moving in S.H.M. From (3), on taking square root, we

get $\dfrac{dx}{dt}=+\sqrt{\left(\dfrac{4g}{l}\right)}\sqrt{\left(\dfrac{9}{16}l^2-x^2\right)}$,

where the +ive sign has been taken because the weight is moving in the direction of x increasing. Separating the variables, we get

$$\sqrt{\left(\frac{l}{4g}\right)}\cdot\frac{dx}{\sqrt{\left(\frac{9l^2}{16}-x^2\right)}}=dt.$$

Integrating from B to D, we get

$$\int_0^{t_2}dt=\sqrt{\left(\frac{l}{4g}\right)}\int_{-l/4}^{3l/4}\frac{dx}{\sqrt{\left(\frac{9}{16}l^2-x^2\right)}}$$

$$\therefore\quad t_2=\sqrt{\left(\frac{l}{4g}\right)}\left[\sin^{-1}\frac{x}{\frac{3}{4}l}\right]_{-l/4}^{3l/4}=\sqrt{\left(\frac{l}{4g}\right)}\left[\sin^{-1}1-\sin^{-1}\left(-\tfrac{1}{3}\right)\right]$$

$$=\sqrt{\left(\frac{l}{4g}\right)}\left[\tfrac{1}{2}\pi+\sin^{-1}\tfrac{1}{3}\right].$$

Hence the total time taken to fall from A to $D=t_1+t_2$

$$=\sqrt{\left(\frac{2l}{g}\right)}+\sqrt{\left(\frac{l}{4g}\right)}\left[\tfrac{1}{2}\pi+\sin^{-1}\tfrac{1}{3}\right]$$

$$=\sqrt{\left(\frac{l}{4g}\right)}\left[\frac{\pi}{2}+\sin^{-1}\tfrac{1}{3}+2\sqrt{2}\right].$$

Now after instantaneous rest at D, the weight begins to move upwards. From D to B it moves in S.H.M. whose equation is (2). At B the string becomes slack and S.H.M. ceases. The velocity of the weight at B is $\sqrt{(2gl)}$ upwards. Above B the weight rises freely under gravity and comes to instantaneous rest at A. Thus it oscillates again and again between A and D.

The time period of one complete oscillation $=2$. time from A to $D=2(t_1+t_2)=\sqrt{\left(\dfrac{l}{4g}\right)}\left\{\pi+4\sqrt{2}+2\sin^{-1}\tfrac{1}{3}\right\}$.

Ex. 63. *A heavy particle of mass m is attached to one end of an elastic string of natural length l ft., whose modulus of elasticity is equal to the weight of the particle and the other end is fixed at O. The particle is let fall from O. Show that a part of the motion is simple harmonic and that the greatest depth of the particle below O is $(2+\sqrt{3})\,l$ ft. Show that this depth is attained in time*

$[\sqrt{2}+\pi-\cos^{-1}(1/\sqrt{3})]\sqrt{(l/g)}$ seconds. [Lucknow 1980]

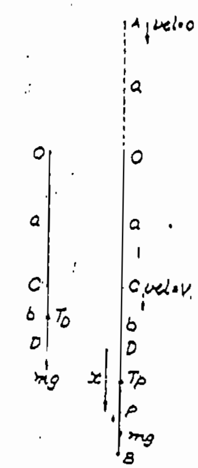
Sol. Proceed as in the preceding example.

Ex. 64. *A particle of mass m is attached to one end of an elastic string of natural length a and modulus of elasticity 2mg, whose other end is fixed at O. The particle is let fall from A, when A is*

vertically above O and OA=a. Show that its velocity will be zero at B, where OB=3a. (Meerut 77, 83)

Calculate also the time from A to B.

Sol. Let $OC=a$, be the natural length of an elastic string suspended from the fixed point O. The modulus of elasticity λ of the string is given to be equal to $2mg$, where m is the mass of the particle attached to the other end of the string.



If D is the position of equilibrium of the particle such that $CD=b$, then at D the tension T_D in the string OD balances the weight of the particle.

$$\therefore\quad mg=T_D=\lambda\,\frac{b}{a}=2mg.\frac{b}{a}$$

or $\quad b=a/2.$

The particle is let fall from A where $OA=a$. Then the motion from A to C will be freely under gravity.

If V is the velocity of the particle gained at the point C, then

$$V^2=0+2g.2a\quad\text{or}\quad V=2\sqrt{(ag)},\qquad\ldots(1)$$

in the downward direction.

As the particle moves below C, the string begins to extend beyond its natural length and the tension begins to operate. The velocity of the particle continues increasing upto D after which it starts decreasing. Suppose that the particle comes to instantaneous rest at B. During the motion below C, let P be the position of the particle at any time t, where $DP=x$. If T_P is the tension in the string OP, we have

$$T_P=\lambda\,\frac{b+x}{a},\ \text{acting vertically upwards.}$$

$\therefore$ The equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-T_P=mg-\lambda\,\frac{b+x}{a}$$

$$=mg-2mg.\frac{\frac{1}{2}a+x}{a}=-\frac{2mg}{a}\,x$$

or $$\frac{d^2x}{dt^2}=-\frac{2g}{a}\,x,\qquad\ldots(2)$$

which represents a S. H. M. with centre at D and holds good for the motion from C to B.

Multiplying both sides of (2) by 2 (dx/dt) and then integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a} x^2 + k, \quad \text{where } k \text{ is a constant.}$$

But at C, $x = -DC = -b = -a/2$ and $(dx/dt)^2 = V^2 = 4ag$.

$$4ag = -\frac{2g}{a} \cdot \frac{a^2}{4} + k \quad \text{or} \quad k = \tfrac{9}{2} ag.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a} x^2 + \frac{9}{2} ag$$

or $$\left(\frac{dx}{dt}\right)^2 = \frac{2g}{a} \left(\frac{9}{4} a^2 - x^2\right). \quad \ldots(3)$$

If the particle comes to instantaneous rest at B where $DB = x_1$, (say), then at B, $x = x_1$ and $dx/dt = 0$. Therefore from (3), we have

$$0 = \frac{2g}{a}\left(\frac{9}{4} a^2 - x_1^2\right), \quad \text{giving } x_1 = \frac{3}{2} a.$$

Now $OB = OC + CD + DB = a + \frac{1}{2}a + \frac{3}{2}a = 3a$, which proves the first part of the question.

To find the time from A to B.

If t_1 is the time from A to C, then from $s = ut + \frac{1}{2}ft^2$,

$$2a = 0 + \tfrac{1}{2} g t_1^2. \quad \therefore \quad t_1 = 2\sqrt{(a/g)}. \quad \ldots(4)$$

Now from (3), we have

$$\frac{dx}{dt} = \sqrt{\left(\frac{2g}{a}\right)} \sqrt{\left(\frac{9}{4} a^2 - x^2\right)},$$

the +ive sign has been taken because the particle is moving in the direction of x increasing

or $$dt = \sqrt{\left(\frac{a}{2g}\right)} \cdot \frac{dx}{\sqrt{(\frac{9}{4}a^2 - x^2)}}$$

Integrating from C to B, the time t_2 from C to B is given by

$$t_2 = \sqrt{\left(\frac{a}{2g}\right)} \int_{x=-a/2}^{3a/2} \frac{dx}{\sqrt{(\frac{9}{4}a^2 - x^2)}}$$

$$= \sqrt{\left(\frac{a}{2g}\right)} \cdot \left[\sin^{-1}\left(\frac{x}{3a/2}\right)\right]_{-a/2}^{3a/2}$$

$$= \sqrt{\left(\frac{a}{2g}\right)} \cdot \left[\sin^{-1} 1 - \sin^{-1}\left(-\frac{1}{3}\right)\right]$$

$$= \sqrt{\left(\frac{a}{2g}\right)} \cdot \left[\frac{\pi}{2} + \sin^{-1}\left(\frac{1}{3}\right)\right].$$

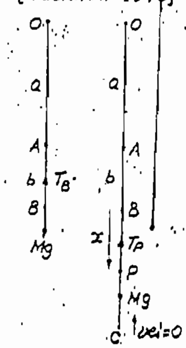
$\therefore$ the time from A to $B = t_1 + t_2$

$$= 2\sqrt{(a/g)} + \sqrt{(a/2g)} \cdot [\pi/2 + \sin^{-1}(1/3)]$$

$$= \tfrac{1}{2}\sqrt{(a/2g)}\, [4\sqrt{2} + \pi + 2 \sin^{-1}(1/3)].$$

Ex. 65. *Two bodies of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest; M' falls off, show that the distance of M from the upper end of the string at time t is $a + b + c \cos\{\sqrt{(g/b)}\, t\}$, where a is the unstretched length of the string, b and c the distances by which it would be stretched when supporting M and M' respectively.*

[Lucknow 1978]



Sol. Let $OA = a$ be the natural length of an elastic string suspended from the fixed point O. If B is the position of equilibrium of the particle of mass M attached to the lower end of the string and $AB = b$, then

$$Mg = \lambda \frac{AB}{a} = \lambda \frac{b}{a} \quad \ldots(1)$$

Similarly $$M'g = \lambda \frac{c}{a} \quad \ldots(2)$$

Adding (1) and (2) we have

$$(M + M')\, g = \lambda \frac{b + c}{a}$$

Thus the string will be stretched by the distance $b + c$ when supporting both the masses M and M' at the lower end. Let OC be the stretched length of the string when both the masses M and M' are attached to its lower end. Then

$$AC = b + c \text{ and so } BC = AC - AB = b + c - b = c.$$

Now when M' falls off at C, the mass M will begin to move towards B starting with velocity zero at C. Let P be the position of the particle of mass M at any time t, where $BP = x$.

If T_P be the tension in the string OP, then

$$T_P = \lambda \frac{b + x}{a}, \text{ acting vertically upwards.}$$

$\therefore$ the equation of motion of the particle of mass M at P is

$$M \frac{d^2x}{dt^2} = Mg - T_P = Mg - \lambda \frac{b + x}{a}$$

$$= Mg - \lambda \frac{b}{a} - \frac{\lambda x}{a}$$

$$=Mg-Mg-\frac{Mg}{b}x, \quad \left[\because \text{ from (1), } Mg=\frac{\lambda b}{a}\right]$$

$$=-\frac{Mg}{b}x,$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{g}{b}x, \qquad \ldots(3)$$

which represents a S. H. M. with centre at B and amplitude BC.

Multiplying both sides of (3) by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{b}x^2+k, \text{ where } k \text{ is a constant.}$$

But at the point C, $x=BC=c$ and $dx/dt=0$.

$$\therefore \quad 0=-(g/b)c^2+k \text{ or } k=(g/b)c^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{g}{b}(c^2-x^2)$$

or
$$\frac{dx}{dt}=-\sqrt{\left(\frac{g}{b}\right)}\sqrt{(c^2-x^2)},$$

the $-$ive sign has been taken since the particle is moving in the direction of x decreasing.

$$\therefore \quad dt=-\sqrt{\left(\frac{b}{g}\right)}\frac{dx}{\sqrt{(c^2-x^2)}}, \text{ separating the variables.}$$

Integrating, $t=\sqrt{(b/g)}\cos^{-1}(x/c)+D$, where D is a constant.

But at C, $t=0$ and $x=c$; $\therefore$ $D=0$.

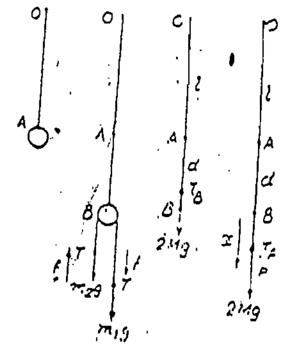
$$\therefore \quad t=\sqrt{(b/g)}\cos^{-1}(x/c)$$

or
$$x=BP=a\cos\{\sqrt{(g/b)}\,t\}.$$

$\therefore$ the required distance of the particle of mass M at time t from the point O

$$=OP=OA+AB+BP=a+b+c\cos\{\sqrt{(g/b)}\,t\}.$$

Ex. 66. *A smooth light pulley is suspended from a fixed point by a spring of natural length l and modulus of elasticity mg. If masses m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley, show that the pulley executes simple harmonic motion about a centre whose depth below the point of suspension is $l\{1+(2M/n)\}$, where M is the harmonic mean between m_1 and m_2.* [Meerut 1981, 84, 85 S]



Sol. Let a smooth light pulley be suspended from a fixed point O by a spring OA of natural length l and modulus of elasticity $\lambda=ng$. Let B be the position of equilibrium of the pulley when masses m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley. Let T be the tension in the inextensible string passing round the pulley. Let us first find the value of T.

Let f be the common acceleration of the particles m_1, m_2 which hang at the ends of a light inextensible string passing round the pulley. If $m_1>m_2$, then the equations of motion of m_1, m_2 are

$$m_1g-T=m_1f \quad \text{and} \quad T-m_2g=m_2f.$$

Solving, we get $T=\dfrac{2m_1m_2}{(m_1+m_2)}g=Mg$,

where $M=\dfrac{2m_1m_2}{m_1+m_2}=$ the harmonic mean between m_1 and m_2.

Now the pressure on the pulley $=2T=2Mg$ and therefore the pulley, which itself is light, behaves like a particle of mass $2M$.

Now the problem reduces to the vertical motion of a mass $2M$ attached to the end A of the string OA whose other end is fixed at O. If B is the equilibrium position of the mass $2M$ and $AB=d$, then the tension T_B in the spring OB is $\lambda(d/l)$, acting vertically upwards.

For equilibrium of the pulley of mass $2M$ at the point B, we have

$$2Mg=T_B=\lambda\frac{d}{l}=ng\frac{d}{l}$$

or
$$d=\frac{2Ml}{n}. \qquad \ldots(1)$$

Now let the particle of mass $2M$ be slightly pulled down and then let go. If P is the position of this particle at time t such that $BP=x$, then the tension in the spring OP

$$=T_P=\lambda\frac{d+x}{l}=ng.\frac{d+x}{l}, \text{ acting vertically upwards.}$$

$\therefore$ The equation of motion of the pulley is given by

$$2M\frac{d^2x}{dt^2}=2Mg-T_P$$

$$=2Mg-ng\frac{d+x}{l}=2Mg-ng\frac{d}{l}-\frac{ng}{l}x=-\frac{ng}{l}x, \quad \text{[by (1)]}$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{ng}{2Ml}x,$$

which represents a simple harmonic motion about the centre B.

Hence the pulley executes simple harmonic motion with centre at the point B whose depth below the point of suspension O is given by

$$OB=OA+AB=l+d$$

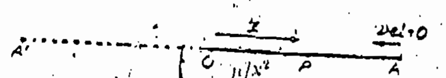
$$=l+\frac{2Ml}{n}=l\left(1+\frac{2M}{n}\right)$$

§ 11. Motion under inverse square law.

A particle moves in a straight line under an attraction towards a fixed point on the line, which varies inversely as the square of the distance from the fixed point. If the particle was initially at rest, to investigate the motion.

[Lucknow 1977, Meerut 83S, 84, 86, 87S]

Let a particle start from rest from a point A such that $OA=a$, where O is the fixed point (*i.e.*, the centre of force) on the line and is taken as origin. Let P be the position of the particle at any time t, such that $OP=x$. Then the acceleration at $P=\mu/x^2$, towards O, where μ is a constant.



$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}. \quad \ldots(1)$$

[$-$ive sign has been taken because d^2x/dt^2 is positive in the direction of x increasing while here μ/x^2 acts in the direction of x decreasing].

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=\frac{2\mu}{x}+A$, where A is constant of integration.

But at A, $x=OA=a$ and $dx/dt=0$.

$$\therefore \quad 0=\frac{2\mu}{a}+A \quad \text{or} \quad A=-\frac{2\mu}{a}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right), \quad \ldots(2)$$

which gives the velocity of the particle at any distance x from the centre of force O.

From (2), we have on taking square root

$$\frac{dx}{dt}=-\sqrt{\left(\frac{2\mu}{a}\right)}\cdot\sqrt{\left(\frac{a-x}{x}\right)}.$$

[Here $-$ive sign is taken since the particle is moving in the direction of x decreasing].

Separating the variables, we get

$$dt=-\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\sqrt{\left(\frac{x}{a-x}\right)}\,dx.$$

Integrating, $t=-\sqrt{\left(\frac{a}{2\mu}\right)}\int\sqrt{\left(\frac{x}{a-x}\right)}\,dx+B$, where B is constant of integration.

Putting $x=a\cos^2\theta$, so that $dx=-2a\cos\theta\sin\theta\,d\theta$, we have

$$t=\sqrt{\left(\frac{a}{2\mu}\right)}\int\sqrt{\left(\frac{a\cos^2\theta}{a-a\cos^2\theta}\right)}\cdot 2a\sin\theta\cos\theta\,d\theta+B$$

$$=a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\int 2\cos^2\theta\,d\theta+B=a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\int(1+\cos 2\theta)\,d\theta+B$$

$$=a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\left(\theta+\frac{\sin 2\theta}{2}\right)+B=a\sqrt{\left(\frac{a}{2\mu}\right)}(\theta+\sin\theta\cos\theta)+B$$

$$=a\sqrt{\left(\frac{a}{2\mu}\right)}[\theta+\sqrt{(1-\cos^2\theta)}.\cos\theta]+B.$$

But $x=a\cos^2\theta$ means $\cos\theta=\sqrt{(x/a)}$ and $\theta=\cos^{-1}\sqrt{(x/a)}$.

$$\therefore \quad t=a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left(1-\frac{x}{a}\right)}\cdot\sqrt{\left(\frac{x}{a}\right)}\right]+B.$$

But initially at A, $t=0$ and $x=OA=a$.

$$\therefore \quad 0=a\sqrt{\left(\frac{a}{2\mu}\right)}[0+0]+B \quad \text{or} \quad B=0.$$

$$\therefore \quad t=a\sqrt{\left(\frac{a}{2\mu}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left(1-\frac{x}{a}\right)}\cdot\sqrt{\left(\frac{x}{a}\right)}\right], \quad \ldots(3)$$

which gives the time from the initial position A to any point distant x from the centre of force.

Putting $x=0$ in (3), the time t_1 taken by the particle from A to O is given by

$$t_1 = a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\left[\frac{\pi}{2}+0\right]=\frac{\pi}{2}\sqrt{\left(\frac{a^3}{2\mu}\right)}. \quad \ldots(4)$$

Putting $x=0$ in (2), we see that the velocity at O is infinite and therefore the particle moves to the left of O. But the acceleration on the particle is towards O, so the particle moves to the left of O under retardation which is inversely proportional to the square of the distance from O. The particle will come to instantaneous rest at A', where $OA'=OA=a$, and then retrace its path. Thus, the particle will oscillate between A and A'.

Time of one complete oscillation $=4\times$(time from A to O)

$$=4t_1=2\pi\sqrt{(a^3/2\mu)}.$$

§ 12. Motion of a particle under the attraction of the earth. Newton's law of gravitation. When a particle moves under the attraction of the earth, the acceleration acting on it towards the centre of the earth will be as follows :

1. When the particle moves (upwards or downwards) outside the surface of the earth, the acceleration varies inversely as the square of the distance of the particle from the centre of the earth.
2. When the particle moves inside the earth through a hole made in the earth, the acceleration varies directly as the distance of the particle from the centre of the earth.
3. The value of the acceleration at the surface of the earth is g.

Illustrative Examples :

Ex. 67. *Show that the time occupied by a body, under the acceleration K/x^2 towards the origin, to fall from rest at distance a to distance x from the attracting centre can be put in the form*

$$\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right].$$

Prove also that the time occupied from $x=3a/4$ to $a/4$ is one-third of the whole time of descent from a to 0.

Sol. For the first part see equation (3) of § 11. (Deduce this equation here).

Thus the time t measured from the initial position $x=a$ to any point at a distance x from the centre O is given by

$$t=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right]. \quad \ldots(1)$$

[Note that here $\mu=K$].

Let t_1 be the whole time of descent from $x=a$ to $x=0$. Then at O, $x=0$, $t=t_1$. Putting these values in the relation (1) connecting x and t, we have

$$t_1=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}0+0\right]=\frac{\pi}{2}\sqrt{\left(\frac{a^3}{2K}\right)}. \quad \ldots(2)$$

Now let t_2 be the time from $x=a$ to $x=3a/4$. Then putting $x=3a/4$ and $t=t_2$ in (1), we get

$$t_2=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\sqrt{\left(\frac{3}{4}\cdot\frac{1}{4}\right)}\right]=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right].$$

Again let t_3 be the time from $x=a$ to $x=a/4$. Then putting $x=a/4$ and $t=t_3$ in (1), we get

$$t_3=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\tfrac{1}{2}+\sqrt{\left(\frac{1}{4}\cdot\frac{3}{4}\right)}\right]=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right].$$

Therefore if t_4 be the time from $x=3a/4$ to $x=a/4$, we have

$$t_4=t_3-t_2=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]=\frac{\pi}{6}\sqrt{\left(\frac{a^3}{2K}\right)}$$

$$=\tfrac{1}{3}\left[\frac{\pi}{2}\sqrt{\left(\frac{a^3}{2K}\right)}\right]=\tfrac{1}{3}t_1, \text{ from (2).}$$

Hence the time from $x=3a/4$ to $x=a/4$ is one-third of the whole time of descent from $x=a$ to $x=0$.

Note. To find the time from $x=3a/4$ to $x=a/4$, we have first found the times from $x=a$ to $x=3a/4$ and from $x=a$ to $x=a/4$ because in the relation (1) connecting x and t the time t has been measured from the point $x=a$.

Ex. 68. *Show that the time of descent to the centre of force, varying inversely as the square of the distance from the centre, through first half of its initial distance is to that through the last half as $(\pi+2):(\pi-2)$.*

[Lucknow 1975; Meerut 83P; Rohilkhand 87]

Sol. Let the particle start from rest from the point A at a distance a from the centre of force O. If x is the distance of the particle from the centre of force at time t, then the equation of motion of the particle at time t is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}.$$

Now proceeding as in § 11, page 126, we find that the time t measured from the initial position $x=a$ to any point distant x from the centre O is given by the equation

$$t=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right]. \quad \ldots(1)$$

[Give the complete proof for deducing this equation here].

Now let B be the middle point of OA. Then at B, $x=a/2$.

Let t_1 be the time from A to B i.e., the time to cover the first half of the initial displacement. Then at B, $x=a/2$ and $t=t_1$. So putting $x=a/2$ and $t=t_1$ in (1), we get

$$t_1=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)+\tfrac{1}{2}\right]=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\frac{\pi}{4}+\frac{1}{2}\right].$$

Again let t_2 be the time from A to O. Then at O, $x=0$ and $t=t_2$. So putting $x=0$ and $t=t_2$ in (1), we get

$$t_2=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}0+0\right]=\sqrt{\left(\frac{a^3}{2\mu}\right)}\cdot\frac{\pi}{2}.$$

Now if t_3 be the time from B to O (i.e., the time to cover the last half of the initial displacement), then

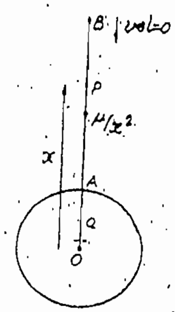
$$t_3=t_2-t_1=\sqrt{\left(\frac{a^3}{2\mu}\right)}\cdot\left[\frac{\pi}{4}-\frac{1}{2}\right].$$

We have $\dfrac{t_1}{t_3}=\dfrac{\frac{1}{4}\pi+\frac{1}{2}}{\frac{1}{4}\pi-\frac{1}{2}}=\dfrac{\pi+2}{\pi-2}$, which proves the required result.

Ex. 69. *If the earth's attraction vary inversely as the square of the distance from its centre and g be its magnitude at the surface, the time of falling from a height h above the surface to the surface is* $\sqrt{\left(\frac{a+h}{2g}\right)}\left[\sqrt{\left(\frac{h}{a}\right)}+\frac{a+h}{a}\sin^{-1}\sqrt{\left(\frac{h}{a+h}\right)}\right]$, *where a is the radius of the earth.* [Meerut 1981, 84, 85, 85S, 90; Lucknow 79; Kanpur 74]

Sol. Let O be the centre of the earth taken as origin. Let OB be the vertical line through O which meets the surface of the earth at A and let $AB=h$; $OA=a$ is the radius of the earth.

A particle falls from rest from B towards the surface of the earth. Let P be the position of the particle at any time t, where $OP=x$. [Note that O is the origin and OP is the direction of x increasing]. According to the Newton's law of gravitation the acceleration of the particle at P is μ/x^2 directed towards O i.e., in the direction of x decreasing. Hence the equation of motion of the particle at P is



$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}\qquad\ldots(1)$$

The equation (1) holds good for the motion of the particle from B to A. At A (i.e., on the surface of the earth $x=a$ and $d^2x/dt^2=-g$. Therefore $-g=-\mu/a^2$ or $\mu=a^2g$. Thus the equation (1) becomes

$$\frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}.$$

Integrating, we get

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+C.\quad \text{At } B,\ x=OB=a+h,\ \frac{dx}{dt}=0.$$

$$\therefore\quad 0=\frac{2a^2g}{a+h}+C\quad\text{or}\quad C=-\frac{2a^2g}{a+h}.$$

Thus, we have

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}-\frac{2a^2g}{a+h}=2a^2g\left(\frac{1}{x}-\frac{1}{a+h}\right).$$

For the sake of convenience let us put $a+h=b$. Then

$$\left(\frac{dx}{dt}\right)^2=2a^2g\left(\frac{1}{x}-\frac{1}{b}\right)=\frac{2a^2g}{b}\left(\frac{b-x}{x}\right).\qquad\ldots(2)$$

The equation (2) gives velocity at any point from B to A.

From (2) on taking square root, we get

$$\frac{dx}{dt}=-a\sqrt{\left(\frac{2g}{b}\right)}\sqrt{\left(\frac{b-x}{x}\right)},$$

where the negative sign has been taken because the particle is moving in the direction of x decreasing.

$$\therefore\quad dt=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\sqrt{\left(\frac{x}{b-x}\right)}\,dx.\qquad\ldots(3)$$

Let t_1 be the time from B to A. Then integrating (3) from B to A, we get

$$\int_0^{t_1}dt=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_b^a\sqrt{\left(\frac{x}{b-x}\right)}\,dx,$$

$$\therefore\quad t_1=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_b^a\sqrt{\left(\frac{x}{b-x}\right)}\,dx.$$

Put $x=b\cos^2\theta$, so that $dx=-2b\cos\theta\sin\theta\,d\theta$.

$$\therefore\quad t_1=\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_0^{\cos^{-1}\sqrt{(a/b)}}\frac{\cos\theta}{\sin\theta}\,2b\cos\theta\sin\theta\,d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\int_0^{\cos^{-1}\sqrt{(a/b)}}2\cos^2\theta\,d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\int_0^{\cos^{-1}\sqrt{(a/b)}}(1+\cos 2\theta)\,d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\Big[\theta+\tfrac{1}{2}\sin 2\theta\Big]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\Big[\theta+\sin\theta\cos\theta\Big]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\Big[\theta+\cos\theta\sqrt{(1-\cos^2\theta)}\Big]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$= \sqrt{\left(\frac{b}{2g}\right)} \frac{b}{a}\left[\cos^{-1} \sqrt{\left(\frac{a}{b}\right)} + \sqrt{\left(\frac{a}{b}\right)} \sqrt{\left(1-\frac{a}{b}\right)}\right]$$

$$= \sqrt{\left(\frac{b}{2g}\right)} \left[\frac{b}{a}\cos^{-1} \sqrt{\left(\frac{a}{b}\right)} + \sqrt{\left(\frac{b}{a}\right)} \sqrt{\left(1-\frac{a}{b}\right)}\right]$$

$$= \sqrt{\left(\frac{a+h}{2g}\right)} \left[\frac{a+h}{a}\cos^{-1} \sqrt{\left(\frac{a}{a+h}\right)} + \sqrt{\left(\frac{a+h}{a}\right)} \sqrt{\left(1-\frac{a}{a+h}\right)}\right]$$

[replacing b by $a+h$]

$$= \sqrt{\left(\frac{a+h}{2g}\right)} \left[\frac{a+h}{a}\sin^{-1} \sqrt{\left(1-\frac{a}{a+h}\right)} + \sqrt{\left(\frac{a+h}{a}\right)} \sqrt{\left(\frac{h}{a+h}\right)}\right]$$

$$= \sqrt{\left(\frac{a+h}{2g}\right)} \left[\frac{a+h}{a}\sin^{-1} \sqrt{\left(\frac{h}{a+h}\right)} + \sqrt{\left(\frac{h}{a}\right)}\right].$$

Ex. 70. *A particle falls towards the earth form infinity; show that its velocity on reaching the surface of the earth is the same as that which it would have acquired in falling with constant acceleration g through a distance equal to the earth's radius.*

[Kanpur 1975; Agra 87]

Sol. Let a be the radius of the earth and O be the centre of the earth taken as origin. Let the vertical line through O meet the earth's surface at A. [Draw figure as in Ex. 69].

A particle falls from rest from infinity towards the earth. Let P be the position of the particle at any time t, where $OP=x$. [Note that O is the origin and OP is the direction of x increasing.] According to Newton's law of gravitation the acceleration of the particle at P is μ/x^2 towards O i.e., in the direction of x decreasing. Hence the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \qquad \ldots(1)$$

The equation (1) holds good for the motion of the particle upto A. At A (i.e., on the surface of the earth),

$$x = a \text{ and } \frac{d^2x}{dt^2} = -g$$

$\therefore \quad -g = -\mu/a^2$ or $\mu = a^2 g$. Thus the equation (1) becomes

$$\frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2}.$$

Multiplying both sides by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + C.$$

But initially when $x=\infty$, the velocity $dx/dt=0$. Therefore $C=0$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x}. \qquad \ldots(2)$$

Putting $x=a$ in (2), the velocity V at the earth's surface is given by

$$V^2 = 2a^2 g/a = 2ag \quad \text{or} \quad V = \sqrt{(2ag)}. \qquad \ldots(3)$$

If v_1 is the velocity acquired by the particle in falling a distance equal to the earth's radius a with constant acceleration g, then

$$v_1^2 = 0 + 2ag \quad \text{or} \quad v_1 = \sqrt{(2ag)}. \qquad \ldots(4)$$

From (3) and (4), we have $V = v_1$, which proves the required result.

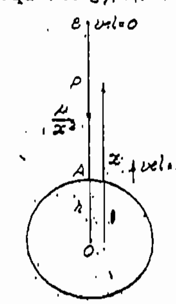
Ex. 71. *If h be the height due to the velocity v at the earth's surface supposing its attraction constant and H the corresponding height when the variation of gravity is taken into account, prove that* $\frac{1}{h} - \frac{1}{H} = \frac{1}{r}$, *where r is the radius of the earth.*

[Kanpur 1978; Meerut 82, 85P; Rohilkhand 85]

Sol. If h is the height of the particle due to the velocity v at the earth's surface, supposing its attraction constant (i.e., taking the acceleration due to gravity as constant and equal to g), then from the formula $v^2 = u^2 + 2fs$, we have

$$0^2 = v^2 - 2gh.$$

$$\therefore \quad v^2 = 2gh. \qquad \ldots(1)$$



When the variation of gravity is taken into account, let P be the position of the particle at any time t measured from the instant the particle is projected vertically upwards from the earth's surface with velocity v, and let $OP=x$.

The acceleration of the particle at P is μ/x^2 directed towards O.

$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \qquad \ldots(2)$$

[Here the $-$ive sign is taken since the acceleration acts in the direction of x decreasing.]

But at A i.e., on the surface of the earth,

$$x = OA = r \text{ and } \frac{d^2x}{dt^2} = -g.$$

$\therefore$ from (2), we have $-g=-\mu/r^2$ or $\mu=gr^2$.

Substituting in (2), we have

$$\frac{d^2x}{dt^2}=-\frac{gr^2}{x^2}. \quad \ldots(3)$$

Multiplying both sides of (3) by $2(dx/dt)$ and then integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+A$, where A is a constant of integration.

But at the point A, $x=OA=r$ and $dx/dt=v$, which is the velocity of projection at A.

$$\therefore \quad v^2=\frac{2gr^2}{r}+A \text{ or } A=v^2-2gr.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+v^2-2gr. \quad \ldots(4)$$

Suppose the particle in this case rises upto the point B, where $AB=H$. Then at the point B, $x=OB=OA+AB=r+H$ and $dx/dt=0$.

$\therefore$ from (4), we have $0=\frac{2gr^2}{r+H}+v^2-2gr$

or $$v^2=-\frac{2gr^2}{r+H}+2gr=\frac{2grH}{r+H}. \quad \ldots(5)$$

Equating the values of v^2 from (1) and (5), we have

$$2gh=\frac{2grH}{r+H} \text{ or } \frac{1}{h}=\frac{r+H}{rH}$$

or $$\frac{1}{h}=\frac{1}{H}+\frac{1}{r} \text{ or } \frac{1}{h}-\frac{1}{H}=\frac{1}{r}.$$

Ex. 72. *A particle is shot upwards from the earth's surface with a velocity of one mile per second. Considering variations in gravity, find roughly in miles the greatest height attained.*

Sol. [Refer fig. of Ex. 71].

Let r be the radius of the earth. Suppose the particle is projected vertically upwards from the surface of the earth with velocity u and it rises to a height H above the surface of the earth. Let P be the position of the particle at any time t and x the distance of P from the centre of the earth. Since P is outside the surface of the earth, therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}.$$

But on the surface of the earth, $x=r$ and $d^2x/dt^2=-g$. Therefore $-g=-(\mu/r^2)$ or $\mu=gr^2$.

$\therefore$ the equation of motion of P becomes

$$\frac{d^2x}{dt^2}=-\frac{gr^2}{x^2}. \quad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. 't', we get $\left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+C$, where C is constant of integration.

When $x=r$, $dx/dt=u$. Therefore $u^2=2gr+C$ or $C=u^2-2gr$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+u^2-2gr. \quad \ldots(2)$$

Since the particle rises to a height H above the surface of the earth, therefore $dx/dt=0$ when $x=r+H$.

Putting these values in (2), we get

$$0=\frac{2gr^2}{r+H}+u^2-2gr$$

or $$0=2gr^2+u^2(r+H)-2gr(r+H)$$

or $$u^2r+u^2H-2grH=0$$

or $$H(2gr-u^2)=u^2r.$$

$$H=\frac{u^2r}{(2gr-u^2)}.$$

But according to the question, $u=1$ mile/second. Also $r=$ the radius of the earth $=4000$ miles, and

$$g=32 \text{ ft./second}^2=\frac{32}{3\times 1760} \text{ miles/sec}^2.$$

Hence, $$H=\frac{4000}{\frac{2\times 32\times 400}{3\times 1760}-1} \text{ miles}=\frac{1}{\frac{2}{165}-\frac{1}{4000}} \text{ miles}$$

$$=\frac{165}{2}\left[1-\frac{165}{8000}\right]^{-1} \text{ miles}=\frac{165}{2}\left[1+\frac{165}{8000}\right] \text{ miles approximately,}$$

[expanding by binomial theorem and neglecting higher powers]

$$=\left[\frac{165}{2}+\frac{(165)^2}{16000}\right] \text{ miles}=82\cdot 5 \text{ miles}+1\cdot 5 \text{ miles nearly}$$

$=84$ miles approximately.

Remark. If the particle is projected from the surface of the earth with a velocity 1 kilometre per second, then for the calculation work we shall take $r=6380$ km. and $g=9\cdot 8$ metre/sec^2 $=10^{-3}\times 9\cdot 8$ km./sec^2. The answer in this case is $51\cdot 43$ km approximately.

Ex. 73. *A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is*

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right],$$

where a is the radius of the earth.

[Meerut 1979, 86S, 88P; Kanpur 76, 87; Agra 84, 85, 88; Rohilkhand 88]

Sol. [Refer fig. of Ex. 71]

Let O be the centre of the earth and A the point of projection on the earth's surface.

If P is the position of the particle at any time t, such that $OP=x$, then the acceleration at $P=\mu/x^2$ directed towards O.

$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}. \quad ...(1)$$

But at the point A, on the surface of the earth, $x=a$ and $d^2x/dt^2=-g$.

$$\therefore \quad -g=-\mu/a^2 \quad \text{or} \quad \mu=a^2g.$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}.$$

Multiplying by $2\,(dx/dt)$ and intergating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+C, \text{ where } C \text{ is a constant.}$$

But when $x\to\infty$, $dx/dt\to 0$. $\therefore$ $C=0$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x} \quad \text{or} \quad \frac{dx}{dt}=\frac{a\sqrt{(2g)}}{\sqrt{x}} \quad ...(2)$$

[Here +ive sign is taken because the particle is moving in the direction of x increasing.]

Separating the variables, we have

$$dt=\frac{1}{a\sqrt{(2g)}}\sqrt{(x)}\,dx.$$

Integrating between the limits $x=a$ to $x=a+h$, the required time t to reach a height h is given by

$$t=\frac{1}{a\sqrt{(2g)}}\int_a^{a+h}\sqrt{(x)}\,dx=\frac{1}{a\sqrt{(2g)}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h}$$

$$=\frac{1}{3a}\sqrt{\left(\frac{2}{g}\right)}\left[(a+h)^{3/2}-a^{3/2}\right]=\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$

Ex. 74. *Calculate in miles per second the least velocity which will carry the particle from earth's surface to infinity.* [Agra 1977]

Sol. The least velocity of projection from the earth's surface to carry the particle to infinity is that for which the velocity of the particle tends to zero as the distance of the particle from the earth's surface tends to infinity. Now proceed as in Ex. 73.

The velocity at a distance x from the centre of the earth is given by $\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}$.

$\therefore$ putting $x=a$, the least velocity V at the earth's surface which will carry the particle to infinity is given by $V=\sqrt{(2ag)}$.

But $a=4000$ miles $=4000\times 3\times 1760$ ft. and $g=32$ ft/sec².

$$\therefore \quad V=\sqrt{[2\times 4000\times 3\times 1760\times 32]} \text{ ft./sec.}$$

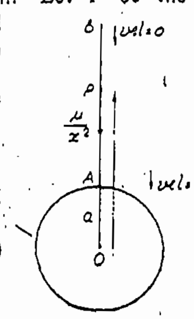
$$=8\times 200\times 4\sqrt{(33)} \text{ ft./sec.}$$

$$=\frac{8\times 200\times 4\times\sqrt{33}}{3\times 1760} \text{ miles/sec.}$$

$=7$ miles/sec. approximately.

Ex. 75. *Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance, show that a particle falling from rest at a distance b (b>a) from the centre of the earth would on reaching the centre acquire a velocity* $\sqrt{[ga\,(3b-2a)/b]}$ *and the time to travel from the surface to the centre of the earth is* $\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left[\frac{b}{(3b-2a)}\right]}$, *where a is the radius of the earth and g is the acceleration due to gravity on the earth's surface.* [I.F.S. 1976; Meerut 81S; 83S; Agra 84, 86]

Sol. Let the particle fall from rest from the point B such that $OB=b$, where O is the centre of the earth. Let P be the position of the particle at any time t measured from the instant it starts falling from B and let $OP=x$.



Acceleration at $P=\mu/x^2$ towards O. The equation of motion of P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2},$$

which holds good for the motion from B to A i.e., outside the surface of the earth.

But at the point A (on the earth's surface) $x=a$ and $d^2x/dt^2=-g$.

$$\therefore \quad -g=-\mu/a^2 \quad \text{or} \quad \mu=a^2g.$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}. \quad ...(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+A$, where A is a constant.

But at B, $x=OB=b$ and $dx/dt=0$.

$$\therefore \quad 0=\frac{2a^2g}{b}+A \text{ or } A=-\frac{2a^2g}{b}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=2a^2g\left(\frac{1}{x}-\frac{1}{b}\right). \qquad \ldots(2)$$

If V is the velocity of the particle at the point A, then at A, $x=OA=a$ and $(dx/dt)^2=V^2$.

$$\therefore \quad V^2=2a^2g\left(\frac{1}{a}-\frac{1}{b}\right). \qquad \ldots(3)$$

Now the particle starts moving through a hole from A to O with velocity V at A.

Let x, $(x<a)$, be the distance of the particle from the centre of the earth at any time t measured from the instant the particle starts penetrating the earth at A. The acceleration at this point will be λx towards O, where λ is a constant.

The equation of motion (inside the earth) is $\dfrac{d^2x}{dt^2}=-\lambda x$, which holds good for the motion from A to O.

At A, $x=a$ and $d^2x/dt^2=-g$. $\therefore$ $\lambda=g/a$.

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{g}{a}x.$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{a}x^2+B, \text{ where } B \text{ is a constant.} \qquad \ldots(4)$$

But at A, $x=OA=a$ and $\left(\dfrac{dx}{dt}\right)^2=V^2=2a^2g\left(\dfrac{1}{a}-\dfrac{1}{b}\right)$, from (3).

$$\therefore \quad 2a^2g\left(\frac{1}{a}-\frac{1}{b}\right)=-\frac{g}{a}a^2+B \quad \text{or} \quad B=ag\left(\frac{3b-2a}{b}\right).$$

Substituting the value of B in (4), we have

$$\left(\frac{dx}{dt}\right)^2=ag\left(\frac{3b-2a}{b}\right)-\frac{g}{a}x^2. \qquad \ldots(5)$$

Putting $x=0$ in (5), we get the velocity on reaching the centre of the earth as $\sqrt{[ga(3b-2a)/b]}$.

Again from (5), we have

$$\left(\frac{dx}{dt}\right)^2=\frac{g}{a}\left[a^2\frac{(3b-2a)}{b}-x^2\right]$$

$$=\frac{g}{a}(c^2-x^2), \text{ where } c^2=\frac{a^2}{b}(3b-2a).$$

$$\therefore \quad \frac{dx}{dt}=-\sqrt{\left(\frac{g}{a}\right)}\cdot\sqrt{(c^2-x^2)},$$ the $-$ive sign being taken because the particle is moving in the direction of x decreasing

or $$dt=-\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{dx}{\sqrt{(c^2-x^2)}}, \text{ separating the variables.}$$

Integrating from A to O, the required time t is given by

$$t=-\sqrt{\left(\frac{a}{g}\right)}\int_{x=a}^{0}\frac{dx}{\sqrt{(c^2-x^2)}}$$

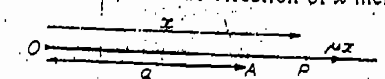
$$=\sqrt{\left(\frac{a}{g}\right)}\int_0^a\frac{dx}{\sqrt{(c^2-x^2)}}=\sqrt{\left(\frac{a}{g}\right)}\left[\sin^{-1}\frac{x}{c}\right]_0^a$$

$$=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left(\frac{a}{c}\right)=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left[\frac{a}{a\sqrt{\{(3b-2a)/b\}}}\right],$$ substituting for c

$$=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left(\frac{b}{3b-2a}\right)}.$$

§ 13. *A particle moves under an acceleration varying as the distance and directed away from a fixed point, to investigate the motion.*

Sol. Let O be the fixed point and x the distance of the particle from O, at any time t. Then the acceleration of the particle at this point is μx in the direction of x increasing.



$\therefore$ the equation of motion of the particle is $\dfrac{d^2x}{dt^2}=\mu x$, $\qquad \ldots(1)$

where the $+$ive sign has been taken since the acceleration acts in the direction of x increasing.

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$(dx/dt)^2=\mu x^2+A, \text{ where } A \text{ is a constant.}$$

Suppose the particle starts from rest at a distance a from O, i.e., $dx/dt=0$ at $x=a$. Then $0=\mu a^2+A$, or $A=-\mu a^2$.

$$\therefore \quad (dx/dt)^2=\mu(x^2-a^2), \qquad \ldots(2)$$

which gives the velocity at any distance x from O.

From (2), on extracting square root, we have

$$dx/dt=\sqrt{\mu}\sqrt{(x^2-a^2)}$$

[$+$ive sign being taken because the particle moves in the direction of x increasing].

or $$dt=\frac{1}{\sqrt{\mu}}\cdot\frac{dx}{\sqrt{(x^2-a^2)}}.$$

Integrating, $t=\frac{1}{\sqrt{\mu}}\cosh^{-1}\frac{x}{a}+B.$

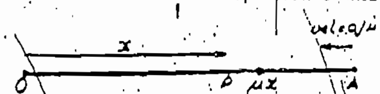
But when $t=0$; $x=a$. $\therefore$ $B=0$.

$\therefore$ $t=\frac{1}{\sqrt{\mu}}\cosh^{-1}\frac{x}{a}$ or $x=a\cosh(\sqrt{\mu}t)$, ...(3)

which gives the position of the particle at time t.

Ex. 76. *If a particle is projected towards the centre of repulsion, varying as the distance from the centre, from a distance a from it with a velocity $a\sqrt{\mu}$; prove that the particle will approach the centre but will never reach it.* [Lucknow 1978; Alld. 80; Agra 84]

Sol. Let the particle be projected from the point A with velocity $a\sqrt{\mu}$ towards the centre of repulsion O and let $OA=a$.



If P is the position of the particle at time t such that $OP=x$, then at P, the acceleration on the particle is μx in the direction PA.

$\therefore$ the equation of motion of the particle is

$$\frac{d^2x}{dt^2}=\mu x.$$ [+ive sign is taken because the acceleration is in the direction of x increasing]

Multiplying by $2(dx/dt)$ and integrating w.r.t. 't', we have
$(dx/dt)^2=\mu x^2+C$, where C is a constant.

But at A, $x=a$ and $(dx/dt)^2=a^2\mu$. $\therefore$ $C=0$.

$\therefore$ $(dx/dt)^2=\mu x^2$ or $dx/dt=-\sqrt{\mu}x$. ...(1)

[−ive sign is taken because the particle is moving in the direction of x decreasing].

The equation (1) shows that the velocity of the particle will be zero when $x=0$ and not before it and so the particle will approach the centre O.

From (1), we have $dt=-\frac{1}{\sqrt{\mu}}\frac{dx}{x}$.

Integrating between the limits $x=a$ to $x=0$, the time t_1 from A to O is given by

$$t_1=-\frac{1}{\sqrt{\mu}}\int_a^0\frac{dx}{x}=\frac{1}{\sqrt{\mu}}\left[\log x\right]_0^a=\frac{1}{\sqrt{\mu}}(\log a-\log 0)$$
$=\infty$ [$\because$ $\log 0=-\infty$]

Hence the particle will take an infinite time to reach the centre O or in other words it will never reach the centre O.

§ 14. *A particle moves in such a way that its acceleration varies inversely as the cube of the distance from a fixed point and is directed towards the fixed point; discuss the motion.*

[Lucknow 1976; Agra 79]

Let O be the fixed point and x the distance of the particle from O, at any time t. Then the equation of motion of the particle is $$\frac{d^2x}{dt^2}=-\frac{\mu}{x^3}.$$

[The −ive sign has been taken because the force is given to be attractive.]

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2=\frac{\mu}{x^2}+A.$$

Suppose the particle starts from rest at a distance a from O, i.e., $dx/dt=0$ at $x=a$.

Then $0=\frac{\mu}{a^2}+A$ or $A=-\frac{\mu}{a^2}$.

$$\therefore\left(\frac{dx}{dt}\right)^2=\mu\left(\frac{1}{x^2}-\frac{1}{a^2}\right), \quad ...(2)$$

which gives the velocity at any distance x from the centre of force O.

From (2), we have $\frac{dx}{dt}=-\frac{\sqrt{\mu}}{a}\frac{\sqrt{(a^2-x^2)}}{x}$

[the −ive sign has been taken since the particle is moving in the direction of x decreasing.]

or $dt=-\frac{a}{\sqrt{\mu}}\cdot\frac{x\,dx}{\sqrt{(a^2-x^2)}}$, separating the variables

$=\frac{a}{2\sqrt{\mu}}(a^2-x^2)^{-1/2}(-2x)\,dx.$

Integrating, $t=\frac{a}{\sqrt{\mu}}\cdot\sqrt{(a^2-x^2)}+B.$

But initially when $t=0$, $x=a$. $\therefore$ $B=0$.

$\therefore$ $t=\frac{a}{\sqrt{\mu}}\sqrt{(a^2-x^2)}$. ...(3)

which gives the position of the particle at any time t.

Ex. 77. *A particle moves in a straight line towards a centre of force μ/(distance)^3 starting from rest at a distance a from the*

centre of force; show that the time of reaching a point distant b from the centre of force is $a\sqrt{\left(\frac{a^2-b^2}{\mu}\right)}$, and that its velocity then is $\sqrt{[\mu(a^2-b^2)]}/ab$. Also show that the time to reach the centre is $a^2/\sqrt{\mu}$.
(Kanpur 1987)

Sol. Let the particle start at rest from A and at time t let it be at P, where $OP=x$; O being the centre of force.

Given that the acceleration at P is μ/x^3 towards O, we have

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^3}. \quad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. 't', we have: $\left(\frac{dx}{dt}\right)^2=\frac{\mu}{x^2}+C.$

When $x=a$, $dx/dt=0$, so that $C=-\mu/a^2$.

Hence $\left(\frac{dx}{dt}\right)^2=\mu\left(\frac{1}{x^2}-\frac{1}{a^2}\right)=\mu\left(\frac{a^2-x^2}{a^2x^2}\right)$.

$$\therefore \frac{dx}{dt}=-\frac{\sqrt{[\mu(a^2-x^2)]}}{ax}, \quad \ldots(2)$$

the negative sign being taken because the particle is moving towards x decreasing.

Putting $x=b$ in (2), the velocity at $x=b$ is $\sqrt{[\mu(a^2-b^2)]}/ab$, in magnitude. This proves the second result.

If t_1 is the time from $x=a$ to $x=b$, then integrating (2) after seperating the variables, we get

$$t_1=-\frac{a}{\sqrt{\mu}}\int_a^b \frac{x}{\sqrt{(a^2-x^2)}}\,dx=\frac{a}{2\sqrt{\mu}}\int_a^b \frac{-2x}{\sqrt{(a^2-x^2)}}\,dx$$

$$=\frac{a}{2\sqrt{\mu}}\left[2\sqrt{(a^2-x^2)}\right]_a^b=\frac{a\sqrt{(a^2-b^2)}}{\sqrt{\mu}}.$$

This proves the first result.

And if T be the time to reach the centre O, where $x=0$, then

$$T=\frac{a}{2\sqrt{\mu}}\int_a^0 \frac{-2x}{\sqrt{(a^2-x^2)}}\,dx=\frac{a}{2\sqrt{\mu}}\left[2\sqrt{(a^2-x^2)}\right]_a^0=\frac{a^2}{\sqrt{\mu}}.$$

§ 15. Motion under miscellaneous laws of forces.

Now we shall give a few examples in which the particle moves under different laws of acceleration.

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Ex. 78. *A particle whose mass is m is acted upon by a force $m\mu\left[x+\frac{a^4}{x^3}\right]$ towards origin; if it starts from rest at a distance a, show that it will arrive at origin in time $\pi/(4\sqrt{\mu})$.*

[Lucknow 1981; Meerut 87; Kanpur 84; Agra 77, 85]

Sol. Given $\frac{d^2x}{dt^2}=-\mu\left[x+\frac{a^4}{x^3}\right]$, $\quad\ldots(1)$

the $-$ive sign being taken because the force is attractive.

Integrating it after multiplying throughout by $2(dx/dt)$, we get

$$\left(\frac{dx}{dt}\right)^2=\mu\left[-x^2+\frac{a^4}{x^2}\right]+C.$$

When $x=a$, $dx/dt=0$, so that $C=0$.

$$\therefore \left(\frac{dx}{dt}\right)^2=\mu\left[\frac{a^4-x^4}{x^2}\right]$$

or
$$\frac{dx}{dt}=-\frac{\sqrt{\mu}\sqrt{(a^4-x^4)}}{x}, \quad \ldots(2)$$

the $-$ive sign is taken because the particle is moving in the direction of x decreasing.

If t_1 be the time taken to reach the origin, then integrating (2), we get

$$t_1=-\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{(a^4-x^4)}}\,dx=\frac{1}{\sqrt{\mu}}\int_0^a \frac{x\,dx}{\sqrt{(a^4-x^4)}}$$

Put $x^2=a^2\sin\theta$ so that $2x\,dx=a^2\cos\theta\,d\theta$. When $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore t_1=\frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta}=\frac{1}{2\sqrt{\mu}}\int_0^{\pi/2}d\theta=\frac{1}{2\sqrt{\mu}}\left[\theta\right]_0^{\pi/2}$$

$$=\frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2}=\frac{\pi}{4\sqrt{\mu}}.$$

Ex. 79. *A particle moves in a straight line with an acceleration towards a fixed point in the straight line, which is equal to $\mu/x^2-\lambda/x^3$ at a distance x from the given point; the particle starts from rest at a distance a; show that it oscillates between this distance and the distance $\frac{\lambda a}{(2\mu a-\lambda)}$ and the periodic time is $\frac{2\pi\mu a^3}{(2a\mu-\lambda)^{3/2}}$.*

Sol. Let O be the fixed point taken as origin and A the starting point such that $OA=a$. At any time t let P be the position of the particle, where $OP=x$. Equation of motion of the particle is

$$\frac{d^2x}{dt^2}=-\left(\frac{\mu}{x^2}-\frac{\lambda}{x^3}\right). \text{ [given]} \quad \ldots(1)$$

Integrating, we get $\left(\frac{dx}{dt}\right)^2=\frac{2\mu}{x}-\frac{\lambda}{x^2}+C.$

When $x=a$, $\frac{dx}{dt}=0$, so that $C=-\frac{2\mu}{a}+\frac{\lambda}{a^2}$.

$$\therefore \left(\frac{dx}{dt}\right)^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right)-\lambda\left(\frac{1}{x^2}-\frac{1}{a^2}\right)$$

$$=\left(\frac{1}{x}-\frac{1}{a}\right)\left(2\mu-\frac{\lambda}{x}-\frac{\lambda}{a}\right)$$

$$=\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{2a\mu-\lambda}{a}-\frac{\lambda}{x}\right)$$

$$=\lambda\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{2a\mu-\lambda}{\lambda a}-\frac{1}{x}\right). \quad \ldots(2)$$

The particle comes to rest where $dx/dt=0$, *i.e.*, where

$$\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{2a\mu-\lambda}{\mu a}-\frac{1}{x}\right)=0.$$

One solution of this equation is $\dfrac{1}{x}-\dfrac{1}{a}=0$ *i.e.*, $x=a$, which gives the initial position. Another solution is $\dfrac{2a\mu-\lambda}{\lambda a}-\dfrac{1}{x}=0$ *i.e.*, $x=\dfrac{\lambda a}{(2a\mu-\lambda)}$ which gives the other position of instantaneous rest.

Hence the particle oscillates between $x=a$ and $x=\dfrac{\lambda a}{(2a\mu-\lambda)}$.

This proves one result. To prove the other result, put $\dfrac{\lambda a}{2a\mu-\lambda}=b$, so that the equation (2) becomes

$$\left(\frac{dx}{dt}\right)^2=\lambda\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{x}\right)=\frac{\lambda}{ab}\frac{(a-x)(x-b)}{x^2}$$

or $$\frac{dx}{dt}=-\sqrt{\left(\frac{\lambda}{ab}\right)}\cdot\frac{\sqrt{\{(a-x)(x-b)\}}}{x}$$

[the $-$ive sign is taken because the particle is moving in the direction of x decreasing.]

or $$dt=-\sqrt{\left(\frac{ab}{\lambda}\right)}\cdot\frac{x\,dx}{\sqrt{\{(a-x)(x-b)\}}}.$$

Integrating between the limits $x=a$ to $x=b$, the time t_1 from one position of rest to the other position of rest is given by

$$t_1=-\sqrt{\left(\frac{ab}{\lambda}\right)}\int_a^b\frac{x\,dx}{\sqrt{\{(a-x)(x-b)\}}}$$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_b^a\frac{x\,dx}{\sqrt{[-ab-\{x^2-(a+b)x\}]}}$$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_b^a\frac{x\,dx}{\sqrt{[\frac{1}{4}(a-b)^2-\{x-\frac{1}{2}(a+b)\}^2]}}$$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_{-(a-b)/2}^{(a-b)/2}\frac{\{\frac{1}{2}(a+b)+y\}\,dy}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}},$$

putting $x-\frac{1}{2}(a+b)=y$ so that $dx=dy$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_{-(a-b)/2}^{(a-b)/2}\frac{\frac{1}{2}(a+b)}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}}\,dy$$

$$+\sqrt{\left(\frac{ab}{\lambda}\right)}\int_{-(a-b)/2}^{(a-b)/2}\frac{y}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}}\,dy$$

$$=2\sqrt{\left(\frac{ab}{\lambda}\right)}\cdot\frac{1}{2}(a+b)\int_0^{(a-b)/2}\frac{1}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}}\,dy,$$

the second integral vanishes because the integrand is an odd function of y.

$$=(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}\left[\sin^{-1}\left\{\frac{y}{\frac{1}{2}(a-b)}\right\}\right]_0^{(a-b)/2}$$

$$=(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}\left[\sin^{-1}1-\sin^{-1}0\right]=\frac{\pi}{2}(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}.$$

$\therefore$ the periodic time of one complete oscillation

$$=2t_1=2\cdot\frac{\pi}{2}(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}$$

$$=\pi\left(a+\frac{\lambda a}{2a\mu-\lambda}\right)\sqrt{\left\{\frac{a}{\lambda}\cdot\frac{\lambda a}{2a\mu-\lambda}\right\}}$$

$$=\pi\frac{2a^2\mu}{(2a\mu-\lambda)}\cdot\frac{a}{\sqrt{(2a\mu-\lambda)}}=\frac{2\pi\mu a^3}{(2a\mu-\lambda)^{3/2}}.$$

Remark. To evaluate the integral giving the time t_1, we can also make the substitution $x=a\cos^2\theta+b\sin^2\theta$, so that $dx=-2(a-b)\sin\theta\cos\theta\,d\theta$. Also $\theta=0$ when $x=a$ and $\theta=\pi/2$ when $x=b$.

Ex. 80. *A particle moves in a straight line under a force to a point in it, varying as* $(distance)^{-4/3}$. *Show that the velocity in falling from rest at infinity to a distance a is equal to that acquired in falling from rest at a distance a to a distance a/8.* [Kanpur 1977]

Sol. If x is the distance of the particle from the fixed point at time t, then the equation of motion of the particle is

$$\frac{d^2x}{dt^2}=-\mu x^{-4/3}. \quad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=\frac{6\mu}{x^{1/3}}+A. \quad \ldots(2)$$

If the particle falls from rest at infinity, *i.e.*, $dx/dt=0$ when $x=\infty$, we have from (2), $A=0$.

$$\therefore (dx/dt)^2=6\mu/x^{1/3}.$$

If v_1 is the velocity of the particle at $x=a$, then

$$v_1^2=6\mu/a^{1/3}. \quad \ldots(3)$$

Again if the particle falls from rest at a distance a, *i.e.*, if $dx/dt=0$ when $x=a$, we have, from (2)

$$0=\frac{6\mu}{a^{1/3}}+A \quad \text{or} \quad A=-\frac{6\mu}{a^{1/3}}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=6\mu\left(\frac{1}{x^{1/3}}-\frac{1}{a^{1/3}}\right).$$

If in this case v_2 is the velocity of the particle at $x=a/8$, then

$$v_2^2=6\mu\left[\left(\frac{8}{a}\right)^{1/3}-\frac{1}{a^{1/3}}\right]=6\mu\left(\frac{2}{a^{1/3}}-\frac{1}{a^{1/3}}\right)=\frac{6\mu}{a^{1/3}}. \quad \text{...(4)}$$

From (3) and (4), we observe that $v_1=v_2$, which proves the required result.

Ex. 81. *Find the time of descent to the centre of force, when the force varies as (distance)$^{-5/3}$, and show that the velocity at the centre is infinite.*

Sol. Let O be the centre of force taken as the origin. Suppose a particle starts at rest from A, where $OA=a$. The particle moves towards O on account of a centre of attraction at O. Let P be the position of the particle at any time t, where $OP=x$. The acceleration of the particle at P is $\mu x^{-5/3}$ directed towards O. Therefore the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\mu x^{-5/3}. \quad \text{...(1)}$$

Multiplying both sides of (1) by $2\,(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{2\mu x^{-2/3}}{-2/3}+k=\frac{3\mu}{x^{2/3}}+k, \text{ where } k \text{ is a constant.}$$

At A, $x=a$ and $dx/dt=0$, so that $(3\mu/a^{2/3})+k=0$

or $\qquad k=-3\mu/a^{2/3}.$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{3\mu}{x^{2/3}}-\frac{3\mu}{a^{2/3}}=\frac{3\mu\,(a^{2/3}-x^{2/3})}{a^{2/3}\,x^{2/3}}, \quad \text{...(2)}$$

which gives the velocity of the particle at any distance x from the centre of force O. Putting $x=0$ in (2), we see that at O, $(dx/dt)^2=\infty$. Therefore the velocity of the particle at the centre is infinite.

Taking square root of (2), we get

$$\frac{dx}{dt}=-\sqrt{(3\mu)}\sqrt{\left(\frac{a^{2/3}-x^{2/3}}{a^{2/3}\,x^{2/3}}\right)},$$ where the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we get

$$dt=-\frac{a^{1/3}}{\sqrt{(3\mu)}}\cdot\frac{x^{1/3}}{\sqrt{(a^{2/3}-x^{2/3})}}\,dx. \quad \text{...(3)}$$

Let t_1 be the time from A to O. Then at A, $t=0$ and $x=a$ while at O, $t=t_1$ and $x=0$. So integrating (3) from A to O, we have

$$\int_0^{t_1} dt=-\frac{a^{1/3}}{\sqrt{(3\mu)}}\int_a^0 \frac{x^{1/3}}{\sqrt{(a^{2/3}-x^{2/3})}}\,dx$$

$$=\frac{a^{1/3}}{\sqrt{(3\mu)}}\int_0^a \frac{x^{1/3}\,dx}{\sqrt{(a^{2/3}-x^{2/3})}}.$$

Putting $x=a\sin^3\theta$, so that $dx=3a\sin^2\theta\cos\theta\,d\theta$. When $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore \quad t_1=\frac{a^{1/3}}{\sqrt{(3\mu)}}\int_0^{\pi/2}\frac{a^{1/3}\sin\theta}{a^{1/3}\cos\theta}\,3a\sin^2\theta\cos\theta\,d\theta$$

$$=\frac{3a^{4/3}}{\sqrt{(3\mu)}}\int_0^{\pi/2}\sin^3\theta\,d\theta=\frac{3a^{4/3}}{\sqrt{(3\mu)}}\cdot\frac{2}{3.1}=\frac{2a^{4/3}}{\sqrt{(3\mu)}}.$$

Hence the time of descent to the centre of force is $2a^{4/3}/\sqrt{(3\mu)}$.

Ex. 82. *A particle starts from rest at a distance a from the centre of force which attracts inversely as the distance. Prove that the time of arriving at the centre is $a\sqrt{(\pi/2\mu)}$.*

[Meerut 1980, 84, 85, 88P]

Sol. If x is the distance of the particle from the centre of force at time t, then the equation of motion is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x}.$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have $(dx/dt)^2=-2\mu\log x+A$, where A is a constant.

But initially at $x=a$, $dx/dt=0$;

$\therefore \quad 0=-2\mu\log a+A \quad$ or $\quad A=2\mu\log a.$

$\therefore \quad (dx/dt)^2=2\mu\,(\log a-\log x)=2\mu\log(a/x)$

or $\qquad dx/dt=-\sqrt{(2\mu)}\sqrt{\{\log(a/x)\}},$

where the $-$ive sign has been taken since the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt=-\frac{1}{\sqrt{(2\mu)}}\frac{dx}{\sqrt{\{\log(a/x)\}}}.$$

Integrating from $x=a$ to $x=0$, the required time t_1 to reach the centre is given by

$$t_1=-\frac{1}{\sqrt{(2\mu)}}\int_{x=a}^0\frac{dx}{\sqrt{\{\log(a/x)\}}}.$$

Put $\log\left(\dfrac{a}{x}\right)=u^2$ *i.e.*, $x=ae^{-u^2}$, so that $dx=-2ae^{-u^2}u\,du$.

When $x=a$, $u=0$ and when $x\to0$, $u\to\infty$.

$\therefore \quad t_1=\dfrac{2}{\sqrt{(2\mu)}}\displaystyle\int_0^\infty e^{-u^2}\,du.$ But $\displaystyle\int_0^\infty e^{-u^2}\,du=\frac{\sqrt{\pi}}{2}$ (Remember)

$$\therefore \quad t_1 = \frac{2a}{\sqrt{(2\mu)}} \cdot \frac{\sqrt{\pi}}{2} = a\sqrt{\left(\frac{\pi}{2\mu}\right)}.$$

Ex. 83. *A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu\,(a^5/x^2)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance a from O, show that it will arrive at O with a velocity $a\sqrt{(6\mu)}$ after time* $\dfrac{8}{15}\sqrt{\left(\dfrac{6}{\mu}\right)}$.

[Agra 1980, 84; Meerut 86, 87P, 90S]

Sol. Take the centre of force O as origin. Suppose a particle starts from rest at A, where $OA = a$. It moves towards O because of a centre of attraction at O. Let P be the position of the particle after any time t, where $OP = x$. The acceleration of the particle at P is $\mu a^{5/3}\, x^{-2/3}$ directed towards O. Therefore the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu a^{5/3}\, x^{-2/3}. \qquad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu a^{5/3}\, x^{1/3}}{1/3} + k = -6\mu a^{5/3}\, x^{1/3} + k,$$

where k is a constant.

At A, $x = a$ and $dx/dt = 0$, so that

$$-6\mu a^{5/3}\, a^{1/3} + k = 0 \quad \text{or} \quad k = 6\mu a^2.$$

$$\therefore \quad (dx/dt)^2 = -6\mu a^{5/3}\, x^{1/3} + 6\mu a^2 = 6\mu a^{5/3}\,(a^{1/3} - x^{1/3}), \qquad \ldots(2)$$

which gives the velocity of the particle at any distance x from the centre of force. Suppose the particle arrives at O with the velocity v_1. Then at O, $x = 0$ and $(dx/dt)^2 = v_1^2$. So from (2), we have

$$v_1^2 = 6\mu a^{5/3}\,(a^{1/3} - 0) = 6\mu a^2 \quad \text{or} \quad v_1 = a\sqrt{(6\mu)}.$$

Now taking square root of (2), we get

$$dx/dt = -\sqrt{(6\mu a^{5/3})}\sqrt{(a^{1/3} - x^{1/3})},$$

where the $-$ive sign has been taken because the particle moves in the direction of x decreasing.

Separating the variables, we get

$$dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}} \cdot \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}. \qquad \ldots(3)$$

Let t_1 be the time from A to O. Then integrating (3) from A to O, we have

$$\int_0^{t_1} dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}} \int_a^0 \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}$$

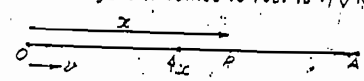
$$= \frac{1}{\sqrt{(6\mu a^{5/3})}} \int_0^a \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}.$$

Put $x = a \sin^6 \theta$, so that $dx = 6a \sin^5 \theta \cos \theta \, d\theta$. When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \pi/2$.

$$\therefore \quad t_1 = \frac{1}{\sqrt{(6\mu a^{5/3})}} \int_0^{\pi/2} \frac{6a \sin^5 \theta \cos \theta \, d\theta}{a^{1/6} \cos \theta}$$

$$= \sqrt{\left(\frac{6}{\mu}\right)} \int_0^{\pi/2} \sin^5 \theta \, d\theta = \sqrt{\left(\frac{6}{\mu}\right)} \cdot \frac{4.2}{5.3.1} = \frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}.$$

Ex. 84. *A particle starts with a given velocity v and moves under a retardation equal to k times the space described. Show that the distance traversed before it comes to rest is $v/\sqrt{k}$.*



Sol. Suppose the particle starts from O with velocity v and moves in the straight line OA. Let P be the position of the particle after any time t, where $OP = x$. Then the retardation of the particle at P is kx i.e., the acceleration of the particle at P is kx and is directed towards O i.e., in the direction of x decreasing. Therefore the equation of motion of the particle at P is

$$d^2x/dt^2 = -kx. \qquad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. t, we have $(dx/dt)^2 = -kx^2 + C$, where C is a constant.

At O, $x = 0$ and $dx/dt = v$, so that $v^2 = C$.

$$\therefore \quad (dx/dt)^2 = v^2 - kx^2, \qquad \ldots(2)$$

which gives the velocity of the particle at a distance x from O.

From (2), $dx/dt = 0$ when $v^2 - kx^2 = 0$ i.e., when $x = v/\sqrt{k}$.

Hence the distance traversed before the particle comes to rest is $v/\sqrt{k}$.

Ex. 85. *Assuming that at a distance x from a centre of force, the speed v of a particle, moving in a straight line is given by the equation $x = ae^{bv^2}$, where a and b are constants. Find the law and the nature of the force.*

Sol. Given, $x = ae^{bv^2}$. Therefore $e^{bv^2} = x/a$

or $\qquad bv^2 = \log(x/a) = \log x - \log a. \qquad \ldots(1)$

Differentiating both sides of (1) w.r.t. x, we get

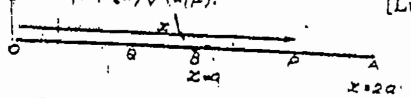
$$2bv \frac{dv}{dx} = \frac{1}{x} \quad \text{or} \quad v\frac{dv}{dx} = \frac{1}{2b} \cdot \frac{1}{x}.$$

$\therefore$ the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = \frac{1}{2b} \cdot \frac{1}{x}. \qquad \left[\text{Note that } v\frac{dv}{dx} = \frac{d^2x}{dt^2}\right]$$

Hence the acceleration varies inversely as the distance of the particle from the centre of force. Also the force is repulsive or attractive according as b is positive or negative.

Ex. 86. *A particle of mass m moving in a straight line is acted upon by an attractive force which is expressed by the formula $m\mu a^2/x^2$ for values of $x \geqslant a$, and by the formula $m\mu x/a$ for $x \leqslant a$, where x is the distance from a fixed origin in the line. If the particle starts at a distance $2a$ from the origin, prove that it will reach the origin with velocity $(2\mu a)^{1/2}$. Prove further that the time taken to reach the origin is $(1+\frac{1}{4}\pi)\sqrt{(a/\mu)}$.* [Lucknow 1981]



Sol. Let O be the origin and A the point from which the particle starts. We have $OA=2a$ and let $OB=a$, so that B is the middle point of OA.

Motion from A to B. The particle starts from rest at A and it moves towards B. Let P be its position at any time t, where $OP=x$. According to the question the acceleration of P is $\mu a^2/x^2$ and is directed towards O i.e. in the direction of x decreasing. Therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=-\frac{\mu a^2}{x^2}. \qquad \text{...(1)}$$

Multiplying (1) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=\frac{2\mu a^2}{x}+C.$$

When $x=2a$, $dx/dt=0$, so that $C=-2\mu a^2/2a$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{2\mu a^2}{x}-\frac{2\mu a^2}{2a}=2a^2\mu\left[\frac{1}{x}-\frac{1}{2a}\right]=a\mu\,\frac{2a-x}{x}, \qquad \text{...(2)}$$

which gives the velocity of the particle at any position between A and B. Suppose the particle reaches B with the velocity v_1. Then at B, $x=a$ and $(dx/dt)^2=v_1^2$. So from (2), we get

$$v_1^2=a\mu\,\frac{2a-a}{a}=a\mu \quad \text{or} \quad v_1=\sqrt{(a\mu)},$$ its direction being towards the origin O.

Now taking square root of (2), we get

$$\frac{dx}{dt}=-\sqrt{(a\mu)}\sqrt{\left(\frac{2a-x}{x}\right)},$$ where the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we get

$$dt=-\frac{1}{\sqrt{(a\mu)}}\sqrt{\left(\frac{x}{2a-x}\right)}\,dx. \qquad \text{...(3)}$$

Let t_1 be the time from A to B. Then at A, $x=2a$ and $t=0$ while at B, $x=a$ and $t=t_1$. So integrating (3) from A to B, we get

$$\int_0^{t_1} dt=-\frac{1}{\sqrt{(a\mu)}}\int_{2a}^{a}\sqrt{\left(\frac{x}{2a-x}\right)}\,dx.$$

Put $x=2a\cos^2\theta$, so that $dx=-4a\cos\theta\sin\theta\,d\theta$. When $x=2a$, $\theta=0$ and when $x=a$, $\theta=\pi/4$.

$$\therefore \quad t_1=-\frac{1}{\sqrt{(a\mu)}}\int_0^{\pi/4}\frac{\cos\theta}{\sin\theta}\cdot(-4a\cos\theta\sin\theta)\,d\theta$$

$$=\sqrt{\left(\frac{a}{\mu}\right)}\int_0^{\pi/4}2\cos^2\theta\,d\theta=2\sqrt{\left(\frac{a}{\mu}\right)}\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta$$

$$=2\sqrt{\left(\frac{a}{\mu}\right)}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}=2\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{4}+\frac{1}{2}\right]=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}+1\right].$$

Motion from B to O. Now the particle starts from B towards O with velocity $\sqrt{(a\mu)}$ gained by it during its motion from A to B. Let Q be its position after time t since it starts from B and let $OQ=x$. Now according to the question the acceleration of Q is $\mu x/a$ directed towards O. Therefore the equation of motion of Q is

$$\frac{d^2x}{dt^2}=-\frac{\mu x}{a}. \qquad \text{...(4)}$$

Multiplying both sides of (4) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{\mu}{a}x^2+D.$$

At B, $x=a$ and $(dx/dt)^2=v_1^2=a\mu$, so that $a\mu=-a\mu+D$ or $D=2a\mu$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=-\frac{\mu}{a}x^2+2a\mu=\frac{\mu}{a}(2a^2-x^2), \qquad \text{...(5)}$$

which gives the velocity of the particle at any position between B and O. Let v_2 be the velocity of the particle at O. Then putting $x=0$ and $(dx/dt)^2=v_2^2$ in (5), we get

$$v_2^2=\frac{\mu}{a}(2a^2-0)=2a\mu \quad \text{or} \quad v_2=\sqrt{(2a\mu)}.$$

Hence the particle reaches the origin with the velocity $\sqrt{(2a\mu)}$.

Now taking square root of (5), we get

$$\frac{dx}{dt}=-\sqrt{\left(\frac{\mu}{a}\right)}\sqrt{(2a^2-x^2)},$$ where the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt=-\sqrt{\left(\frac{a}{\mu}\right)}\frac{dx}{\sqrt{(2a^2-x^2)}}. \quad \ldots(6)$$

Let t_2 be the time from B to O. Then at B, $t=0$ and $x=a$ while at O, $x=0$ and $t=t_2$. So integrating (6) from B to O, we get

$$\int_0^{t_2} dt=-\sqrt{\left(\frac{a}{\mu}\right)}\int_a^0 \frac{dx}{\sqrt{(2a^2-x^2)}}$$

i.e.,
$$t_2=\sqrt{\left(\frac{a}{\mu}\right)}\left[\cos^{-1}\frac{x}{a\sqrt{2}}\right]_a^0$$

$$=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}-\frac{\pi}{4}\right]=\sqrt{\left(\frac{a}{\mu}\right)}\frac{\pi}{4}.$$

Hence the whole time taken to reach the origin $O=t_1+t_2$

$$=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}+1\right]+\sqrt{\left(\frac{a}{\mu}\right)}\frac{\pi}{4}=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{3\pi}{4}+1\right].$$

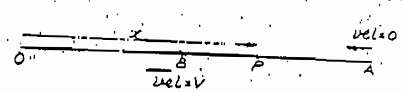
Ex. 87. *A particle moves along the axis of x starting from rest at $x=a$. For an interval t_1 from the beginning of the motion the acceleration is $-\mu x$, for a subsequent time t_2 the acceleration is μx, and at the end of this interval the particle is at the origin; prove that*

$$\tan(\sqrt{\mu}t_1).\ \tanh(\sqrt{\mu}t_2)=1.$$

[I.F.S. 1976; Meerut 82S, 90P]

Sol. Let the particle moving along the axis of x start from rest at A such that $OA=a$.

Let $-\mu x$ be the acceleration for an interval t_1 from A to B and μx that for an interval t_2 from B to O, where $OB=b$.



For motion from A to B, the equation of motion is

$$\frac{d^2x}{dt^2}=-\mu x. \quad \ldots(1)$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$(dx/dt)^2=-\mu x^2+A, \text{ where } A \text{ is a constant.}$$

But at $x=a$, $dx/dt=0$. $\therefore$ $0=-\mu a^2+A$ or $A=\mu a^2$.

$$\therefore \quad (dx/dt)^2=\mu\,(a^2-x^2) \quad \ldots(2)$$

or
$$dx/dt=-\sqrt{\mu}\sqrt{(a^2-x^2)}$$

[the $-$ive sign is taken because the particle is moving in the direction of x decreasing.]

or
$$dt=-\frac{1}{\sqrt{\mu}}\cdot\frac{dx}{\sqrt{(a^2-x^2)}}, \quad \text{[separating the variables].}$$

Integrating between the limits $x=a$ to $x=b$, the time t_1 from A to B is given by

$$t_1=-\frac{1}{\sqrt{\mu}}\int_{x=a}^{b}\frac{dx}{\sqrt{(a^2-x^2)}}=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_{x=a}^{b}=\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}.$$

$\therefore$ $\cos(\sqrt{\mu}t_1)=b/a$ and $\sin(\sqrt{\mu}t_1)=\sqrt{\{1-\cos^2(\sqrt{\mu}t_1)\}}$

$$=\sqrt{\left(1-\frac{b^2}{a^2}\right)}=\frac{\sqrt{(a^2-b^2)}}{a}.$$

Dividing, $\tan(\sqrt{\mu}t_1)=\dfrac{\sqrt{(a^2-b^2)}}{b}$. $\quad \ldots(3)$

If V is the velocity at B where $x=b$, then from (2),

$$V^2=\mu\,(a^2-b^2). \quad \ldots(4)$$

For motion from B to O, the velocity at B is V and the particle moves towards O under the acceleration μx.

$\therefore$ the equation of motion is $\dfrac{d^2x}{dt^2}=\mu x$. $\quad \ldots(5)$

Integrating, $(dx/dt)^2=\mu x^2+B$, where B is a constant.

But at the point B, $x=b$ and $(dx/dt)^2=V^2=\mu\,(a^2-b^2)$.

$\therefore$ $\mu\,(a^2-b^2)=\mu b^2+B$ or $B=\mu\,(a^2-2b^2)$.

$\therefore$ $\left(\dfrac{dx}{dt}\right)^2=\mu\,[x^2+(a^2-2b^2)]$ or $\dfrac{dx}{dt}=-\sqrt{\mu}\sqrt{[x^2+(a^2-2b^2)]}$

or
$$dt=-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{[x^2+(a^2-2b^2)]}}$$

Integrating between the limits $x=b$ to $x=0$, the time t_2 from B to O is given by

$$t_2=-\frac{1}{\sqrt{\mu}}\int_{x=b}^{0}\frac{dx}{\sqrt{[x^2+(a^2-2b^2)]}}$$

$$=-\frac{1}{\sqrt{\mu}}\left[\sinh^{-1}\frac{x}{\sqrt{(a^2-2b^2)}}\right]_b^0=\frac{1}{\sqrt{\mu}}\sinh^{-1}\frac{b}{\sqrt{(a^2-2b^2)}}$$

$\therefore$ $\sinh(\sqrt{\mu}t_2)=\dfrac{b}{\sqrt{(a^2-2b^2)}}$ so that

$$\cosh(\sqrt{\mu}t_2)=\sqrt{\{1+\sinh^2(\sqrt{\mu}t_2)\}}$$

$$=\sqrt{\left(1+\frac{b^2}{a^2-2b^2}\right)}=\sqrt{\left(\frac{a^2-b^2}{a^2-2b^2}\right)}.$$

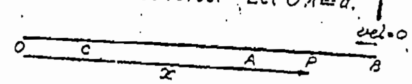
Dividing, $\tanh(\sqrt{\mu}t_2)=\dfrac{b}{\sqrt{(a^2-b^2)}}$. $\quad \ldots(6)$

Multiplying (3) and (6), we have

$$\tan(\sqrt{\mu}t_1) \cdot \tanh(\sqrt{\mu}t_2) = 1.$$

Ex. 88. *A particle starts from rest at a distance b from a fixed point, under the action of a force through the fixed point, the law of which at a distance x is $\mu\left[1-\frac{a}{x}\right]$ towards the point when $x>a$ but $\mu\left[\frac{a^2}{x^2}-\frac{a}{x}\right]$ from the same point when $x<a$; prove that particle will oscillate through a space $\left[\frac{b^2-a^2}{b}\right]$.*

Sol. Let the particle start from rest at B, where $OB=b$, and move towards the centre of force. Let $OA=a$.



Motion from B to A *i.e.*, when $x>a$.

Since the law of force, when $x>a$, is $\mu(1-a/x)$ towards O, therefore the equation of motion is

$$\frac{d^2x}{dt^2}=-\mu\left(1-\frac{a}{x}\right).$$

Multiplying both sides by $2(dx/dt)$ and integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=-2\mu(x-a\log x)+C$, where C is a constant.

But at B, $x=OB=b$ and $dx/dt=0$. $\therefore$ $C=2\mu(b-a\log b)$.

$$\therefore \left(\frac{dx}{dt}\right)^2=2\mu(b-a\log b-x+a\log x). \quad \text{...(1)}$$

If V is the velocity at the point A where $x=OA=a$, then from (1), we have $V^2=2\mu(b-a-a\log b+a\log a)$. ...(2)

Motion from A towards O *i.e.*, when $x<a$.

The velocity of the particle at A is V and it moves towards O under the law of force $\mu\left(\frac{a^2}{x^2}-\frac{a}{x}\right)$ at the distance x from the fixed point O.

$\therefore$ the equation of motion is $\frac{d^2x}{dt^2}=\mu\left[\frac{a^2}{x^2}-\frac{a}{x}\right]$.

Multiplying both sides by $2(dx/dt)$ and integrating, we have $\left(\frac{dx}{dt}\right)^2=2\mu\left(-\frac{a^2}{x}-a\log x\right)+D$, where D is a constant.

But at the point A, $x=a$ and $(dx/dt)^2=V^2=2\mu(b-a-a\log b+a\log a)$.

$\therefore$ $D=2\mu(b-a-a\log b+a\log a)+2\mu(a+a\log a)$
$=2\mu(b-a\log b+2a\log a)=2\mu\{b+a\log(a^2/b)\}$.

$$\therefore \left(\frac{dx}{dt}\right)^2=-2\mu\left(\frac{a^2}{x}+a\log x\right)+2\mu\left\{b+a\log\left(\frac{a^2}{b}\right)\right\}. \quad \text{...(3)}$$

If the particle comes to rest at the point C, where $x=c$, then putting $x=c$ and $dx/dt=0$ in (3), we get

$$2\mu\left(\frac{a^2}{c}+a\log c\right)=2\mu\left\{b-a\log\left(\frac{a^2}{b}\right)\right\}$$

or

$$\frac{a^2}{c}+a\log c=\frac{a^2}{(a^2/b)}+a\log\left(\frac{a^2}{b}\right).$$

$$\therefore \quad c=a^2/b \quad i.e., \quad OC=a^2/b.$$

Since B and C are the positions of instantaneous rest of the particle, therefore the particle oscillates through the space BC.

We have $BC=OB-OC=b-\frac{a^2}{b}=\frac{b^2-a^2}{b}$,

which proves the required result.

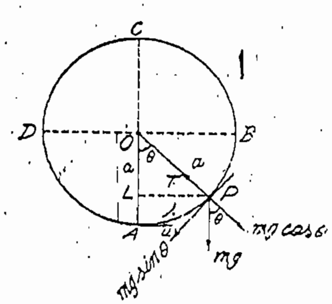
3

Constrained Motion

§ 1. **Introduction.** *The motion of a particle is called constrained motion, if it is compelled to move along a given curve or surface.*

Here in this chapter we shall consider the motion on smooth plane curves, vertical circle and cycloid only.

§ 2. **Motion in a vertical circle.** *A heavy particle is tied to one end of a light inextensible string whose other end is attached to a fixed point. It is projected horizontally with a given velocity u from its vertical position of equilibrium; to discuss the subsequent motion.* [Meerut 77, 79, 88; Agra 1976; Lucknow 79; Kanpur 81; Allahabad 78, 79; Rohilkhand 86]



Let one end of a string of length a be attached to the fixed point O and a particle of mass m be attached at the other end A. Let OA be the vertical position of equilibrium of the string. Let the particle be projected horizontally from A with velocity u. Since the string is inextensible the particle starts moving in a circle whose centre is O and radius a. If P is the position of the particle at time t such that $\angle AOP = \theta$ and arc $AP = s$, the forces acting on the particle at P are :

(i) weight mg of the particle acting vertically downwards, and (ii) tension T in the string acting along PO.

If v be the velocity of the particle at P, the tangential and normal accelerations of P are

$\dfrac{d^2s}{dt^2}$ (in the direction of s increasing)

and $\dfrac{v^2}{\rho}$ (along inwards drawn normal at P).

$\therefore$ the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad \ldots(1)$$

and $$m\frac{v^2}{\rho} = T - mg\cos\theta. \qquad \ldots(2)$$

Also $\quad s = \text{arc } AP = a\theta.$

$$\therefore \quad v = \frac{ds}{dt} = a\frac{d\theta}{dt}$$

and $$\frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2} \qquad \ldots(3)$$

$\therefore$ from (1) and (3), we have

$$a\frac{d^2\theta}{dt^2} = -g\sin\theta.$$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating w.r.t. 't', we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A,$$

where A is constant of integration.

But initially at A, $\theta = 0$, $v = u$.

$\therefore \quad A = u^2 - 2ag\cos 0 = u^2 - 2ag.$

$$\therefore \quad v^2 = u^2 - 2ag + 2ag\cos\theta. \qquad \ldots(4)$$

Now for a circle $\rho = a$ (radius).

$\therefore$ from (2), we have

$$T = \frac{m}{a}v^2 + mg\cos\theta = \frac{m}{a}(v^2 + ag\cos\theta).$$

Substituting the value of v^2 from (4), we have

$$T = \frac{m}{a}(u^2 - 2ag + 3ag\cos\theta). \qquad \ldots(5)$$

If the velocity $v = 0$ at $\theta = \theta_1$ then from (4), we have

$$0 = u^2 - 2ag + 2ag\cos\theta_1$$

or $$\cos\theta_1 = \frac{2ag - u^2}{2ag}. \qquad \ldots(6)$$

If h_1 is the height from the lowest point A of the point where the velocity vanishes, then

$$h_1 = OA - a\cos\theta_1 = a - a.\frac{2ag - u^2}{2ag}$$

or $$h_1 = \frac{u^2}{2g} \qquad \ldots(7)$$

Again if the tension $T=0$, at $\theta=\theta_2$, then from (5), we have

$$0=u^2-2ag+3ag\cos\theta_2,$$

$$\therefore\quad \cos\theta_2=\frac{2ag-u^2}{3ag}. \qquad \ldots(8)$$

If h_2 is the height from the lowest point A of the point where the tension vanishes, then

$$h_2=OA-a\cos\theta_2=a-a.\frac{2ag-u^2}{3ag}$$

or

$$h_2=\frac{u^2+ag}{3g}. \qquad \ldots(9)$$

Now the following cases may arise here.

Case I. The velocity v vanishes before the tension T.

This is possible if and only if $h_1<h_2$

or $\quad \dfrac{u^2}{2g}<\dfrac{u^2+ag}{3g}$ or $3u^2<2(u^2+ag)$

or $\quad u^2<2ag$ or $u<\sqrt{(2ag)}$.

But when $u<\sqrt{(2ag)}$, we have from (6), $\cos\theta_1=+$ive *i.e.*, θ_1 is an acute angle.

Thus if the particle is projected with the velocity $u<\sqrt{(2ag)}$, then it will oscillate about A and will not rise upto the horizontal diameter through O.

[Gorakhpur 1978]

Case II. The velocity v and the tension T vanish simultaneously:

This is possible if and only if $h_1=h_2$

i.e., $\quad \dfrac{u^2}{2g}=\dfrac{u^2+ag}{3g}$ *i.e.*, $u^2=2ag$ *i.e.*, $u=\sqrt{(2ag)}$.

Also when $u=\sqrt{(2ag)}$, we have from (6) and (8), $\theta_1=\pi/2=\theta_2$.

Thus if the particle is projected with the velocity $u=\sqrt{(2ag)}$, then it will rise upto the level of the horizontal diameter through O and will oscillate about A in the semi-circular arc BAD.

Case III. Condition for describing the complete circle.

[Meerut 1988]

At the highest point C, we have $\theta=\pi$. Therefore from (4) and (5), we have at C, $\quad v^2=u^2-4ag$

and $\quad T=\dfrac{m}{a}(u^2-5ag)$.

If $u^2>5ag$ *i.e.*, if $u>\sqrt{(5ag)}$, then neither the velocity v nor the tension T is zero at the highest point C, and so the particle will go on describing the complete circle.

And if $u^2=5ag$ *i.e.*, if $u=\sqrt{(5ag)}$, then at the highest point C the tension T vanishes whereas the velocity does not vanish. Hence in this case the string will become momentarily slack at C and the particle will go on describing the complete circle.

Thus the condition for describing the complete circle by the particle is that $u\geqslant\sqrt{(5ag)}$. In other words the least velocity of projection for describing the complete circle is $\sqrt{(5ag)}$.

[Meerut 1973; Gorakhpur 77]

Case IV. The tension T vanishes before the velocity v.

This is possible if and only if $h_1>h_2$

i.e., $\quad \dfrac{u^2}{2g}>\dfrac{u^2+ag}{3}$, *i.e.*, $u^2>2ag$ *i.e.*, $u>\sqrt{(2ag)}$.

When $u>\sqrt{(2ag)}$, we have from (8), $\cos\theta_2=-$ive showing that θ_2 must be $>90°$.

Now at the point where the tension T is zero, the string becomes slack. Since the velocity v is not zero at that point, therefore the particle will leave the circular path and trace a parabolic path while moving freely under gravity.

Thus if the particle is projected with the velocity u such that $\sqrt{(2ag)}<u<\sqrt{(5ag)}$, then it will leave the circular path at a point somewhere between B and C and trace out a parabolic path.

[Meerut 1974; 79]

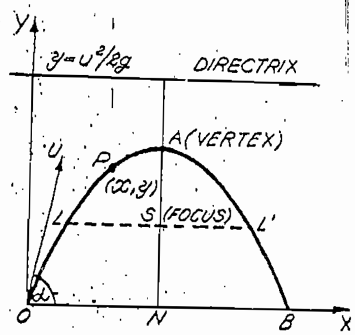
§ 3. *A particle is projected, along the inside of a smooth fixed hollow sphere (or circle) from its lowest point; to discuss the motion.*

The discussion is exactly the same as in § 2 with the difference that in this case the tension T is replaced by the reaction R between the particle and the sphere (or circle).

§ 4. Some important results of the motion of a projectile to be used in this chapter. Suppose a particle of mass m is projected in vaccum, in a vertical plane through the point of projection, with velocity u in a direction making an angle α with the horizontal. Then the path of the projectile is a parabola.

The following results about the motion of the projectile to be used in this chapter should be remembered.

Take the point of projection O as the origin, the horizontal line OX in the plane of projection as the x-axis, and the vertical line OY as the y-axis. Then the initial horizontal velocity of the projectile is $u \cos \alpha$, and the initial vertical velocity is $u \sin \alpha$.



The equation of the trajectory *i.e.*, the equation of the parabolic path is

$$y=x \tan \alpha-\tfrac{1}{2} g \frac{x^2}{u^2 \cos^2 \alpha}.$$

The length of the latus rectum LSL' of the above parabolic path is

$$\frac{2}{g} u^2 \cos^2 \alpha=\frac{2}{g} (\text{horizontal velocity})^2.$$

If H is the maximum height NA attained by the projectile above the point of projection O, then considering the vertical motion from O to A and using the formula $v^2=u^2+2fs$, we have

$$0=u^2 \sin^2 \alpha-2gH$$

or $$H=\frac{u^2 \sin^2 \alpha}{2g}.$$

Thus the maximum height of the projectile above the point of projection is $\dfrac{u^2 \sin^2 \alpha}{2g}$.

Also remember that the velocity of a projectile at any point P of its path is that due to a fall from the directrix to that point.

Illustrative Examples

Ex. 1. *A heavy particle of weight W, attached to a fixed point by a light inextensible string, describes a circle in a vertical plane. The tension in the string has the values mW and nW respectively*

when the particle is at the highest and lowest point in the path. Show that $n=m+6$. [Agra 1976, 79; Lucknow 80; Allahabad 77; Rohilkhand 86]

Sol. Let M be the mass of the particle. Then
$W=Mg$ i.e., $M=W/g$.

Proceeding as in § 2, the tension T in the string in any position is given by

$$T=\frac{M}{a} (u^2-2ag+3ag \cos \theta)$$ [See eqn. (5) of § 2 and deduce it here]

or $$T=\frac{W}{ag} (u^2-2ag+3ag \cos \theta). \quad ...(1)$$

Now mW is given to be the tension in the string at the highest point and nW that at the lowest point. Therefore $T=mW$ when $\theta=\pi$ and $T=nW$ when $\theta=0$. So from (1), we have

$$mW=\frac{W}{ag} (u^2-2ag+3ag \cos \pi) \text{ giving } m=\frac{1}{ag} (u^2-5ag) \quad ...(2)$$

and $$nW=\frac{W}{ag} (u^2-2ag+3ag \cos 0) \text{ giving } n=\frac{1}{ag} (u^2+ag). \quad ...(3)$$

Subtracting (2) from (3), we have
$n-m=6$ or $n=m+6$.

Ex. 2 (a). *A heavy particle hanging vertically from a point by a light inextensible string of length l is started so as to make a complete revolution in a vertical plane. Prove that the sum of the tensions at the end of any diameter is constant.*
[Rohilkhand 1977; Agra 80, 85; Meerut 76; Kanpur 83]

Sol. Proceeding as in § 2, the tension T in the string in any position is given by

$$T=\frac{m}{l} (u^2-2lg+3lg \cos \theta), \quad ...(1)$$

where θ is the angle which the string makes with OA.

Now take any diameter of the circle. If at one end of this diameter we have $\theta=\alpha$, then at the other end we shall have $\theta=\pi+\alpha$. Let T_1 and T_2 be the tensions at these ends i.e., $T=T_1$ when $\theta=\alpha$ and $T=T_2$ when $\theta=\pi+\alpha$. Then from (1), we have

$$T_1=\frac{m}{l} (u^2-2lg+3lg \cos \alpha) \quad ...(2)$$

and $$T_2=\frac{m}{l} \{u^2-2lg+3lg \cos (\pi+\alpha)\}$$

or $$T_2=\frac{m}{l}(u^2-2lg-3lg\cos\alpha). \qquad ...(3)$$

Adding (2) and (3), we have

$$T_1+T_2=2\frac{m}{l}(u^2-2lg)$$

which is constant, as it is independent of α.

Hence the sum of the tensions at the ends of any diameter is constant.

Ex. 2 (b). *A particle makes complete revolutions in a vertical circle. If ω_1, ω_2 be the greatest and least angular velocities and R_1, R_2 the greatest and least reactions, prove that when the particle projected from the lowest point of the circle makes an angle θ at the centre, its angular velocity is*

$$\sqrt{[\omega_1^2\cos^2\tfrac{1}{2}\theta+\omega_2^2\sin^2\tfrac{1}{2}\theta]} \qquad \text{[Meerut 1975, 82]}$$

and that the reaction is

$$R_1\cos^2\tfrac{1}{2}\theta+R_2\sin^2\tfrac{1}{2}\theta. \qquad \text{[Rohilkhand 1979]}$$

Sol. Proceed as in § 2. Replace the tension T by the reaction R.

Let u be the velocity of projection at the lowest point. For making complete circles, we must have $u^2\geq 5ag$.

If v be the velocity of the particle at any time t, then proceeding as in § 2, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=u^2-2ag+2ag\cos\theta,$$

and $$R=\frac{m}{a}(u^2-2ag+3ag\cos\theta). \qquad ...(1)$$

If ω be the angular velocity of the particle at time t, then $\omega=d\theta/dt$. So from (1), we have ...(2)

$$a^2\omega^2=u^2-2ag+2ag\cos\theta. \qquad ...(3)$$

From the equation (3) we observe that the angular velocity ω is greatest when $\cos\theta=1$ i.e., $\theta=0$ and is least when $\cos\theta=-1$ i.e., $\theta=\pi$. So putting $\theta=0$, $\omega=\omega_1$ and $\theta=\pi$, $\omega=\omega_2$ in (3), we get

$$a^2\omega_1^2=u^2 \quad\text{and}\quad a^2\omega_2^2=u^2-4ag. \qquad ...(4)$$

Now from (3), we have

$$a^2\omega^2=u^2-2ag(1-\cos\theta)=\tfrac{1}{2}[2u^2-4ag(1-\cos\theta)]$$
$$=\tfrac{1}{2}[2u^2-(u^2-a^2\omega_2^2)(1-\cos\theta)] \quad [\because \text{ from (4)}, 4ag=u^2-a^2\omega_2^2]$$
$$=\tfrac{1}{2}[2u^2-u^2(1-\cos\theta)+a^2\omega_2^2(1-\cos\theta)]$$
$$=\tfrac{1}{2}[u^2(1+\cos\theta)+a^2\omega_2^2(1-\cos\theta)]$$

$$=\tfrac{1}{2}[a^2\omega_1^2(1+\cos\theta)+a^2\omega_2^2(1-\cos\theta)] \quad [\because \text{ from (4)}, u^2=a^2\omega_1^2]$$
$$=\tfrac{1}{2}[2a^2\omega_1^2\cos^2\tfrac{1}{2}\theta+2a^2\omega_2^2\sin^2\tfrac{1}{2}\theta].$$

$$\therefore\quad \omega^2=\omega_1^2\cos^2\tfrac{1}{2}\theta+\omega_2^2\sin^2\tfrac{1}{2}\theta$$

or $$\omega=\sqrt{[\omega_1^2\cos^2\tfrac{1}{2}\theta+\omega_2^2\sin^2\tfrac{1}{2}\theta]}.$$

From the equation (2) we observe that the reaction R is greatest when $\cos\theta=1$ i.e., $\theta=0$ and is least when $\cos\theta=-1$ i.e., $\theta=\pi$. So putting $\theta=0$, $R=R_1$ and $\theta=\pi$, $R=R_2$ in (2), we get $R_1=(m/a)(u^2+ag)$ and $R_2=(m/a)(u^2-5ag)$. ...(5)

Now from (2), we have

$$R=(m/a)[u^2-2ag+3ag\cos\theta]$$
$$=\tfrac{1}{2}(m/a)[2u^2-4ag+6ag\cos\theta]$$
$$=\tfrac{1}{2}(m/a)[(u^2+ag)(1+\cos\theta)+(u^2-5ag)(1-\cos\theta)] \quad \text{[Note]}$$
$$=\tfrac{1}{2}[R_1(1+\cos\theta)+R_2(1-\cos\theta)] \quad \text{[From (5)]}$$
$$=\tfrac{1}{2}[2R_1\cos^2\tfrac{1}{2}\theta+2R_2\sin^2\tfrac{1}{2}\theta]=R_1\cos^2\tfrac{1}{2}\theta+R_2\sin^2\tfrac{1}{2}\theta.$$

Ex. 3. *A heavy particle hangs from a fixed point O, by a string of length a. It is projected horizontally with a velocity $v^2=(2+\sqrt{3})ag$; show that the string becomes slack when it has described an angle $\cos^{-1}(-1/\sqrt{3})$.* [Meerut 1973, 78, 82, 84]

Sol. Refer fig. of § 2, page 156.

The equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad ...(1)$$

and $$m\frac{v^2}{a}=T-mg\cos\theta. \qquad ...(2)$$

Also $$s=a\theta. \qquad ...(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\,(d\theta/dt)$ and then integrating w.r.t. t, we have $v^2=\left(a\dfrac{d\theta}{dt}\right)^2=2ag\cos\theta+A$,

where A is the constant of integration.

But initially at A, $\theta=0$ and $v^2=(2+\sqrt{3})ag$.

$\therefore\quad (2+\sqrt{3})ag=2ag\cos 0+A$, giving $A=\sqrt{3}ag$.

$\therefore\quad v^2=2ag\cos\theta+\sqrt{3}ag$.

Substituting this value of v^2 in (2), we have

$$T=\frac{m}{a}[v^2+ag\cos\theta]$$
$$=\frac{m}{a}[\sqrt{3}ag+3ag\cos\theta]. \qquad ...(4)$$

The string becomes slack when $T=0$.

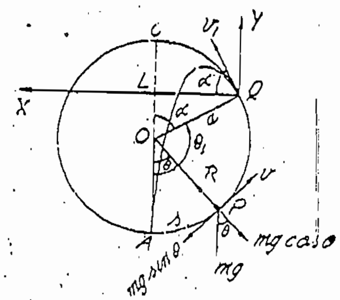
$\therefore$ from (4), we have

$$0=\frac{m}{a}\left[\sqrt{3}ag+3ag\cos\theta\right]$$

or $\cos\theta=-1/\sqrt{3}$ or $\theta=\cos^{-1}(-1/\sqrt{3})$.

Ex. 4. *A particle inside and at the lowest point of a fixed smooth hollow sphere of radius a is projected horizontally with velocity $\sqrt{(\frac{7}{2}ag)}$. Show that it will leave the sphere at a height $\frac{3}{2}a$ above the lowest point and its subsequent path meets the sphere again at the point of projection.* [Meerut 1979; Kanpur 77]

Sol. A particle is projected from the lowest point A of a sphere with velocity $u=\sqrt{(\frac{7}{2}ag)}$ to move along the inside of the sphere. Let P be the position of the particle at any time t where arc $AP=s$ and $\angle AOP=\theta$. If v be the velocity of the particle at P, the equations of motion along the tangent and normal are



$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad ...(1)$$

and

$$m\frac{v^2}{a}=R-mg\cos\theta. \quad ...(2)$$

Also $s=a\theta$. ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and then integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A$$

But at the point A, $\theta=0$ and $v=u=\sqrt{(\frac{7}{2}ag)}$.

$\therefore$ $A=\frac{7}{2}ag-2ag=\frac{3}{2}ag$.

$\therefore$ $v^2=\frac{3}{2}ag+2ag\cos\theta$. ...(4)

Now from (2) and (4), we have

$$R=\frac{m}{a}[v^2+ag\cos\theta]=\frac{m}{a}\left[\frac{3}{2}ag+2ag\cos\theta+ag\cos\theta\right]$$
$$=3mg\left(\tfrac{1}{2}+\cos\theta\right).$$

If the particle leaves the sphere at the point Q, where $\theta=\theta_1$, then $0=3mg(\frac{1}{2}+\cos\theta_1)$ or $\cos\theta_1=-\frac{1}{2}$.

If $\angle COQ=\alpha$, then $\alpha=\pi-\theta_1$.

$\therefore$ $\cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=\frac{1}{2}$. ...(5)

$\therefore$ $AL=AO+OL=a+a\cos\alpha=a+\frac{a}{2}=\frac{3a}{2}$

i.e., the particle leaves the sphere at a height $\frac{3}{2}a$ above the lowest point.

If v_1 is the velocity of the particle at the point Q, then putting $v=v_1$, $R=0$ and $\theta=\theta_1$ in (2), we get

$$v_1^2=-ag\cos\theta_1=-ag.(-\tfrac{1}{2})=\tfrac{1}{2}ag.$$

$\therefore$ the particle leaves the sphere at the point Q with velocity $v_1=\sqrt{(\frac{1}{2}ag)}$ making an angle α with the horizontal and subsequently describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{1}{2}\frac{gx^2}{v_1^2\cos^2\alpha}$$

or $$y=x.\sqrt{3}-\frac{gx^2}{2.\frac{1}{2}ag.\frac{1}{4}}$$ [$\because$ $\cos\alpha=\frac{1}{2}$ and so $\sin\alpha=\sqrt{(1-\cos^2\alpha)}=\sqrt{3}/2$. Thus $\tan\alpha=\sqrt{3}$.]

or $$y=\sqrt{3}x-\frac{4x^2}{a}. \quad ...(6)$$

From the figure, for the point A, $x=QL=a\sin\alpha=a\sqrt{3}/2$

and $y=-LA=-\frac{3}{2}a$.

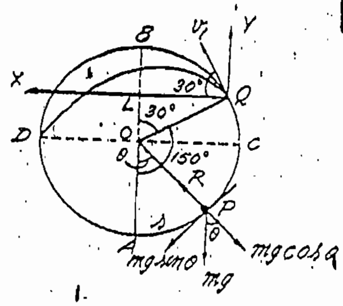
If we put $x=a\sqrt{3}/2$ in the equation (6), we get

$$y=a\frac{\sqrt{3}}{2}.\sqrt{3}-\frac{4}{a}.\frac{3a^2}{4}=\frac{3a}{2}-3a=-\frac{3}{2}a.$$

Thus the co-ordinates of the point A satisfy the equation (6). Hence the particle, after leaving the sphere at Q, describes a parabolic path which meets the sphere again at the point of projection A.

Ex. 5. *Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius a from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical. Show that it will strike the hoop again at an extremity of the horizontal diameter.*

Sol. Let a particle of mass m be projected with velocity u from the lowest point A of a smooth circular hoop of radius a along the interior of the hoop. If P is its position at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion along the tangent and normal are



$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad \ldots(1)$$

and
$$m\frac{v^2}{a}=R-mg\cos\theta. \qquad \ldots(2)$$

Also
$$s=a\theta. \qquad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and then integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the point A, $\theta=0$ and $v=u$. $\therefore$ $A=u^2-2ag$.

$$\therefore \quad v^2=u^2-2ag+2ag\cos\theta. \qquad \ldots(4)$$

From (2) and (4), we have

$$-R=\frac{m}{a}(v^2+ag\cos\theta)$$

$$=\frac{m}{a}(u^2-2ag+3ag\cos\theta).$$

If the particle leaves the circular hoop at the point Q where $\theta=150°$, then

$$0=\frac{m}{a}(u^2-2ag+3ag\cos 150°)$$

or
$$0=u^2-2ag-\frac{3\sqrt{3}}{2}ag.$$

$$u=[\tfrac{1}{2}ag(4+3\sqrt{3})]^{1/2}.$$

Hence the particle will leave the circular hoop at an angular distance of 30° from the vertical if the initial velocity of projection is $u=[\tfrac{1}{2}ag(4+3\sqrt{3})]^{1/2}$.

Again $OL=OQ\cos 30°=a(\sqrt{3}/2)$ and $QL=OQ\sin 30°=a/2$.

If v_1 is the velocity of the particle at the point Q, then $v=v_1$ when $\theta=150°$. Therefore from (4), we have

$$v_1^2=\tfrac{1}{2}ag(4+3\sqrt{3})-2ag+2ag\cos 150°=\tfrac{1}{2}ag\sqrt{3}$$

so that $\quad v_1=(\tfrac{1}{2}ag\sqrt{3})^{1/2}.$

Thus the particle leaves the circular hoop at the point Q with velocity $v_1=(\tfrac{1}{2}\sqrt{3}ag)^{1/2}$ at an angle 30° to the horizontal and subsequently it describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan 30°-\frac{gx^2}{2v_1^2\cos^2 30°}=\frac{x}{\sqrt{3}}-\frac{gx^2}{2.\tfrac{1}{2}\sqrt{3}ag.(\sqrt{3}/2)^2}$$

or
$$y=\frac{x}{\sqrt{3}}-\frac{4x^2}{3\sqrt{3}a}. \qquad \ldots(5)$$

For the point D which is the extremity of the horizontal diameter CD, we have

$$x=QL+OD=\tfrac{1}{2}a+a=3a/2,\ y=-LO=-a\sqrt{3}/2.$$

Clearly the co-ordinates of the point D satisfy the equation (5). Hence the particle after leaving the circular hoop at Q strikes the hoop again at an extremity of the horizontal diameter.

Ex. 6. *A particle is projected along the inner side of a smooth vertical circle of radius a, the velocity at the lowest point being u. Show that if $2ga<u^2<5ag$, the particle will leave the circle before arriving at the highest point and will describe a parabola whose latus rectum is*

$$\frac{2(u^2+2ag)^3}{27a^2g^3}.$$

(Meerut 1986S, 90P)

Sol. For figure refer Ex. 5. Proceeding as in Ex. 5, the velocity v and the reaction R at any time t are given by

$$v^2=u^2-2ag+2ag\cos\theta \qquad \ldots(1)$$

and
$$R=\frac{m}{a}(u^2-2ag+3ag\cos\theta). \qquad \ldots(2)$$

If the particle leaves the circle at Q, where $\angle AOQ=\theta_1$, then from (2), we have

$$0=\frac{m}{a}(u^2-2ag+3ag\cos\theta_1)$$

or $\cos\theta_1 = -\dfrac{u^2-2ag}{3ag}$.

Since $2ag < u^2 < 5ag$, therefore $\cos\theta_1$ is negative and its absolute value is <1. Therefore θ_1 is real and $\frac{1}{2}\pi < \theta_1 < \pi$.

Thus the particle leaves the circle before arriving at the highest point. If v_1 is the velocity of the particle at the point Q, then $v = v_1$ when $\theta = \theta_1$. Therefore from (1), we have

$$v_1^2 = u^2 - 2ag + 2ag \cos\theta_1$$

$$= (u^2-2ag) - 2ag\left(\frac{u^2-2ag}{3ag}\right)$$

$$= (u^2-2ag)(1-\tfrac{2}{3}) = \tfrac{1}{3}(u^2-2ag).$$

If $\angle BOQ = \alpha$, then $\alpha = \pi - \theta_1$.

$\therefore \cos\alpha = \cos(\pi - \theta_1) = -\cos\theta_1 = \dfrac{u^2-2ag}{3ag}$.

Thus the particle leaves the circle at the point Q with velocity $v_1 = \sqrt{[\frac{1}{3}(u^2-2ag)]}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

The latus rectum of the parabola

$$= \frac{2}{g} v_1^2 \cos^2\alpha = \frac{2}{g}\cdot\frac{1}{3}(u^2-2ag)\cdot\left(\frac{u^2-2ag}{3ag}\right)^2 = \frac{2(u^2-2ag)^3}{27a^2g^3}.$$

Ex. 7. *A heavy particle is attached to a fixed point by a fine string of length a; the particle is projected horizontally from the lowest point with velocity $\sqrt{[ag(2+3\sqrt{3}/2)]}$. Prove that the string would first become slack when inclined to the upward vertical at an angle of 30°; will become tight again when horizontal.*

[Meerut 1978]

Sol. Refer figure of Ex. 5 page 166. Taking $R = T$ (i.e., the tension in the string), the equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad \text{...(1)}$$

and $$m\frac{v^2}{a} = T - mg\cos\theta \qquad \text{...(2)}$$

Also $s = a\theta$.

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But at the point A, $\theta = 0$ and $v = \sqrt{[ag(2+3\sqrt{3}/2)]}$.

$\therefore ag(2+3\sqrt{3}/2) = 2ag + A$ or $A = \frac{3}{2}\sqrt{3}ag$.

$\therefore v^2 = ag(2\cos\theta + \frac{3}{2}\sqrt{3})$. ...(4)

From (2) and (4), we have

$$T = \frac{m}{a}(v^2 + ag\cos\theta) = \frac{m}{a}\cdot\left[ag(2\cos\theta + \tfrac{3}{2}\sqrt{3}) + ag\cos\theta\right]$$

$$= mg(3\cos\theta + \tfrac{3}{2}\sqrt{3}). \qquad \text{...(5)}$$

If the string becomes slack at the point Q, where $\theta = \theta_1$, then at Q, $T = 0 = mg(3\cos\theta_1 + \frac{3}{2}\sqrt{3})$

giving $\cos\theta_1 = -\sqrt{3}/2$ i.e., $\theta_1 = 150°$.

Hence the string becomes slack when inclined to the upward vertical at an angle of $180° - 150°$ i.e., 30°.

If v_1 is the velocity of the particle at Q, then $v = v_1$ when $\theta = 150°$. Therefore from (4), we have

$$v_1^2 = ag(2\cos 150° + \tfrac{3}{2}\sqrt{3}) = \tfrac{1}{2}\sqrt{3}ag.$$

Hence the particle leaves the circular path at the point Q with velocity $v_1 = (\frac{1}{2}ag\sqrt{3})^{1/2}$ at an angle of 30° to the horizontal and subsequently it describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as coordinate axes is

$$y = x\tan 30° - \frac{gx^2}{2v_1^2\cos^2 30°} = \frac{x}{\sqrt{3}} - \frac{gx^2}{2\cdot\frac{1}{2}\sqrt{3}ag\cdot(\sqrt{3}/2)^2}$$

or $$y = \frac{x}{\sqrt{3}} - \frac{4x^2}{3\sqrt{3}a}. \qquad \text{...(6)}$$

The co-ordinates of the point D, which is an extremity of the horizontal diameter CD, are given by

$x = QL + OD = \frac{1}{2}a + a = 3a/2$ and $y = -LO = -a\sqrt{3}/2$.

Clearly the co-ordinates of the point D satisfy the equation (6) showing that the parabolic trajectory meets the circle again at D. When the particle is at D, the string again becomes tight because $OD = a =$ the length of the string.

Hence the string becomes slack when inclined to the upward vertical at an angle of 30° and becomes tight again when horizontal.

Ex. 8. *A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$, prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point.* [Gorakhpur 1979; Meerut 77; 85S]

Also find the height of the highest point of the parabola subsequently described.

Sol. Refer figure of Ex. 4 page 164. Take $R=T$ (*i.e.*, the tension in the string).

Let a particle tied to a cord OA of length l be struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$. If P is the position of the particle at time t such that $\angle AOP=\theta$, then the equations of motion are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad \ldots(1)$$

and
$$m\frac{v^2}{l}=T-mg\cos\theta. \quad \ldots(2)$$

Also
$$s=l\theta. \quad \ldots(3)$$

From (1) and (3), we have $l\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2l\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(l\frac{d\theta}{dt}\right)^2=2lg\cos\theta+A.$$

But at the point A, $\theta=0$ and $v=2\sqrt{(gl)}$.

$\therefore$ $4gl=2lg+A$ so that $A=2gl$.

$\therefore$ $v^2=2lg(\cos\theta+1)$. ...(4)

From (2) and (4), we have

$$T=\frac{m}{l}(v^2+gl\cos\theta)=mg(3\cos\theta+2). \quad \ldots(5)$$

If the cord becomes slack at the point Q, where $\theta=\theta_1$, then from (5), we have

$$T=0=mg(3\cos\theta_1+2)$$

giving $\cos\theta_1=-2/3$.

If $\angle COQ=\alpha$, then $\alpha=\pi-\theta_1$ and $\cos\alpha=2/3$.

If v_1 is the velocity of the particle at Q, then $v=v_1$ where $\theta=\theta_1$. Therefore from (4), we have

$$v_1^2=2lg(1+\cos\theta_1)=2lg(1-\tfrac{2}{3})=2lg/3.$$

Now $OL=l\cos\alpha=\frac{2}{3}l$.

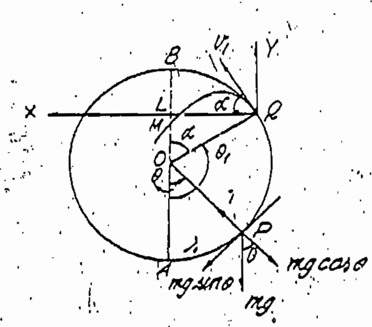
Thus the particle leaves the circular path at the point Q at a height $2l/3$ above the fixed point O with velocity $v_1=\sqrt{(2lg/3)}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

Max. height H of the particle above Q

$$=\frac{v_1^2\sin^2\alpha}{2g}=\frac{v_1^2}{2g}(1-\cos^2\alpha)=\frac{\frac{2}{3}lg}{2g}\left(1-\frac{4}{9}\right)=\frac{5l}{27}.$$

$\therefore$ Height of the highest point of the parabolic path above the fixed point $O=OL+H=\frac{2}{3}l+\frac{5l}{27}=\frac{23l}{27}$.

Ex. 9. *A heavy particle hangs by an inextensible string of length a from a fixed point and is then projected horizontally with a velocity $\sqrt{(2gh)}$. If $\frac{5a}{2}>h>a$, prove that the circular motion ceases when the particle has reached the height $\frac{1}{3}(a+2h)$. Prove also that the greatest height ever reached by the particle above the point of projection is $\dfrac{(4a-h)(a+2h)^2}{27a^2}$.* [Meerut 1984S]



Sol. Let a particle of mass m be attached to one end of a string of length a whose other end is fixed at O. The particle is projected horizontally with a velocity $u=\sqrt{(2gh)}$ from A. If P is the position of the particle at time t such that $\angle AOP=\theta$ and $AP=s$, then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad \ldots(1)$$

and
$$m\frac{v^2}{a}=T-mg\cos\theta \quad \ldots(2)$$

Also
$$s=a\theta. \quad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the point A, $\theta=0$, and $v=u=\sqrt{(2gh)}$.

$\therefore$ $A=2gh-2ag$.

$\therefore \quad v^2=2ag\cos\theta+2gh-2ag.$ ...(4)

From (2) and (4), we have

$$T=\frac{m}{a}(v^2+ag\cos\theta)=\frac{m}{a}(3ag\cos\theta+2gh-2ag).$$

If the particle leaves the circular path at Q where $\theta=\theta_1$, then $T=0$ when $\theta=\theta_1$.

$\therefore \quad 0=\frac{m}{a}(3ag\cos\theta_1+2gh-2ag)$ or $\cos\theta_1=-\frac{2h-2a}{3a}$.

Since $\frac{5}{2}a>h>a$ i.e., $5a>2h>2a$, therefore $\cos\theta_1$ is negative and its absolute value is <1. So θ_1 is real and $\frac{1}{2}\pi<\theta_1<\pi$.

Thus the particle leaves the circular path at Q before arriving at the highest point.

Height of the point Q above A

$$=AL=AO+OL=a+a\cos(\pi-\theta_1)=a-a\cos\theta_1$$

$$=a+a.\frac{2h-2a}{3a}=\frac{1}{3}(a+2h)$$

i.e., the particle leaves the circular path when it has reached a height $\frac{1}{3}(a+2h)$ above the point of projection.

If v_1 is the velocity of the particle at the point Q, then from (4), we have

$$v_1^2=2ag\cos\theta_1+2gh-2ag$$

$$=-2ag.\frac{(2h-2a)}{3a}+2g(h-a)$$

$$=2g(h-a)(1-\tfrac{2}{3})=\tfrac{2}{3}g(h-a).$$

If $\angle LOQ=\alpha$, then $\alpha=\pi-\theta_1$.

$$\therefore \quad \cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=\frac{2(h-a)}{3a}.$$

Thus the particle leaves the circular path at the point Q with velocity $v_1=\sqrt{\{\frac{2}{3}g(h-a)\}}$ at an angle $\alpha=\cos^{-1}\{2(h-a)/3a\}$ to the horizontal and will subsequently describe a parabolic path.

Maximum height of the particle above the point Q

$$=H=\frac{v_1^2\sin^2\alpha}{2g}=\frac{v_1^2}{2g}(1-\cos^2\alpha)=\frac{1}{3}(h-a).\left[1-\frac{4}{9a^2}(h-a)^2\right]$$

$$=\frac{1}{27a^2}(h-a)[9a^2-4(h^2-2ah+a^2)]$$

$$=\frac{(h-a)}{27a^2}\left[5a^2+8ah-4h^2\right]=\frac{1}{27a^2}(h-a)(a+2h)(5a-2h).$$

$\therefore$ Greatest height ever reached by the particle above the point of projection A

$$=AL+H=\frac{1}{3}(a+2h)+\frac{1}{27a^2}(h-a)(a+2h)(5a-2h)$$

$$=\frac{1}{27a^2}(a+2h)[9a^2+(h-a)(5a-2h)]$$

$$=\frac{1}{27a^2}(a+2h)[4a^2+7ah-2h^2]$$

$$=\frac{1}{27a^2}(a+2h)(a+2h)(4a-h)=\frac{1}{27a^2}(4a-h)(a+2h)^2.$$

Ex. 10. *A particle is projected along the inside of a smooth fixed sphere, from its lowest point, with a velocity equal to that due to falling freely down the vertical diameter of the sphere. Show that the particle will leave the sphere and afterwards pass vertically over the point of projection at a distance equal to $\frac{16}{27}$ of the diameter.*

Sol. Refer figure of Ex. 9 page 171. Replace T by R (i.e., reaction).

Here the velocity of projection $u=\sqrt{(2g.2a)}=\sqrt{(4ag)}$ i.e., the particle is projected from the lowest point A with velocity $u=2\sqrt{(ag)}$ inside a smooth sphere of radius a. If P is the position of the particle at time t such that $\angle AOP=\theta$, then the equations of motion are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad ...(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta. \quad ...(2)$$

Also $$s=a\theta. \quad ...(3)$$

From (1) and (3), we have $a\frac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\frac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the lowest point A, $\theta=0$ and $v=2\sqrt{(ag)}$.

$\therefore \quad A=4ag-2ag=2ag$

$\therefore \quad v^2=2ag\cos\theta+2ag.$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}\,(ag\cos\theta+v^2)$$

$$=\frac{m}{a}\,(3ag\cos\theta+2ag). \qquad \ldots(5)$$

Here $2ag<u^2<5ag$, therefore the particle will leave the sphere at an angle θ_1 where $\pi/2<\theta_1<\pi$.

If the particle leaves the sphere at the point Q, where $\theta=\theta_1$, then from (5), we have

$$R=0=\frac{m}{a}\,(3ag\cos\theta_1+2ag) \text{ giving } \cos\theta_1=-2/3.$$

If v_1 is the velocity of the particle at Q, then from (4), we have

$$v_1^2=2ag\cos\theta_1+2ag=2ag\,(\cos\theta_1+1)$$

or $\quad v_1^2=2ag\,(-\tfrac{2}{3}+1)=\tfrac{2}{3}ag.$

If $\angle BOQ=\alpha$, then $\alpha=\pi-\theta_1$.

$\therefore \quad \cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=\tfrac{2}{3}.$

Hence the particle leaves the spere at the point Q with velocity $v_1=\sqrt{(\tfrac{2}{3}ag)}$ at an angle $\alpha=\cos^{-1}(\tfrac{2}{3})$ to the horizontal and subsequently it describes a parabolic path.

Equation of the trajectory described by the particle after leaving the sphere at Q w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{2v_1^2\cos^2\alpha}$$

or $\quad y=x\cdot\dfrac{\sqrt{5}}{2}-\dfrac{gx^2}{2\cdot\tfrac{2}{3}ag\cdot\tfrac{4}{9}}$

$[\because \cos\alpha=\tfrac{2}{3}, \quad \therefore \sin\alpha=\sqrt{(1-\cos^2\alpha)}=\sqrt{5}/3$ and $\tan\alpha=\sin\alpha/\cos\alpha=\sqrt{5}/2]$

or $\quad y=\dfrac{\sqrt{5}}{2}\,x-\dfrac{27}{16a}\,x^2. \qquad \ldots(6)$

If the particle passes vertically over the point of projection A at the point M, then the x-co-ordinate of M is given by $x=QL=a\sin\alpha=a\sqrt{5}/3$. Let the y-coordinate of M be y_1.

The point M *i.e.*, $(a\sqrt{5}/3, y_1)$ lies on the trajectory (6).

$$\therefore \quad y_1=\frac{\sqrt{5}}{2}\cdot\frac{\sqrt{5}}{3}\,a-\frac{27}{16a}\cdot\frac{5a^2}{9}=\frac{5a}{6}-\frac{15a}{16}=-\frac{5a}{48}.$$

Since the y-coordinate of M is negative, therefore the point M is below the x-axis QX.

The required height $=AM=AO+OL+y_1=a+a\cos\alpha+y_1$

$$=a+\frac{2}{3}\,a-\frac{5a}{48}=\frac{25a}{16}=\frac{25}{32}\,(2a).$$

Hence the required height is equal to $\tfrac{25}{32}$ of the diameter of the sphere.

Ex. 11. *A particle is projected from the lowest point inside a smooth circle of radius a with a velocity due to a height h above the centre. Find the point where it leaves the circle and show that it will afterwards pass through*

(a) the centre if $h=\tfrac{1}{2}(a\sqrt{3})$,

and (b) the lowest point if $h=3a/4$. [Rohilkhand 1985]

Sol. Refer figure of Ex. 9 on page 171. Take $T=R$ (*i.e.*, reaction).

Here the velocity of projection u is equal to that due to a height h above the centre *i.e.*, due to a height $(h+a)$ above the lowest point A.

$$\therefore \quad u=\sqrt{\{2g\,(h+a)\}}.$$

Let the particle be projected from the lowest point A with velocity u along the inside of a smooth circle of radius a. If P is its position at time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion along the tangent and normal are

$$m\,\frac{d^2s}{dt^2}=-mg\sin\theta \qquad \ldots(1)$$

and $\quad m\,\dfrac{v^2}{a}=R-mg\cos\theta. \qquad \ldots(2)$

Also $\quad s=a\theta \qquad \ldots(3)$

From (1) and (3), we have $a\,\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\,\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the point A, $\theta=0$ and $v^2=u^2=2g\,(h+a)$.

$\therefore \quad A=2g(h+a)-2ag=2gh.$

$\therefore \quad v^2=2ag\cos\theta+2gh. \qquad \ldots(4)$

From (2), we have

$$R=\frac{m}{a}\,(v^2+ag\cos\theta)$$

$$=\frac{m}{a}\,(3\,ag\cos\theta+2gh). \qquad \ldots(5)$$

If the particle leaves the circle at the point Q, where $\theta=\theta_1$, then from (5), we have

$$R=0=\frac{m}{a}\,(3ag\cos\theta_1+2h)$$

giving $\cos\theta_1=-\dfrac{2h}{3a}$.

If v_1 is the velocity of the particle at Q, then from (4), we have $v_1^2=2ag\cos\theta_1+2gh=2ag\left(\dfrac{-2h}{3a}\right)+2gh=\dfrac{2}{3}gh$.

If $\angle BOQ=\alpha$, then $\alpha=\pi-\theta_1$.

$\therefore\quad \cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=(2h/3a)$,

and $\quad OL=a\cos\alpha=2h/3$.

Hence the particle leaves the circle at the point Q at height $2h/3$ above the centre O with velocity $v_1=\sqrt{(2gh/3)}$ at an angle $\alpha=\cos^{-1}(2h/3a)$ to the horizontal and then it describes a parabolic path.

Equation of the trajectory of the parabola described by the particle after leaving the circle at Q w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{2v_1^2\cos^2\alpha}$$

or $$y=x\tan\alpha-\frac{gx^2}{2\cdot\frac{2}{3}gh\cos^2\alpha}$$

or $$y=x\tan\alpha-\frac{3x^2}{4h\cos^2\alpha}\qquad\ldots(6)$$

(a) The co-ordinates of the centre O w.r.t. QX and QY as co-ordinate axes are given by

$$x=QL=a\sin\alpha\text{ and }y=-OL=-a\cos\alpha.$$

If the particle passes through the centre O i.e., the point $(a\sin\alpha, -a\cos\alpha)$, then the point O will lie on the curve (6).

$$\therefore\quad -a\cos\alpha=a\sin\alpha\,.\tan\alpha-\frac{3a^2\sin^2\alpha}{4h\cos^2\alpha}$$

or $$\frac{3a\sin^2\alpha}{4h\cos^2\alpha}=\frac{\sin^2\alpha}{\cos\alpha}+\cos\alpha=\frac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha}=\frac{1}{\cos\alpha}$$

or $$3a\sin^2\alpha=4h\cos\alpha$$

or $$3a(1-\cos^2\alpha)=4h\cos\alpha$$

or $$3a\left(1-\frac{4h^2}{9a^2}\right)=4h\cdot\frac{2h}{3a}\qquad\left[\because\ \cos\alpha=\frac{2h}{3a}\right]$$

or $$3a=\frac{h^2}{a}\left(\frac{8}{3}+\frac{4}{3}\right)=\frac{4h^2}{a}$$

or $$h^2=\tfrac{3}{4}a^2.$$

$\therefore\quad h=\frac{1}{2}(a\sqrt{3})$.

(b) The co-ordinates of the lowest point A w.r.t. QX and QY as co-ordinate axes are given by $x=QL=a\sin\alpha$

and $$y=-LA=-(LO+OA)$$
$$=-(a\cos\alpha+a)=-a(\cos\alpha+1).$$

If the particle after leaving the circle at Q, passes through the lowest point A $[a\sin\alpha, -a(\cos\alpha+1)]$, then the point A will lie on (6).

$$\therefore\quad -a(\cos\alpha+1)=a\sin\alpha\cdot\tan\alpha-\frac{3a^2\sin^2\alpha}{4h\cos^2\alpha}$$

or $$\frac{3a\sin^2\alpha}{4h\cos^2\alpha}=\frac{\sin^2\alpha}{\cos\alpha}+\cos\alpha+1$$
$$=\frac{\sin^2\alpha+\cos^2\alpha+\cos\alpha}{\cos\alpha}=\frac{1+\cos\alpha}{\cos\alpha}.$$

or $$3a\sin^2\alpha=4h\cos\alpha\,(1+\cos\alpha)$$

or $$3a(1-\cos^2\alpha)=4h\cos\alpha\,(1+\cos\alpha)$$

or $$3a(1-\cos\alpha)(1+\cos\alpha)=4h\cos\alpha\,(1+\cos\alpha)$$

or $$3a(1-\cos\alpha)=4h\cos\alpha\qquad[\because\ 1+\cos\alpha\neq0]$$

or $$3a\left(1-\frac{2h}{3a}\right)=4h\cdot\frac{2h}{3a}\qquad\left[\because\ \cos\alpha=\frac{2h}{3a}\right]$$

or $$3a(3a-2h)=8h^2\text{ or }9a^2-6ah-8h^2=0$$

or $$(3a+2h)(3a-4h)=0.$$

$\therefore\quad 3a-4h=0.\qquad[\because\ 3a+2h\neq0]$

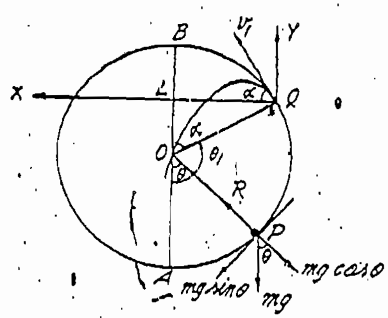
or $\quad h=3a/4.$

Ex. 12. *A particle is projected along the inside of a smooth vertical circle of radius a from the lowest point. Show that the velocity of projection required in order that after leaving the circle, the particle may pass through the centre is* $\sqrt{(\frac{1}{2}ag)}\,.(\sqrt{3}+1)$.

[Meerut 1988]

Sol. Let the particle be projected from the lowest point A along the inside of a smooth vertical circle of radius a, with velocity u. If P is the position of the particle at time t such that $\angle AOP=\theta$ and arc $AP=s$, the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta,\qquad\ldots(1)$$



and $\quad m\dfrac{v^2}{a}=R-mg\cos\theta,$ ...(2)

Also $\quad s=a\theta.$ ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the lowest point A, $\theta=0$ and $v=u$. $\therefore$ $A=u^2-2ag.$

$\therefore$ $v^2=2ag\cos\theta+u^2-2ga.$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(v^2+ag\cos\theta)=\frac{m}{a}(u^2-2ag+3ag\cos\theta). \quad ...(5)$$

If the particle leaves the circle at Q, where $\theta=\theta_1$, then from (5),

$$0=\frac{m}{a}(u^2-2ag+3ag\cos\theta_1)$$

or $$\cos\theta_1=-\left(\frac{u^2-2ag}{3ag}\right).$$

If $\angle BOQ=\alpha$, then $\alpha=\pi-\theta_1$.

$\therefore$ $\cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=\dfrac{u^2-2ag}{3ag}.$

If v_1 is the velocity at Q, then putting $v=v_1$, $R=0$ and $\theta=\theta_1$ in (2), we have

$$v_1^2=-ag\cos\theta_1=-ag\cos(\pi-\alpha)=ag\cos\alpha.$$

Thus the particle leaves the circle at Q with velocity $v_1=\sqrt{(ag\cos\alpha)}$ at angle $\alpha=\cos^{-1}\left(\dfrac{u^2-2ag}{3ag}\right)$ to the horizontal and subsequently it describes a parabolic path.

The equation of the parabolic trajectory w.r.t QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{2v_1^2\cos^2\alpha}=x\tan\alpha-\frac{2x^2}{2ag\cos^3\alpha} \quad ...(6)$$
$$[\because \; v_1^2=ag\cos\alpha]$$

The coordinates of the centre O w.r.t. QX and QY as coordinate axes are given by

$$x=QL=a\sin\alpha \text{ and } y=-LO=-a\cos\alpha.$$

If after leaving the circle at Q the particle passes through the centre O $(a\sin\alpha, -a\cos\alpha)$, then the point O lies on the curve (6).

$\therefore$ $\quad -a\cos\alpha=a\sin\alpha.\tan\alpha-\dfrac{ga^2\sin^2\alpha}{2ag\cos^3\alpha}.$

or $\quad \dfrac{\sin^2\alpha}{2\cos^3\alpha}=\dfrac{\sin^2\alpha}{\cos\alpha}+\cos\alpha=\dfrac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha}=\dfrac{1}{\cos\alpha}$

or $\quad \sin^2\alpha=2\cos^2\alpha$ or $1-\cos^2\alpha=2\cos^2\alpha$ or $3\cos^2\alpha=1$

or $\quad \cos^2\alpha=1/3$ or $\cos\alpha=1/\sqrt{3}.$

$\therefore$ $\quad \dfrac{u^2-2ag}{3ag}=\dfrac{1}{\sqrt{3}}$ $\quad \left[\because \; \cos\alpha=\dfrac{u^2-2ag}{3ag}\right]$

or $\quad u^2-2ag=\sqrt{3}ag.$

or $\quad u^2=(2+\sqrt{3})\,ag=\left(\dfrac{4+2\sqrt{3}}{2}\right)ag=\dfrac{ag}{2}(1+\sqrt{3})^2.$

$\therefore$ $\quad u=\sqrt{(\tfrac{1}{2}ag)}.(\sqrt{3}+1).$

Thus the particle will pass through the centre if the velocity of projection at the lowest point is $\sqrt{(\tfrac{1}{2}ag)}\,(\sqrt{3}+1)$.

Ex. 13. *A particle tied to a string of length a is projected from its lowest point, so that after leaving the circular path it describes a free path passing through the lowest point. Prove that the velocity of projection is* $\sqrt{(\tfrac{7}{2}ag)}$. [Kanpur 1975]

Sol. Refer figure of Ex. 12, page 178. Take $R=T$ (*i.e.*, the tension in the string).

Let a particle of mass m be attached to one end A of the string OA whose other end is fixed at O. Let the particle be projected from the lowest point A with velocity u. If the particle

leaves the circular path at Q with velocity v_1 at an angle α to the horizontal, then proceed as in Ex. 12 to get

$$v_1=\sqrt{(ag\cos\alpha)} \text{ and } \cos\alpha=\left(\frac{u^2-2ag}{3ag}\right).$$

After Q the particle describes a parabolic path whose equation w.r.t. the horizontal and vertical lines QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{v_1^2\cos^2\alpha}=x\tan\alpha-\frac{gx^2}{2ag\cos^3\alpha} \quad ...(1).$$

$$[\because \; v_1^2=ag\cos\alpha]$$

The co-ordinates of the lowest point A w.r.t. QX and QY as co-ordinate axes are given by

$x=QL=a\sin\alpha$ and $y=-LA=-(LO+OA)$

$$=-(a\cos\alpha+a)=-a(\cos\alpha+1).$$

If the particle passes through the lowest point $A\,[a\sin\alpha,\; -a(\cos\alpha+1)]$, then the point A lies on the curve (1).

$$\therefore \quad -a(\cos\alpha+1)=a\sin\alpha\tan\alpha-\frac{ga^2\sin^2\alpha}{2ag\cos^3\alpha}$$

or $\dfrac{\sin^2\alpha}{2\cos^3\alpha}=\dfrac{\sin^2\alpha}{\cos\alpha}+\cos\alpha+1$

$$=\frac{\sin^2\alpha+\cos^2\alpha+\cos\alpha}{\cos\alpha}=\frac{1+\cos\alpha}{\cos\alpha}$$

or $\sin^2\alpha=2\cos^2\alpha\,(1+\cos\alpha)$

or $(1-\cos^2\alpha)=2\cos^2\alpha\,(1+\cos\alpha)$

or $(1-\cos\alpha)(1+\cos\alpha)=2\cos^2\alpha\,(1+\cos\alpha)$

or $1-\cos\alpha=2\cos^2\alpha \quad [\because \; 1+\cos\alpha\neq 0]$

or $2\cos^2\alpha+\cos\alpha-1=0$ or $(2\cos\alpha-1)(\cos\alpha+1)=0$

or $2\cos\alpha-1=0 \quad [\because \; \cos\alpha+1\neq 0]$

or $\cos\alpha=\frac{1}{2}$

or $\dfrac{u^2-2ag}{3ag}=\frac{1}{2} \quad \left[\because \; \cos\alpha=\dfrac{u^2-2ag}{3ag}\right]$

or $u^2=2ag+\frac{3}{2}ag=\frac{7}{2}ag$ or $u=\sqrt{\left(\frac{7}{2}ag\right)}$.

Ex. 14. *Show that the greatest angle through which a person can oscillate on a swing, the ropes of which can support twice the person's weight at rest is 120°.*

If the ropes are strong enough and he can swing through 180° and if v is his speed at any point, prove that the tension in the rope at that point is $\frac{3mv^2}{2l}$ where m is the mass of the person and l the length of the rope.

Sol. Let u be the velocity of a person of mass m at the lowest point. If v is the velocity of the person and T the tension in the rope of length l at a point P at an angular distance θ from the lowest point, then proceed as in § 2 to get

$$v^2=u^2-2lg+2lg\cos\theta, \quad ...(1)$$

and $$T=\frac{m}{l}(u^2-2lg+3lg\cos\theta). \quad ...(2)$$

Now according to the question the ropes can support twice the person's weight at rest. Therefore the maximum tension the rope can bear is $2mg$. So for the greatest angle through which the person can oscillate, the velocity u at the lowest point should be such that $T=2mg$ when $\theta=0$.

Then from (2), we have

$$2mg=\frac{m}{l}(u^2-2lg+3lg\cos 0)$$

or $2gl=u^2-2lg+3lg$ or $u^2=lg$.

Now from (1), we have

$$v^2=lg-2lg+2lg\cos\theta=2lg\cos\theta-lg=lg(2\cos\theta-1).$$

If $v=0$ at $\theta=\theta_1$, then $0=gl(2\cos\theta_1-1)$

or $\cos\theta_1=\frac{1}{2}$. Therefore $\theta_1=60°$.

Thus the person can swing through an angle of 60° from the vertical on one side of the lowest point. Hence the person can oscillate through an angle of $60°+60°=120°$.

Second part. If the rope is strong enough and the person can swing through an angle of 180° *i.e.*, through an angle of 90° on one side of the lowest point, then $v=0$, at $\theta=90°$.

$\therefore$ from (1), we have

$$0=u^2-2lg+2lg\cos 90° \text{ or } u^2=2lg.$$

Thus if the person's velocity at the lowest point is $\sqrt{(2lg)}$, then he can swing through an angle of 180°.

Then from (1), we have $v^2=2lg-2lg+2lg\cos\theta$

or $$\cos\theta=\frac{v^2}{2lg}.$$

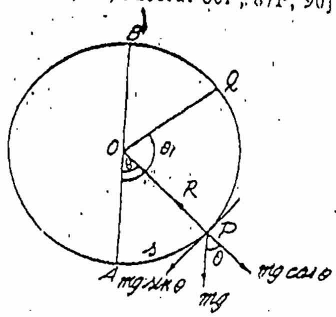
Therefore from (2), the tension in the rope at an angular distance θ where the velocity is v, is given by

$$T=\frac{m}{l}\left[2lg-2lg+3lg.\frac{v^2}{2lg}\right]=\frac{3mv^2}{2l}.$$

Ex. 15. *A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time*

$$\sqrt{(a/g)}.\log(\sqrt{5}+\sqrt{6}).$$

[Agra 1980, 86; Kanpur 79, 81; Meerut 86P, 87P, 90]



Sol. Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the highest point B.

If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad \ldots(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta. \quad \ldots(2)$$

Also $$s=a\theta. \quad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But according to the question $v=0$ at the highest point B, where $\theta=\pi$. $\therefore$ $0=2ag\cos\pi+A$ or $A=2ag$.

$$\therefore \quad v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+2ag. \quad \ldots(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}(v^2+ag\cos\theta)=\frac{m}{a}(2ag+3ag\cos\theta). \quad \ldots(5)$$

If the reaction $R=0$ at the point Q where $\theta=\theta_1$, then from (5), we have

$$0=\frac{m}{a}(2ag+3ag\cos\theta_1)$$

or $$\cos\theta_1=-2/3. \quad \ldots(6)$$

From (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2=2ag(\cos\theta+1)=2ag.2\cos^2\tfrac{1}{2}\theta=2ag\cos^2\tfrac{1}{2}\theta.$$

$\therefore$ $\dfrac{d\theta}{dt}=2\sqrt{(g/a)}\cos\tfrac{1}{2}\theta$, the positive sign being taken before the radical sign because θ increases as t increases

or $dt=\tfrac{1}{2}\sqrt{(a/g)}\sec\tfrac{1}{2}\theta\,d\theta.$

Integrating from $\theta=0$ to $\theta=\theta_1$, the required time t is given by

$$t=\tfrac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\theta_1}\sec\tfrac{1}{2}\theta\,d\theta$$

or $$t=\sqrt{(a/g)}\left[\log(\sec\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\theta)\right]_0^{\theta_1}$$

or $$t=\sqrt{(a/g)}\log(\sec\tfrac{1}{2}\theta_1+\tan\tfrac{1}{2}\theta_1). \quad \ldots(7)$$

From (6), we have

$$2\cos^2\tfrac{1}{2}\theta_1-1=-\tfrac{2}{3}$$

or $$2\cos^2\tfrac{1}{2}\theta_1=1-\tfrac{2}{3}=\tfrac{1}{3}$$

or $$\cos^2\tfrac{1}{2}\theta_1=\tfrac{1}{6} \quad \text{or} \quad \sec^2\tfrac{1}{2}\theta_1=6.$$

$\therefore$ $\sec\tfrac{1}{2}\theta_1=\sqrt{6}$

and $\tan\tfrac{1}{2}\theta_1=\sqrt{(\sec^2\tfrac{1}{2}\theta_1-1)}=\sqrt{(6-1)}=\sqrt{5}.$

Substituting in (7), the required time is given by

$$t=\sqrt{(a/g)}\log(\sqrt{6}+\sqrt{5}).$$

Ex. 16. *A heavy bead slides on a smooth circular wire of radius a. It is projected from the lowest point with a velocity just sufficient to carry it to the heighest point, prove that the radius through the bead in time t will turn through an angle*

$$2\tan^{-1}[\sinh\{t\sqrt{(g/a)}\}]$$

and that the bead will take an infinite time to reach the highest point. [Meerut 1972, 75, 84 P 85P, 87, 87S, 90S; Agra 88]

Sol. Refer figure of Ex. 15 page 182.

The equations of motion of the bead are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta, \quad ...(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta. \quad ...(2)$$

Also $s=a\theta$. ...(3)

From (1) and (3), we have $a\frac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But according to the question at the highest point $v=0$

i.e., when $\theta=\pi$, $v=0$.

$\therefore$ $0=2ag\cos\pi+A$ or $A=2ag$.

$\therefore$ $v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag+2ag\cos\theta=2ag(1+\cos\theta)$

$$=2ag.2\cos^2\tfrac{1}{2}\theta$$

or $a\frac{d\theta}{dt}=2\sqrt{(ag)}.\cos\tfrac{1}{2}\theta$

or $dt=\tfrac{1}{2}\sqrt{(a/g)}.\sec\tfrac{1}{2}\theta.d\theta$.

Integrating, the time t from A to P is given by

$$t=\tfrac{1}{2}\sqrt{(a/g)}.\int_0^\theta\sec\tfrac{1}{2}\theta\,d\theta$$

$$=\tfrac{1}{2}\sqrt{(a/g)}.2\left[\log(\tan\tfrac{1}{4}\theta+\sec\tfrac{1}{2}\theta)\right]_0^\theta$$

$$=\sqrt{(a/g)}\,[\log(\tan\tfrac{1}{2}\theta+\sec\tfrac{1}{2}\theta)-\log 1]$$

$$=\sqrt{(a/g)}.[\log\{\tan\tfrac{1}{2}\theta+\sqrt{(1+\tan^2\tfrac{1}{2}\theta)}\}]$$

$$=\sqrt{(a/g)}.\sinh^{-1}(\tan\tfrac{1}{2}\theta)$$

$[\because \sinh^{-1}x=\log\{x+\sqrt{(1+x^2)}\}]$

or $t\sqrt{(g/a)}=\sinh^{-1}(\tan\tfrac{1}{2}\theta)$

or $\tan\tfrac{1}{2}\theta=\sinh\{t\sqrt{(g/a)}\}$.

$\therefore$ $\theta=2\tan^{-1}[\sinh\{t\sqrt{(g/a)}\}]$.

Again the time to reach the highest point B while starting from A

$$=\tfrac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\pi}\sec\tfrac{1}{2}\theta\,d\theta$$

$$=\tfrac{1}{2}\sqrt{(a/g)}.2\left[\log(\tan\tfrac{1}{4}\theta+\sec\tfrac{1}{2}\theta)\right]_0^\pi$$

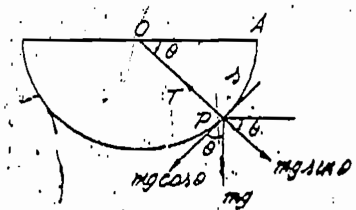
$$=\sqrt{(a/g)}.[\log(\tan\tfrac{1}{2}\pi+\sec\tfrac{1}{2}\pi)-\log(\tan 0+\sec 0)]$$

$$=\sqrt{(a/g)}.[\log\infty-\log 1]=\infty.$$

Therefore the bead takes an infinite time to reach the highest point.

15 M

Ex. 17. *A particle attached to a fixed peg O by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is $g\sqrt{(1+3\sin^2\theta)}$.*



Sol. Let a particle of mass m be attached to a string of length l whose other end is attached to a fixed peg O. Initially let the string be horizontal in the position OA such that $OA=l$. The particle starts from A and moves in a circle whose centre is O and radius is l. Let P be the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$. The forces acting on the particle at P are : (i) its weight mg acting vertically downwards and (ii) the tension T in the string along PO.

$\therefore$ the equations of motion of the particle along the tangent and normal at P are

$$m\frac{d^2s}{dt^2}=mg\cos\theta, \quad ...(1)$$

and $$m\frac{v^2}{l}=T-mg\sin\theta. \quad ...(2)$$

Also $s=l\theta$. ...(3)

From (1) and (3), we have $l\frac{d^2\theta}{dt^2}=g\cos\theta$.

Multiplying both sides by $2l(d\theta/dt)$ and integrating, we have

$$v^2=\left(l\frac{d\theta}{dt}\right)^2=2lg\sin\theta+A.$$

But initially at the point A, $\theta=0$, $v=0$. $\therefore$ $A=0$.

$\therefore$ $v^2=2lg\sin\theta$. ...(4)

The resultant acceleration of the particle at P.

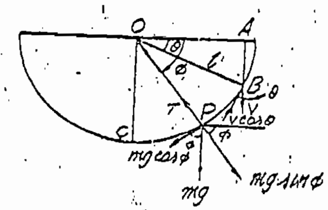
$$=\sqrt{\{(\text{Tangential accel.})^2+(\text{Normal accel.})^2\}}$$

$$=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{l}\right)^2\right]} \quad \left[\because \text{Normal accel}=\frac{v^2}{\rho}=\frac{v^2}{l}\right]$$

$$=\sqrt{\left[(g\cos\theta)^2+\left(\frac{2lg\sin\theta}{l}\right)^2\right]}$$

$$=g\sqrt{(1-\sin^2\theta+4\sin^2\theta)}=g\sqrt{(1+3\sin^2\theta)}.$$

Ex. 18. *A particle attached to a fixed peg O by a string of length l, is let fall from a point in the horizontal line through O at a distance l cos θ from O; show that its velocity when it is vertically below O is* $\sqrt{[2gl(1-\sin^3\theta)]}$.



Sol. Let a particle of mass m be attached to a string of length l whose other end is attached to a fixed peg O. Let the particle fall from a point A in the horizontal line through O such that $OA=l\cos\theta$. The particle will fall under gravity from A to B, where $OB=l$.

$\because$ $OA=l\cos\theta$ and $OB=l$, therefore $\angle AOB=\theta$ and $AB=l\sin\theta$.

$\therefore$ the velocity of the particle at B

$=V=\sqrt{(2g.AB)}=\sqrt{(2gl\sin\theta)}$, vertically downwards.

As the particle reaches B, there is a jerk in the string and the impulsive tension in the string destroys the component of the velocity along OB and the component of the velocity along the tangent at B remains unaltered *i.e.*, the particle moves in the circular path with centre O and radius l with the tangential velocity $V\cos\theta$ at B.

[Note: In the figure write D at the end of the horizontal radius through O].

If P is the position of the particle at any time t such that $\angle DOP=\phi$ and arc $DP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\cos\phi, \qquad \ldots(1)$$

and $$\frac{mv^2}{l}=T-mg\sin\phi. \qquad \ldots(2)$$

Also $$s=l\phi. \qquad \ldots(3)$$

From (1) and (3), we have $l\dfrac{d^2\phi}{dt^2}=g\cos\phi.$

Multiplying both sides by $2l\,(d\phi/dt)$ and integrating, we have

$$v^2=\left(l\frac{d\phi}{dt}\right)^2=2lg\sin\phi+A.$$

But at the point B, $\phi=\theta$ and $v=V\cos\theta$.

$\therefore$ $A=V^2\cos^2\theta-2lg\sin\theta=2gl\sin\theta.\cos^2\theta-2lg\sin\theta$

$=-2lg\sin\theta\,(1-\cos^2\theta)=-2lg\sin^3\theta.$

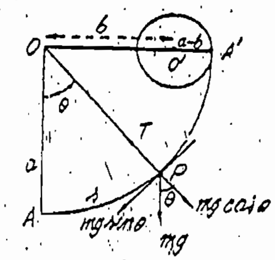
$\therefore$ $v^2=2lg\sin\phi-2lg\sin^3\theta.$

When the particle is at C vertically below O, we have, at C $\phi=\pi/2$. Therefore the velocity v at C is given by

$$v^2=2lg\sin\tfrac{1}{2}\pi-2lg\sin^3\theta=2lg(1-\sin^3\theta).$$

$\therefore$ the required velocity $v=\sqrt{[2lg(1-\sin^3\theta)]}$.

Ex. 19. *A particle is hanging from a fixed point O by means of a string of length a. There is a small nail at O′ in the same horizontal line with O at a distance b (<a) from O. Find the minimum velocity with which the particle should be projected from its lowest point in order that it may make a complete revolution round the nail without the string becoming slack.* [Meerut 1977]



Sol. Let a particle of mass m hang from a fixed point O by means of a string OA of length a. Let O' be a nail in the same horizontal line with O at a distance $OO'=b$ $(<a)$. Let the particle be projected from A with velocity u. It moves in a circle with centre at O and radius as a. If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta, \qquad \ldots(1)$$

and $$m\frac{v^2}{a}=T-mg\cos\theta. \qquad \ldots(2)$$

Also $$s=a\theta. \qquad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=u$. $\therefore$ $A=u^2-2ag$.

$\therefore$ $v^2=u^2-2ag+2ag\cos\theta$. ...(4)

At the point A', $\theta=\pi/2$. If v_1 is the velocity of A', then from (4), we have

$$v_1^2=u^2-2ag \quad \text{or} \quad v_1=\sqrt{(u^2-2ag)}.$$

Since there is a nail at O', the particle will describe a circle with centre at O' and radius as $O'A'=a-b$.

We know that if a particle is attached to a string of length l, the least velocity of projection from the lowest point in order to make a complete circle is $\sqrt{(5gl)}$. Also in this case, using the result (4), the velocity of the particle when it has described an angle θ from the lowest point is given by

$$v^2=5lg-2lg+2lg\cos\theta \quad [\because \text{ here } a=l \text{ and } u^2=5gl]$$
$$=3lg+2lg\cos\theta.$$

At $\theta=\pi/2$, if $v=v_2$, then $v_2=\sqrt{(3lg)}$. $[\because \cos\pi/2=0]$

Thus in order to describe a complete circle of radius l the minimum velocity of the particle at the end of the horizontal diameter should be $\sqrt{(3gl)}$. Therefore in order to describe a complete circle of radius $l=O'A'=a-b$ round O' the minimum velocity of the particle at A' should be $\sqrt{[3g\,(a-b)]}$.

But, as already found out, the velocity of the particle at A' is v_1.

$\therefore$ we must have $v_1\geqslant\sqrt{[3g\,(a-b)]}$

or $\sqrt{(u^2-2ag)}\geqslant\sqrt{[3g\,(a-b)]}$

or $u^2-2ag\geqslant 3g\,(a-b)$

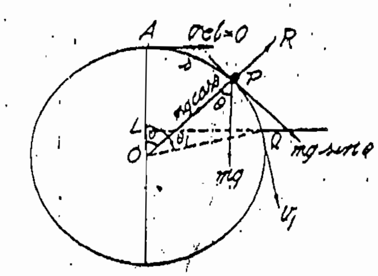
or $u^2\geqslant g\,(5a-3b)$

or $u\geqslant\sqrt{[g(5a-3b)]}$.

Hence the required minimum velocity of projection of the particle at the lowest point is $\sqrt{[g\,(5a-3b)]}$.

§ 5. **Motion on the outside of a smooth vertical circle.** *A particle slides down the outside of a smooth vertical circle starting from rest at the highest point; to discuss the motion.*

[Meerut 1974, 77, 81; Kanpur 76, 80; Agra 78]



Let a particle of mass m slide down the outside of a smooth vertical circle whose centre is O and radius a, starting from rest at the highest point A. Let P be the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$. The forces acting on the particle at P are (i) weight mg acting vertically downwards and (ii) the reaction R acting along the outwards drawn normal OP. If v be the velocity of the particle at P the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta, \quad ...(1)$$

(+ive sign is taken on the R. H. S. because $mg\sin\theta$ acts in the direction of s increasing)

and $$m\frac{v^2}{a}=mg\cos\theta-R. \quad ...(2)$$

[Note that in equation (2) R has been taken with $-$ive sign because it is in the direction of outwards drawn normal and $mg\cos\theta$ with +ive sign because it is in the direction of inwards drawn normal.]

Also $s=a\theta$. ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=0$. $\therefore$ $A=2ag$.

$\therefore$ $v^2=2ag-2ag\cos\theta=2ag\,(1-\cos\theta)$. ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}\left[ag\cos\theta-v^2\right]=\frac{m}{a}\left[3ag\cos\theta-2ag\right]$$
$$=mg\,(3\cos\theta-2). \quad ...(5)$$

If the particle leaves the circle at Q where $\angle AOQ=\theta_1$, then $R=0$ when $\theta=\theta_1$. $\therefore$ Therefore from (5), we have

$$mg\,(3\cos\theta_1-2)=0 \quad \text{or} \quad \cos\theta_1=\tfrac{2}{3}.$$

Vertical depth of the point Q below A

$=AL=OA-OL=a-a\cos\theta_1=a-\frac{2}{3}a=a/3.$

Hence if a particle slides down the outside of a smooth vertical circle, starting from rest at the highest point, it will leave the circle after descending vertically a distance equal to one third of the radius of the circle. [Meerut 1981S]

If v_1 is the velocity of the particle at Q, then $v=v_1$ when $\theta=\theta_1$.

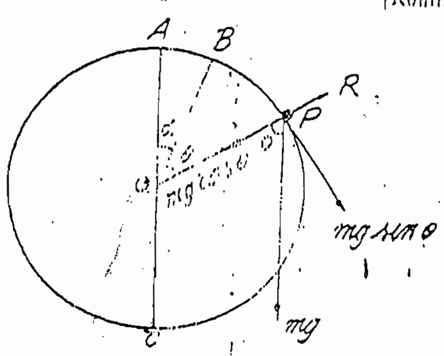
$\therefore$ from (4), we have

$$v_1^2=2ag\,(1-\cos\theta_1)=2ag\,(1-\tfrac{2}{3})=\tfrac{2}{3}ag.$$

The direction of the velocity v_1 is along the tangent to the circle at Q. Therefore the particle leaves the circle at Q with velocity $v_1=\sqrt{(\frac{2}{3}ag)}$ making an angle $\theta_1=\cos^{-1}(\frac{2}{3})$ below the horizontal line through Q. After leaving the circle at Q the particle will move freely under gravity and so it will describe a parabolic path.

Illustrative Examples

Ex. 20. *A particle is placed on the outside of a smooth vertical circle. If the particle starts from a point whose angular distance is α from the highest point of circle, show that it will fly off the curve when $\cos\theta=\frac{2}{3}\cos\alpha$.* [Rohilkhand 1988]



Sol. A particle slides down on the outside of the arc of a smooth vertical circle of radius a, starting from rest at a point B such that $\angle AOB=\alpha$. Let P be the position of the particle at any time t where arc $AP=s$ and $\angle POA=\theta$. The forces acting on the

particle at P are : (i) weight mg acting vertically downwards and (ii) the reaction R along the outwards drawn normal OP.

If v be the velocity of the particle at P, the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta, \qquad \ldots(1)$$

and $$m\frac{v^2}{a}=mg\cos\theta-R. \qquad \ldots(2)$$

Also $$s=a\theta. \qquad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at B, $\theta=\alpha$ and $v=0$. $\therefore$ $A=2ag\cos\alpha$.

$$\therefore\quad v^2=2ag\cos\alpha-2ag\cos\theta. \qquad \ldots(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}\,(-v^2+ag\cos\theta)=\frac{m}{a}\,(-2ag\cos\alpha+3ag\cos\theta)$$

$$=mg\,(-2\cos\alpha+3\cos\theta). \qquad \ldots(5)$$

At the point where the particle flies off the circle, we have $R=0$.

$\therefore$ from (5), we have

$$0=mg\,(-2\cos\alpha+3\cos\theta) \quad \text{or} \quad \cos\theta=\tfrac{2}{3}\cos\alpha.$$

Ex. 21. *A particle is projected horizontally with a velocity $\sqrt{(ag/2)}$ from the highest point of the outside of a fixed smooth sphere of radius a. Show that it will leave the sphere at the point whose vertical distance below the point of projection is $a/6$.* [Allahabad 1976]

Sol. Refer figure of § 5 on page 189.

Let a particle be projected horizontally with a velocity $\sqrt{(ag/2)}$ from the highest point A on the outside of a fixed smooth sphere of radius a. If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta, \qquad \ldots(1)$$

and $\quad m\dfrac{v^2}{a}=mg\cos\theta-R.$ ...(2)

Here v is the velocity of the particle at P.

Also $\quad s=a\theta.$ ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=\sqrt{(ag/2)}$.

$\therefore\ ag/2=-2ag+A$ or $A=\frac{1}{2}ag+2ag=\frac{5}{2}ag.$

$\therefore\ v^2=\frac{5}{2}ag-2ag\cos\theta.$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(ag\cos\theta-v^2)=\frac{m}{a}(3ag\cos\theta-\tfrac{5}{2}ag)$$

or $\quad R=mg\,(3\cos\theta-\frac{5}{2}).$ ...(5)

If the particle leaves the sphere at the point Q where $\theta=\theta_1$, then putting $R=0$ and $\theta=\theta_1$ in (5), we have

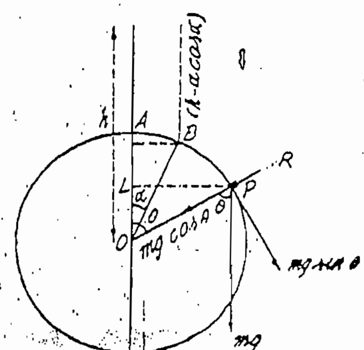
$0=mg\,(3\cos\theta_1-\frac{5}{2})$ or $\cos\theta_1=5/6.$

$\therefore$ Vertical depth of the point Q below the point of projection A

$=AL=OA-OL=a-a\cos\theta_1=a-\frac{5}{6}a=\frac{1}{6}a.$

Ex. 22. *A particle moves under gravity in a vertical circle sliding down the convex side of the smooth circular arc. If the initial velocity is that due to a fall to the starting point from a height h above the centre, show that it will fly off the circle when at a height $\frac{2}{3}h$ above the centre.* [Gorakhpur 1981, Allahabad 87]

Sol. Let a particle start from the point B of a smooth vertical circle where $\angle AOB=\alpha$. The depth of the point B, from the point which is at a height h above the centre O, is $h-a\cos\alpha$.



Therefore the initial velocity of the particle at B

$=u=\sqrt{\{2g\,(h-a\cos\alpha)\}}.$

If P is the position of the particle at time t such that $\angle AOP=\theta$ and arc $AP=s$, the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta,$$

and $\quad m\dfrac{v^2}{a}=mg\cos\theta-R.$ ...(2)

Also $\quad s=a\theta.$ ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at B, $\theta=\alpha$ and $v=\sqrt{\{2g\,(h-a\cos\alpha)\}}.$

$\therefore\ 2g\,(h-a\cos\alpha)=-2ag\cos\alpha+A$ or $A=2gh.$

$\therefore\ v^2=-2ag\cos\theta+2gh.$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(ag\cos\theta-v^2)=\frac{m}{a}(3ag\cos\theta-2gh).$$

The particle will leave the sphere, where $R=0$ i.e., where

$$\frac{m}{a}(3ag\cos\theta-2gh)=0\quad\text{or}\quad\cos\theta=2h/3a.$$

Now the height of the point where the particle flies off the circle, above the centre $O=OL=a\cos\theta=2h/3.$

Ex. 23. *A particle is placed at the highest point of a smooth vertical circle of radius a and is allowed to slide down starting with a negligible velocity. Prove that it will leave the circle after describing vertically a distance equal to one third of the radius. Find the position of the directrix and the focus of the parabola subsequently described and show that its latus-rectum is $\frac{16}{27}a$.*

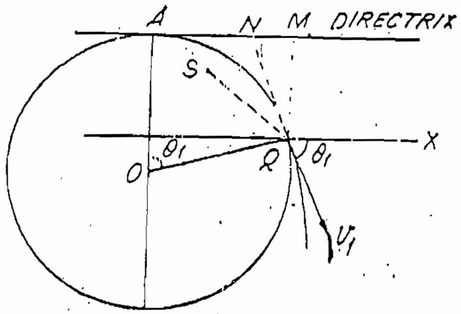
[Meerut 1976, 77, 78, 80, 81, 88P; Lucknow 76, 81; Agra 87]

Sol. For the first part see § 5 on page 189.

From § 5, the particle leaves the sphere at the point Q where $\angle AOQ=\theta_1$ and $\cos\theta_1=\frac{2}{3}$. The velocity v_1 at the point Q is $\sqrt{(2ag/3)}$; its direction is along the tangent to the circle at Q. After leaving the circle at the point Q, the particle describes a parabolic path with the velocity of projection $v_1=\sqrt{(2ag/3)}$ making an angle $\theta_1=\cos^{-1}(2/3)$ below the horizontal line through Q.

Latus rectum of the parabola subsequently described

$$=\frac{2v_1^2\cos^2\theta_1}{g}=\frac{2}{g}\cdot\frac{2ag}{3}\cdot\frac{4}{9}=\frac{16}{27}a.$$



To find the position of the directrix and the focus of the parabola. We know that in a parabolic path of a projectile the velocity at any point of its path is equal to that due to a fall from the directrix to that point.

Therefore if h is the height of the directrix above Q, then the velocity acquired in falling a distance h under gravity $=\sqrt{(2gh)}$.

$\therefore\quad v_1=\sqrt{(2ag/3)}=\sqrt{(2gh)}$

or $\quad h=a/3$ *i.e.*, the height of the directrix above Q is $a/3$.

Hence the directrix is the horizontal line through the highest point of the circle.

Let QM be the perpendicular from Q on the directrix and QN the tangent at Q. If S is the focus of the parabola subsequently described, we have by the geometrical properties of a parabola

$$QS=QM=a/3$$

and $\quad \angle SQN=\angle NQM.$

This gives the position of the focus S of the parabola.

Ex. 24. *A heavy particle is allowed to slide down a smooth vertical circle of radius $27a$ from rest at the highest point. Show that on leaving the circle it moves in a parabola of latus rectum $16a$.*

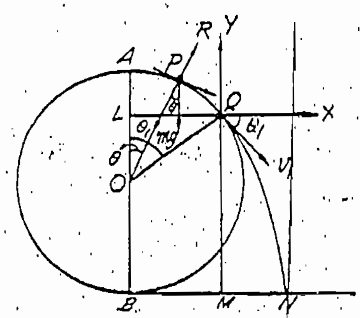
[Lucknow 1975; Kanpur 78, 80, 86]

Sol. Let us take the radius of the circle equal to b so that $b=27a$. Now proceed as in Ex. 23. We get

the latus rectum $=\frac{16b}{27}=\frac{16}{27}(27a)=16a.$

Ex. 25. *A particle slides down the arc of a smooth vertical circle of radius a, being slightly displaced from rest at the highest point. Find where it will leave the circle and prove that it will strike a horizontal plane through the lowest point of the circle at a distance $\frac{5}{27}(\sqrt{5}+4\sqrt{2})a$ from the vertical diameter.*

Sol. Proceeding as in § 5, the particle leaves the circle at the point Q where $\angle AOQ=\theta_1$ and $\cos\theta_1=2/3$. The velocity v_1 of the particle at the point Q is $\sqrt{(2ag/3)}$ and is along the tangent to the circle at the point Q. After leaving the circle at the point Q the motion of the particle is that of a projectile and so it describes a parabolic path with the velocity of projection $v_1=\sqrt{(2ag/3)}$ making an angle $\theta_1=\cos^{-1}(2/3)$ below the horizontal line through Q.



Now the equation of the parabolic path of the particle w.r.t. the horizontal and vertical lines OX and OY as the coordinate axes is

$$y=x\tan(-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(-\theta_1)}\quad [\because \text{ for the motion of the projectile, the angle of projection}=-\theta_1]$$

or $\quad y=-x\tan\theta_1-\frac{gx^2}{2v_1^2\cos^2\theta_1}$

or $\quad y=-x\frac{\sqrt{5}}{2}-\frac{gx^2}{2\cdot\frac{2}{3}ag\cdot\frac{4}{9}}\quad [\because \cos\theta_1=\frac{2}{3}$ gives $\sin\theta_1=\sqrt{(1-\frac{4}{9})}=\sqrt{5}/3$ and $\tan\theta_1=\sqrt{5}/2]$

or $\quad y=-x\frac{\sqrt{5}}{2}-\frac{27}{16a}x^2.\quad \ldots(1)$

Let the particle strike the horizontal plane through the lowest point B at N. If (x_1, y_1) are the coordinates of the point N, then

$$x_1=MN \text{ and } y_1=-QM=-LB=-(LO+OB)$$
$$=-(a\cos\theta_1+a)=-(\tfrac{2}{3}a+a)=-5a/3.$$

The point $N(x_1, y_1)$ lies on the trajectory (1).

$\therefore\quad y_1=-\frac{\sqrt{5}}{2}x_1-\frac{27}{16a}x_1^2.$

or $\qquad -\dfrac{5a}{3}=-\dfrac{\sqrt{5}}{2}x_1-\dfrac{27}{16a}x_1^2$

or $\qquad 81x_1^2+24\sqrt{5}ax_1-80a^2=0.$

$$\therefore \quad x_1=\frac{-24\sqrt{5}a\pm\sqrt{(24\times24\times5a^2+4\times81\times80a^2)}}{2\times81}$$

$$=\frac{-24\sqrt{5}a+120\sqrt{2}a}{162} \quad \text{(leaving the $-$ive sign, since x_1 cannot be negative)}$$

or $\qquad x_1=MN=\dfrac{(-4\sqrt{5}+20\sqrt{2})a}{27}$

$\therefore$ the required distance

$=BN=BM+MN=LQ+MN=a\sin\theta_1+MN$

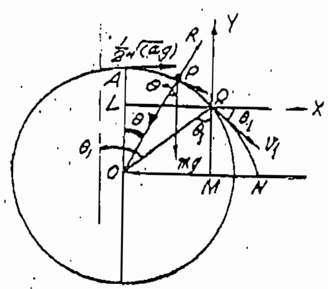
$=a.\dfrac{\sqrt{5}}{3}+\dfrac{(-4\sqrt{5}+20\sqrt{2})a}{27} \qquad [\because \sin\theta_1=\sqrt{5}/3]$

$=\dfrac{5(\sqrt{5}+4\sqrt{2})a}{27}.$

Ex. 26. *A body is projected along the arc of a smooth circle of radius a and from the highest point with velocity* $\frac{1}{2}\sqrt{(ag)}$; *find where it will leave the circle and prove that it will strike a horizontal plane through the centre of the circle at a distance from the centre*

$$\frac{1}{64}\left[9\sqrt{(39)}+7\sqrt{7}\right]a.$$

Sol. Let a body be projected along the outside of a smooth vertical circle of radius a from the highest point A with velocity $\frac{1}{2}\sqrt{(ag)}$. If P is the position of the body at any time t, then the equations of motion of the body are



$$m\frac{d^2s}{dt^2}=mg\sin\theta, \qquad \ldots(1)$$

and $\qquad m\dfrac{v^2}{a}=mg\cos\theta-R. \qquad \ldots(2)$

Also $\qquad s=a\theta. \qquad \ldots(3)$

$\therefore$ From (1) and (3), we have

$$a\frac{d^2\theta}{dt^2}=g\sin\theta.$$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=\frac{1}{2}\sqrt{(ag)}$.

$\therefore \quad \frac{1}{4}ag=-2ag+A$ or $A=\frac{1}{4}ag+2ag=\frac{9}{4}ag.$

$\therefore \quad v^2=\frac{9}{4}ag-2ag\cos\theta=ag\left(\frac{9}{4}-2\cos\theta\right). \qquad \ldots(4)$

From (2) and (4), we have

$R=\dfrac{m}{a}(ag\cos\theta-v^2)=\dfrac{m}{a}(3ag\cos\theta-\frac{9}{4}ag)$

$=3mg(\cos\theta-\frac{3}{4}). \qquad \ldots(5)$

Suppose the body leaves the circle at the point Q, where $\theta=\theta_1$. Then putting $R=0$ and $\theta=\theta_1$ in (5), we have

$0=3mg(\cos\theta_1-\frac{3}{4})$ or $\cos\theta_1=\frac{3}{4}.$

If v_1 is the velocity of the body at Q, then from (4)

$v_1^2=ag\left(\frac{9}{4}-2\cos\theta_1\right)=ag\left(\frac{9}{4}-\frac{3}{2}\right)=\frac{3}{4}ag.$

Hence the body leaves the circle at the point Q with velocity $v_1=\frac{1}{2}\sqrt{(3ag)}$ at an angle $\theta_1=\cos^{-1}\left(\frac{3}{4}\right)$ below the horizontal line through Q and subsequently it describes a parabolic path. The equation of the parabolic trajectory of the body w.r.t. the horizontal and vertical lines QX and QY through Q as the coordinate axes is

$$y=x\tan(-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(-\theta_1)}$$

or $\qquad y=-x\tan\theta_1-\dfrac{gx^2}{2v_1^2\cos^2\theta_1}$

or $\qquad y=-x\dfrac{\sqrt{7}}{3}-\dfrac{gx^2}{2.\frac{3}{4}ag.\frac{9}{16}} \qquad [\because \cos\theta_1=\frac{3}{4}$ gives $\sin\theta_1=\sqrt{(1-\frac{9}{16})}=\sqrt{7}/4$ and $\tan\theta_1=\sqrt{7}/3]$

or $\qquad y=-\dfrac{\sqrt{7}}{3}x-\dfrac{32}{27a}x^2. \qquad \ldots(6)$

Let the particle strike the horizontal plane through the centre O at N. If (x_1, y_1) are the coordinates of the point N, then $x_1=MN$ and $y_1=-QM=-LO=-a\cos\theta_1=-\frac{3}{4}a.$

The point $N(x_1, y_1)$ lies on the trajectory (6).

$\therefore \quad y_1=-\dfrac{\sqrt{7}}{3}x_1-\dfrac{32}{27a}x_1^2$

or $\qquad -\dfrac{3a}{4}=-\dfrac{\sqrt{7}}{3}x_1-\dfrac{32}{27a}x_1^2$

or $\qquad 128x_1^2+36\sqrt{7}ax_1-81a^2=0.$

$$\therefore \quad x_1=\frac{-36\sqrt{7}a\pm\sqrt{(36\times36\times7a^2+4\times128\times81a^2)}}{2\times128}$$

$$=\frac{-36\sqrt{7}a+36\sqrt{(39)}a}{256}$$ [neglecting the $-$ive sign because x_1 cannot be negative]

or $$x_1=MN=\frac{9(\sqrt{39}-\sqrt{7})a}{64}.$$

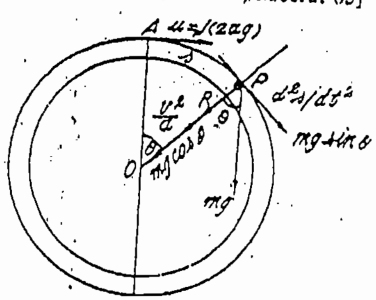
$\therefore$ the required distance $=ON=OM+MN=LQ+MN$

$$=a\sin\theta_1+MN$$

$$=\frac{\sqrt{7}a}{4}+\frac{9(\sqrt{39}-\sqrt{7})a}{64}=\frac{1}{64}\left[9\sqrt{39}+7\sqrt{7}\right]a.$$

Ex. 27. *A heavy particle slides under gravity down the inside of a smooth vertical tube held in a vertical plane. It starts from the highest point with velocity $\sqrt{(2ag)}$ where a is the radius of the circle. Prove that when in the subsequent motion the vertical component of the acceleration is maximum, the pressure on the curve is equal to twice the weight of the particle.*

[Gorakhpur 1978; Meerut 85]



Sol. Let P be the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$. The forces acting on the particle at P are (i) weight mg acting vertically downwards and (ii) the reaction R along PO.

$\therefore$ the equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=mg\sin\theta, \quad \ldots(1)$$

and $$m\frac{v^2}{a}=R+mg\cos\theta. \quad \ldots(2)$$

Also $$s=a\theta. \quad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta$.

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A; $\theta=0$ and $v=\sqrt{(2ag)}$.

$\therefore$ $A=2ag+2ag=4ag$.

$\therefore$ $v^2=4ag-2ag\cos\theta.$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(v^2-ag\cos\theta)$$

or $$R=mg(4-3\cos\theta). \quad \ldots(5)$$

Now $\dfrac{d^2s}{dt^2}$ and $\dfrac{v^2}{a}$ are the accelerations at the point P along the tangent and inward drawn-normal at P. Let f be the vertical component of acceleration at P. Then

$$f=\frac{d^2s}{dt^2}\sin\theta+\frac{v^2}{a}\cos\theta.$$

Substituting from (1) and (4), we have

$$f=g\sin\theta.\sin\theta+\frac{1}{a}(4ag-2ag\cos\theta)\cos\theta$$

$$=g(\sin^2\theta+4\cos\theta-2\cos^2\theta).$$

$$\therefore \quad \frac{df}{d\theta}=g(2\sin\theta\cos\theta-4\sin\theta+4\cos\theta\sin\theta)$$

$$=2g\sin\theta(3\cos\theta-2)$$

and $$\frac{d^2f}{d\theta^2}=g[6(\cos^2\theta-\sin^2\theta)-4\cos\theta]$$

$$=g[6(2\cos^2\theta-1)-4\cos\theta].$$

For a maximum or a minimum of f, we have

$df/d\theta=0$ i.e., $2g\sin\theta(3\cos\theta-2)=0$,

$\therefore$ either $\sin\theta=0$ giving $\theta=0$

or $3\cos\theta-2=0$ giving $\cos\theta=\frac{2}{3}$.

But $\theta=0$ corresponds to the initial position A.

When $\cos\theta=\frac{2}{3}$, $\dfrac{d^2f}{d\theta^2}=g[6(2.\frac{4}{9}-1)-4.\frac{2}{3}]=-\frac{10}{3}g=-$ive.

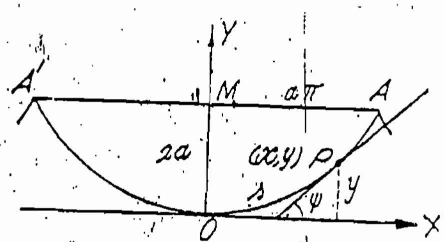
$\therefore$ f is maximum when $\cos\theta=\frac{2}{3}$.

Putting $\cos\theta=2/3$ in (5) the pressure on the curve is given by

$$R=mg(4-3.\tfrac{2}{3})=2mg=2.\text{(weight of the particle)}.$$

Cycloidal Motion

§ 6. Cycloid. *A cycloid is a curve which is traced out by a point on the circumference of a circle as the circle rolls along a fixed straight line.*



In the adjoining figure we have shown an inverted cycloid. The point O is called the vertex of the cycloid. The points A and A' are the cusps and straight line OY is the axis of the cycloid. The line AA' is called the base of the cycloid.

Let $P(x, y)$ be the coordinates of a point on the cycloid w.r.t. OX and OY as coordinate axes and ψ the angle which the tangent at P makes with OX. Then remember the following results:

(i) Parametric equations of the cycloid are given by
$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta);$$
where θ is the parameter and we have $\theta=2\psi$.

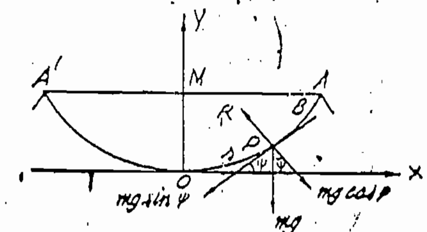
(ii) The intrinsic equation of cycloid is
$$s=4a\sin\psi,\text{ where arc } OP=s.$$

(iii) Arc $OA=4a$ and the height of the cycloid $=OM=2a$. At the point O, $\psi=0$ and $s=0$ while at the cusp A, $\psi=\pi/2$ and $s=4a$.

(vi) For the above cycloid, the relation between s and y is
$$s^2=8ay.$$

§ 7. Motion on a cycloid. *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex downwards. To determine the motion.* [Meerut 1974, 77, 88S; Rohilkhand 81, 88; Agra 76, 85; Kanpur 75, 76, 78; Lucknow 78; Gorakhpur 80; Allahabad 78]

Let O be the vertex of a smooth cycloid and OM its axis. Suppose a particle of mass m slides down the arc of the cycloid starting at rest from a point B where arc $OB=b$. Let P be the position of the particle at any time t where arc $OP=s$ and ψ be the angle which the tangent at P to the cycloid makes with the



tangent at the vertex O. The forces acting on the particle at P are: (i) the weight mg acting vertically downwards and (ii) the normal reaction R acting along the inwards drawn normal at P. Resolving these forces along the tangent and normal at P, the tangential and normal equations of motion of P are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi, \quad \ldots(1)$$

and
$$m\frac{v^2}{\rho}=R-mg\cos\psi. \quad \ldots(2)$$

Here v is the velocity of the particle at P and is along the tangent at P.

[Note that the expression for the tangential acceleration is d^2s/dt^2 and it is positive in the direction of s increasing. In the equation (1) negative sign has been taken because $mg\sin\psi$ acts in the direction of s decreasing. Again the expression for normal acceleration is v^2/ρ and it is positive in the direction of inwards drawn normal. In the equation (2) we have taken R with +ive sign because it is in the direction of inwards drawn normal while negative sign has been fixed before $mg\cos\psi$ because it is in the direction of outwards drawn normal].

Now the intrinsic equation of the cycloid is
$$s=4a\sin\psi. \quad \ldots(3)$$

From (1) and (3), we have
$$\frac{d^2s}{dt^2}=-\frac{g}{4a}s, \quad \ldots(4)$$

which is the equation of a simple harmonic motion with centre at the points $s=0$ *i.e.*, at the point O. Thus the particle will oscillate in S.H.M. about the centre O. The time period T of this S.H.M. is given by
$$T=\frac{2\pi}{\sqrt{(g/4a)}}=\pi\sqrt{(a/g)},$$

which is independent of the amplitude (*i.e.*, the initial displace-

ment b). *Thus from whatever point the particle may be allowed to slide down the arc of a smooth cycloid, the time period remains the same.* Such a motion is called *isochronous* motion.

[Meerut 1977; Rohilkhand 88; Agra 80]

Multiplying both sides of (4) by 2 (ds/dt) and then integrating w.r.t. 't', we get

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the point B, $s=b$ and $v=0$.

Therefore $0=-(g/4a)\,b^2+A$ or $A=(g/4a)\,b^2$.

$$\therefore \quad v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{4a}b^2=\frac{g}{4a}(b^2-s^2), \qquad \ldots(5)$$

which gives us the velocity of the particle at any position 's'. Substituting the value of v^2 in (2), we get R which gives us the pressure at any point on the cycloid.

Taking square root of (5), we get

$$\frac{ds}{dt}=-\sqrt{\left(\frac{g}{4a}\right)}\sqrt{(b^2-s^2)},$$

where the $-$ive sign has been taken because the particle is moving in the direction of s decreasing.

Separating the variables, we get

$$-\frac{ds}{\sqrt{(b^2-s^2)}}=\sqrt{\left(\frac{g}{4a}\right)}\,dt. \qquad \ldots(6)$$

Integrating, we have

$$\cos^{-1}(s/b)=\sqrt{(g/4a)}\,t+C.$$

But initially at B, $s=b$ and $t=0$. Therefore $\cos^{-1}1=0+C$

or $C=0$.

$\therefore \quad \cos^{-1}(s/b)=\sqrt{(g/4a)}\,t,$

or $\quad s=b\cos\sqrt{(g/4a)}\,t,$

which gives a relation between s and t.

If t_1 be the time from B to O, then integrating (6) from B to O, we have

$$-\int_b^0\frac{ds}{\sqrt{(b^2-s^2)}}=\int_0^{t_1}\sqrt{\left(\frac{g}{4a}\right)}\,dt$$

[Note that at B, $s=b$ and $t=0$ while at O, $s=0$ and $t=t_1$]

or $\quad \left[\cos^{-1}\frac{s}{b}\right]_b^0=\sqrt{\left(\frac{g}{4a}\right)}\left[t\right]_0^{t_1}$

or $\quad \cos^{-1}0-\cos^{-1}1=\sqrt{\left(\frac{g}{4a}\right)}\,t_1$

or $\quad \frac{\pi}{2}=\sqrt{\left(\frac{g}{4a}\right)}\,t_1$

or $\quad t_1=\pi\sqrt{(a/g)}.$

Thus time t_1 is independent of the initial displacement b of the particle. *Thus on a smooth cycloid the time of descent to the vertex is independent of the initial displacement of the particle.*

If T is time period of the particle *i.e.*, if T is the time for one complete oscillation, we have

$$T=4\times\text{time from } B \text{ to } O=4t_1=4\pi\sqrt{(a/g)}.$$

Illustrative Examples

Ex. 28. *A particle slides down a smooth cycloid whose axis is vertical and vertex downwards, strating from rest at the cusp. Find the velocity of the particle and the reaction on it at any point of the cycloid.*

[Meerut 1975, 79]

Sol. Refer figure of § 7, on page 201.

Here the particle starts at rest from the cusp A.

The equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \qquad \ldots(1)$$

and $\quad m\frac{v^2}{\rho}=R-mg\cos\psi. \qquad \ldots(2)$

For the cycloid, $s=4a\sin\psi$. $\qquad \ldots(3)$

From (1) and (3), we have

$$\frac{d^2s}{dt^2}=-\frac{g}{4a}s.$$

Multiplying both sides by $2\frac{ds}{dt}$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the cusp A, $s=4a$ and $v=0$.

$\therefore \quad A=\frac{g}{4a}\cdot(4a)^2=4ag.$

$$\therefore \quad v^2=-\frac{g}{4a}s^2+4ag=-\frac{g}{4a}(4a\sin\psi)^2+4ag$$

$$=4ag(1-\sin^2\psi)$$

or $\quad v^2=4ag\cos^2\psi. \qquad \ldots(4)$

Differentiating (3), $\rho=ds/d\psi=4a\cos\psi$.

Substituting for v^2 and ρ in (2), we have

$$R=m\frac{v^2}{\rho}+mg\cos\psi=m\cdot\frac{4ag\cos^2\psi}{4a\cos\psi}+mg\cos\psi$$

or $\quad R=2mg\cos\psi. \qquad \ldots(5)$

The equations (4) and (5) give the velocity and the reaction at any point of the cycloid.

Ex. 29. *A particle oscillates from cusp to cusp of a smooth cycloid whose axis is vertical and vertex lowest. Show that the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P, i.e., $v = V\cos\psi$.*

[Meerut 1975, 81, 82P, 82S; Rohilkhand 78; 86; Allahabad 78]

Sol. Proceed as in Ex. 28.

The velocity v of the particle at any point P of the cycloid is given by $\quad v = 2\sqrt{(ag)}\cos\psi.$ [From equation (4)]

If V is the velocity of the particle at the vertex, where $\psi = 0$, then $\quad V = 2\sqrt{(ag)}\cos 0 = 2\sqrt{(ag)}.$

$\therefore \quad v = V\cos\psi =$ the resolved part of V along the tangent at P. Hence the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P.

Ex. 30. *A heavy particle slides down a smooth cycloid starting from rest at the cusp, the axis being vertical and vertex downwards, prove that the magnitude of the acceleration is equal to g at every point of the path and the pressure when the particle arrives at the vertex is equal to twice the weight of the particle.*

[Meerut 1974, 75, 84S, 87, 90S; Agra 85, 87, 88; Lucknow 77; Gorakhpur 76; Kanpur 79, 86]

Sol. Refer figure of § 7 on page 201.

Here the particle starts at rest from the cusp A.

The equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi, \qquad \ldots(1)$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi. \qquad \ldots(2)$$

For the cycloid, $\quad s = 4a\sin\psi. \qquad \ldots(3)$

From (1) and (3), we have $\dfrac{d^2s}{dt^2} = -\dfrac{g}{4a}s.$

Multiplying both sides by $2\,(ds/dt)$ and integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A.$$

But initially at the cusp A, $s = 4a$ and $v = 0$. $\therefore \quad A = 4ag.$

$$\therefore \quad v^2 = -\frac{g}{4a}s^2 + 4ag = -\frac{g}{4a}(4a\sin\psi)^2 + 4ag = 4ag\,(1 - \sin^2\psi)$$

or $$v^2 = 4ag\cos^2\psi. \qquad \ldots(4)$$

Differentiating (3),

$$\rho = ds/d\psi = 4a\cos\psi.$$

Now at the point P, tangential acceleration

$$= d^2s/dt^2 = -g\sin\psi \qquad \text{[from (1)]}$$

and normal acceleration $= \dfrac{v^2}{\rho} = \dfrac{4ag\cos^2\psi}{4a\cos\psi} = g\cos\psi.$

$\therefore$ the resultant acceleration at any point P

$$= \sqrt{[(\text{tang. accel.})^2 + (\text{normal accel.})^2]}$$
$$= \sqrt{[(-g\sin\psi)^2 + (g\cos\psi)^2]} = g.$$

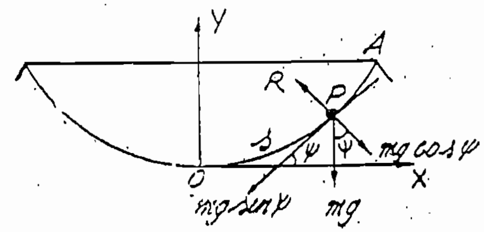
From (2) and (4), we have

$$R = m\cdot\frac{4ag\cos^2\psi}{4a\cos\psi} + mg\cos\psi = 2mg\cos\psi. \qquad \ldots(5)$$

At the vertex O, $\psi = 0$. Therefore putting $\psi = 0$ in (5), the pressure at the vertex $= 2mg =$ twice the weight of the particle.

Ex. 31. *Prove that for a particle, sliding down the arc and starting from the cusp of a smooth cycloid whose vertex is lowest, the vertical velocity is maximum when it has described half the vertical height.*

[Meerut 1972, 88, 90; Agra 78; Kanpur 80, 85, 87; Allahabad 87]



Sol. Let a particle of mass m slide down the arc of a cycloid starting at rest from the cusp A. If P is the position of the particle at any time t, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \qquad \ldots(1)$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi. \qquad \ldots(2)$$

For the cycloid, $\quad s = 4a\sin\psi. \qquad \ldots(3)$

From (1) and (3), we have $\dfrac{d^2s}{dt^2} = -\dfrac{g}{4a}s.$

Multiplying both sides by 2 (ds/dt) and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the cusp A, $s=4a$ and $v=0$. $\therefore$ $A=4ag$.

$$\therefore\ v^2=4ag-\frac{g}{4a}s^2=4ag-\frac{g}{4a}(4a\sin\psi)^2=4ag(1-\sin^2\psi)$$
$$=4ag\cos^2\psi$$

or $v=2\sqrt{(ag)}\cos\psi$, giving the velocity of the particle at the point P its direction being along the tangent at P. Let V be the vertical component of the velocity v at the point P. Then

$$V=v\cos(90°-\psi)=v\sin\psi=2\sqrt{(ag)}\cos\psi.\sin\psi$$

or $V=\sqrt{(ag)}\sin 2\psi$,

which is maximum when $\sin 2\psi=1$ i.e., $2\psi=\pi/2$ i.e., $\psi=\pi/4$.

When $\psi=\pi/4$, $s=4a\sin(\pi/4)=2\sqrt{2}a$.

Putting $s=2\sqrt{2}a$ in the relation $s^2=8ay$, we have

$$(2\sqrt{2}a)^2=8ay \quad\text{or}\quad y=8a^2/8a=a.$$

Thus at the point where the vertical velocity is maximum, we have $y=a$. The vertical depth fallen upto this point $=$ (the y-coordinate of A) $-a=2a-a=a=\frac{1}{2}(2a)$ $=$ half the vertical height of the cycloid.

Ex. 32. *A particle oscillates in a cycloid under gravity, the amplitude of the motion being b, and period being T. Show that its velocity at any time t measured from a position of rest is*

$$\frac{2\pi b}{T}\sin\left(\frac{2\pi t}{T}\right).$$

[Meerut 1977]

Sol. Refer § 7 on page 200.

The equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \qquad ...(1)$$

and

$$m\frac{v^2}{\rho}=R-mg\cos\psi. \qquad ...(2)$$

For the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$, ...(4)

which represents a S. H. M.

$\therefore$ the time period T of the particle is given by $T=2\pi/\sqrt{(g/4a)}$

or $T=4\pi\sqrt{(a/g)}$. ...(5)

Multiplying both sides of (4) by $2\dfrac{ds}{dt}$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A. \qquad ...(6)$$

But the amplitude of the motion is b. So the arcual distance of a position of rest from the vertex O is b i.e., $v=0$ when $s=b$.

$\therefore$ from (6), we have

$$A=\frac{g}{4a}b^2.$$

Substituting this value of A in (6), we have

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}(b^2-s^2). \qquad ...(7)$$

$$\therefore\ \frac{ds}{dt}=-\frac{1}{2}\sqrt{\left(\frac{g}{a}\right)}\sqrt{(b^2-s^2)}$$

($-$ive sign is taken because the particle is moving in the direction of s decreasing)

or $$dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(b^2-s^2)}}$$

Integrating, $t=2\sqrt{(a/g)}.\cos^{-1}(s/b)+B$.

But $t=0$ when $s=b$. $\therefore$ $B=0$.

$\therefore$ $t=2\sqrt{(a/g)}\cos^{-1}(s/b)$

or $$s=b\cos\left\{\frac{t}{2}\sqrt{\left(\frac{g}{a}\right)}\right\}.$$

Substituting this value of s in (7), we have

$$v^2=\frac{g}{4a}\left[b^2-b^2\cos^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}\right]$$
$$=\frac{g}{4a}b^2\sin^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}$$

or $$v=\frac{b}{2}\sqrt{(g/a)}\sin\left\{\frac{t}{2}\sqrt{(g/a)}\right\}.$$

From (5), $\sqrt{(g/a)}=\dfrac{4\pi}{T}$.

$\therefore$ the velocity of the particle at any time t measured from the position of rest is given by

$$v=\frac{b}{2}\cdot\frac{4\pi}{T}\sin\left(\frac{t}{2}\cdot\frac{4\pi}{T}\right)=\left(\frac{2\pi b}{T}\right)\sin\left(\frac{2\pi t}{T}\right).$$

Ex. 33. *A particle starts from rest at the cusp of a smooth cycloid whose axis is vertical and vertex downwards. Prove that*

when it has fallen through half the distance measured along the arc to the vertex, two-thirds of the time of descent will have elapsed.

[Meerut 1976, 83P; Rohilkhand 78; Agra 77, 79; Gorakhpur 77, 79, 81; Kanpur 88]

Sol. Refer figure of § 7 on page 201.

Let a particle of mass m start from rest from the cusp A of the cycloid. If P is the position of the particle after time t such that arc $OP=s$, the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi, \quad \ldots(1)$$

and

$$m\frac{v^2}{\rho}=R-mg\cos\psi. \quad \ldots(2)$$

For the cycloid, $s=4a\sin\psi$. $\quad\ldots(3)$

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and then integrating, we have

$$\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

Initially at the cusp A, $s=4a$ and $\dfrac{ds}{dt}=0$.

$$\therefore\ A=\frac{g}{4a}\cdot(4a)^2=4ag.$$

$$\therefore\ \left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+4ag=\frac{g}{4a}(16a^2-s^2) \quad\ldots(4)$$

or $\quad ds/dt=-\frac{1}{2}\sqrt{(g/a)}\cdot\sqrt{(16a^2-s^2)}$,

the $-$ive sign is taken because the particle is moving in the direction of s decreasing.

Separating the variables, we have

$$dt=-2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{(16a^2-s^2)}}. \quad\ldots(5)$$

If t_1 is the time from the cusp A (*i.e.*, $s=4a$) to the vertex O (*i.e.*, $s=0$), then integrating (5),

$$t_1=-2\sqrt{(a/g)}\int_{4a}^{0}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{4a}\right]_{4a}^{0}=2\sqrt{(a/g)}\,\frac{\pi}{2}=\pi\sqrt{(a/g)}.$$

Again if t_2 is the time taken to move from the cusp A (*i.e.* $s=4a$) to half the distance along the arc to the vertex *i.e.*, to $s=2a$, then integrating (5)

$$t_2=-2\sqrt{(a/g)}\int_{s=4a}^{2a}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}\cdot\left[\cos^{-1}\frac{s}{4a}\right]_{4a}^{2a}$$

$$=2\sqrt{(a/g)}\,[\cos^{-1}\tfrac{1}{2}-\cos^{-1}1]=2\sqrt{(a/g)}\cdot(\pi/3)=(2/3)\,t_1.$$

Ex. 34. *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.*

[Meerut 1976, 83, 85S, 87P, 88P; Agra 76, 78; Lucknow 78, 80; Kanpur 79, 80, 85, 87]

Sol. Let a particle start from rest from the cusp A of the cycloid. Proceeding as in the last example the velocity v of the particle at any point P, at time t, is given by

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}(16a^2-s^2). \quad \text{[Refer equation (4) of the last example]}$$

or $\dfrac{ds}{dt}=-\frac{1}{2}(g/a)\sqrt{(16a^2-s^2)}$, the $-$ive sign is taken because the particle is moving in the direction of s decreasing.

$$\therefore\ dt=-2\sqrt{(a/g)}\,\frac{ds}{\sqrt{(16a^2-s^2)}}. \quad\ldots(1)$$

The vertical height of the cycloid is $2a$. At the point where the particle has fallen down the first half of the vertical height of the cycloid, we have $y=a$. Putting $y=a$ in the equation $s^2=8ay$, we get $s^2=8a^2$ or $s=2\sqrt{2}a$.

$\therefore$ integrating (1) from $s=4a$ to $s=2\sqrt{2}a$, the time t_1 taken in falling down the first half of the vertical height of the cycloid is given by

$$t_1=-2\sqrt{(a/g)}\int_{s=4a}^{2\sqrt{2}a}\frac{ds}{\sqrt{(16a^2-s^2)}}=2\sqrt{(a/g)}\left[\cos^{-1}(s/4a)\right]_{4a}^{2\sqrt{2}a}$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\frac{2\sqrt{2}a}{4a}-\cos^{-1}1\right]=2\sqrt{(a/g)}\left[\cos^{-1}\frac{1}{\sqrt{2}}-\cos^{-1}1\right]$$

$$=2\sqrt{(a/g)}\,[\tfrac{1}{4}\pi-0]=\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

Again integrating (1) from $s=2\sqrt{2}a$ to $s=0$, the time t_2 taken in falling down the second half of the vertical height of the cycloid is given by

$$t_2=-2\sqrt{(a/g)}\int_{s=2\sqrt{2}a}^{0}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}\cdot\left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2}a}^{0}=2\sqrt{(a/g)}\left[\cos^{-1}0-\cos^{-1}\frac{1}{\sqrt{2}}\right]$$

$$=2\sqrt{(a/g)}\,[\tfrac{1}{2}\pi-\tfrac{1}{4}\pi]=\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

Hence $t_1=t_2$ i.e., the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Ex. 35. *A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{(a/g)}\tan^{-1}[\sqrt{(4ag)}/V]$.*

[Meerut 1971, 78, 81, 84, 85, 90P; Rohilkhand 80; Gorakhpur 76; Allahabad 76; Agra 86]

Sol. Refer figure of § 7 on page 201.

Let a particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc. If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP=s$, then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \qquad \ldots(1)$$

and

$$m\frac{v^2}{\rho}=R-mg\cos\psi. \qquad \ldots(2)$$

For the cycloid, $s=4a\sin\psi$. $\ldots(3)$

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the cusp A, $s=4a$ and $(ds/dt)^2=V^2$.

$\therefore$ $V^2=-(g/4a)\cdot 16a^2+A$ or $A=V^2+4ag$.

$$\therefore\quad v^2=\left(\frac{ds}{dt}\right)^2=V^2+4ag-\frac{g}{4a}s^2=\left(\frac{g}{4a}\right)\left[\frac{4a}{g}(V^2+4ag)-s^2\right]$$

or

$$\frac{ds}{dt}=-\tfrac{1}{2}\sqrt{(g/a)}\sqrt{\left[\frac{4a}{g}(V^2+4ag)-s^2\right]}$$

($-$ive sign is taken because the particle is moving in the direction of s decreasing)

or

$$dt=-2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}.$$

Integrating, the time t_1 from the cusp A to the vertex O is given by

$$t_1=-2\sqrt{(a/g)}\int_{s=4a}^{0}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$=2\sqrt{(a/g)}\int_0^{4a}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$=2\sqrt{(a/g)}\left[\sin^{-1}\frac{s}{2\sqrt{(a/g)}\sqrt{(V^2+4ag)}}\right]_0^{4a}$$

$$=2\sqrt{(a/g)}\cdot\sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}$$

$$=2\sqrt{(a/g)}\cdot\theta, \qquad \ldots(4)$$

where $\theta=\sin^{-1}\left\{\dfrac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}$.

We have $\sin\theta=\dfrac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}$.

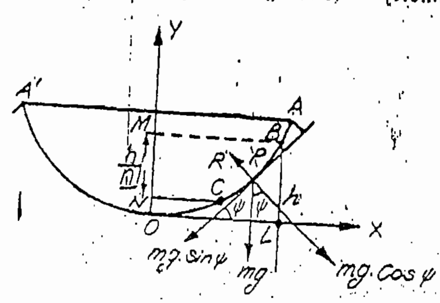
$$\therefore\quad \cos\theta=\sqrt{(1-\sin^2\theta)}=\sqrt{\left[1-\frac{4ag}{V^2+4ag}\right]}=\frac{V}{\sqrt{(V^2+4ag)}}.$$

$$\therefore\quad \tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{2\sqrt{(ag)}}{V}=\frac{\sqrt{(4ag)}}{V}$$

or $\theta=\tan^{-1}[\sqrt{(4ag)}/V]$.

$\therefore$ from (4), the time of reaching the vertex is $=2\sqrt{(a/g)}\cdot\tan^{-1}[\sqrt{(4ag)}/V]$.

Ex. 36 (a). *If a particle starts from rest at a given point of a cycloid with its axis vertical and vertex downwards, prove that it falls $1/n$ of the vertical distance to the lowest point in time $2\sqrt{(a/g)}\cdot\sin^{-1}(1/\sqrt{n})$, where a is the radius of the generating circle.* [Rohilkhand 1977]



Sol. Let a particle start from rest at a given point B of a cycloid with its axis vertical and vertex downwards. Let h be the vertical height of the point B above the vertex O.

If arc $OB = s_1$, then from $s^2 = 8ay$, we have $s_1^2 = 8ah$.

If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP = s$, then the equations of motion along the tangent and normal at P are

$$m \frac{d^2 s}{dt^2} = -mg \sin \psi \qquad \ldots(1)$$

and
$$m \frac{v^2}{\rho} = R - mg \cos \psi. \qquad \ldots(2)$$

For the cycloid, $s = 4a \sin \psi$. $\qquad \ldots(3)$

From (1) and (3), we have $\dfrac{d^2 s}{dt^2} = -\dfrac{g}{4a} s$.

Multiplying both sides by $2\,(ds/dt)$ and integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a} s^2 + A.$$

But at the point B, $s = s_1$ and $v = 0$.

$$\therefore \quad 0 = -\frac{g}{4a} s_1^2 + A \quad \text{or} \quad A = \frac{g}{4a} s_1^2.$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a} s^2 + \frac{g}{4a} s_1^2 = \frac{g}{4a}(s_1^2 - s^2)$$

or $\quad ds/dt = -\tfrac{1}{2}\sqrt{(g/a)} \cdot \sqrt{(s_1^2 - s^2)}$.

(negative sign is taken since the particle is moving in the direction of s decreasing)

or, $\quad dt = -2\sqrt{(a/g)} \dfrac{ds}{\sqrt{(s_1^2 - s^2)}}$. $\qquad \ldots(4)$

Integrating, we have

$$t = 2\sqrt{(a/g)} \cos^{-1}(s/s_1) + A.$$

But at the point B, $s = s_1$ and $t = 0$.

$\therefore \quad 0 = 2\sqrt{(a/g)} \cos^{-1} 1 + A$ or $A = 0$. $\qquad [\because \cos^{-1} 1 = 0]$

$\therefore \quad t = 2\sqrt{(a/g)} \cos^{-1}(s/s_1)$

$$= 2\sqrt{\left(\frac{a}{g}\right)} \cos^{-1}\left[\frac{\sqrt{(8ay)}}{\sqrt{(8ah)}}\right] \qquad [\because s^2 = 8ay \text{ and } s_1^2 = 8ah]$$

$$= 2\sqrt{(a/g)} \cos^{-1} \sqrt{(y/h)}. \qquad \ldots(5)$$

Let C be the point at a vertical depth h/n below the point B. Then the height of C above $O = ON = h - (h/n) = h(1 - 1/n)$. Thus for the point C, we have $y = h(1 - 1/n)$:

If t_1 be the time taken by the particle from B to C, then putting $t = t_1$ and $y = h(1 - 1/n)$ in (5), we get

$$t_1 = 2\sqrt{(a/g)} \cos^{-1}\sqrt{[\{h(1 - 1/n)\}/h]} = 2\sqrt{(a/g)} \cos^{-1}\sqrt{(1 - 1/n)}$$
$$= 2\sqrt{(a/g)} \sin^{-1}\sqrt{\{1 - (1 - 1/n)\}} \qquad [\because \cos^{-1} x = \sin^{-1}\sqrt{(1 - x^2)}]$$
$$= 2\sqrt{(a/g)} \sin^{-1}(1/\sqrt{n}).$$

Ex. 36 (b). *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting from rest at a given point of the cycloid. Prove that the time occupied in falling down the first half of the vertical height to the lowest point is equal to the time of falling down the second half.*

Sol. Proceed as in Ex. 36 (a) by taking $n = 2$.

Thus here if C be the point at a vertical depth $h/2$ below the point B, then at C, we have $y = h/2$. If t_1 be the time taken by the particle from B to C, then putting $t = t_1$ and $y = h/2$ in the result (5) of Ex. 36 (a), we get

$$t_1 = 2\sqrt{(a/g)} \cos^{-1}\sqrt{(\tfrac{1}{2}h/h)} = 2\sqrt{(a/g)} \cos^{-1}(1/\sqrt{2})$$
$$= 2\sqrt{(a/g)} \cdot \tfrac{1}{4}\pi = \tfrac{1}{2}\pi\sqrt{(a/g)}.$$

Again if t_2 be the time taken by the particle from B to O, then putting $t = t_2$ and $y = 0$ in (5), we get

$$t_2 = 2\sqrt{(a/g)} \cos^{-1} 0 = 2\sqrt{(a/g)} \cdot \tfrac{1}{2}\pi = \pi\sqrt{(a/g)}.$$

Since $t_2 = 2t_1$, therefore the time from B to C is equal to the time from C to O.

Ex. 37. *Two particles are let drop from the cusp of a cycloid down the curve at an interval of time t; prove that they will meet in time $2\pi\sqrt{(a/g)} + (t/2)$.* [Kanpur 1981, 83; Rohilkhand 79; Lucknow 79; Gorakhpur 81; Meerut 85P]

Sol. Refer the figure of § 7 on page 201.

Suppose a particle starts at rest from the cusp A. At any time T, the equation of motion of the particle along the tangent is given by

$$m \frac{d^2 s}{dT^2} = -mg \sin \psi.$$

For the cycloid, $\quad s = 4a \sin \psi$.

$$\therefore \quad \frac{d^2 s}{dT^2} = -\frac{g}{4a} s.$$

Multiplying both sides by $2\,(ds/dT)$ and integrating, we have

$$v^2 = \left(\frac{ds}{dT}\right)^2 = -\frac{g}{4a} s^2 + A.$$

The particle is dropped from the cusp. Therefore $v = 0$ when $s = 4a$.

$$\therefore \quad 0 = -\frac{g}{4a}(4a)^2 + A \quad \text{or} \quad A = 4ag.$$

$$\therefore \quad \left(\frac{ds}{dT}\right)^2=-\frac{g}{4a}s^2+4ag=\frac{g}{4a}(16a^2-s^2)$$

or $$ds/dT=-\tfrac{1}{2}\sqrt{(g/a)}\sqrt{(16a^2-s^2)}$$

(–ive sign is taken because the particle is moving in the direction of s decreasing)

or $$dT=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(16a^2-s^2)}}.$$

Integrating, $T=2\sqrt{(a/g)}\cos^{-1}\left(\dfrac{s}{4a}\right)+B.$

But at the cusp A, $T=0$, $s=4a$. $\quad\therefore \quad B=0$.

$$\therefore \quad T=2\sqrt{(a/g)}\cos^{-1}(s/4a)$$

or $$\cos^{-1}(s/4a)=\tfrac{1}{2}T\sqrt{(g/a)}.$$

$$\therefore \quad s=4a\cos[\tfrac{1}{2}T\sqrt{(g/a)}]. \qquad \ldots(1)$$

Thus if a particle starts at rest from the cusp A; the equation (1) gives the arcural distance (*i.e.*, distance measured along the arc) of the particle from the vertex O at any time T measured from the instant the particle starts from the cusp A.

Let the two particles meet after time t_1 measured from the instant the first particle was dropped. Since the two particles are dropped at an interval of time t, therefore the second particle will be in motion for time (t_1-t) before it meets the first particle.

Let s_1 be the distance along the arc of the first particle at time t_1 measured from the instant it starts from the cusp A and s_2 that of the second particle at time t_1-t measured from the instant it starts from the cusp A. Then from (1), we have

$$s_1=4a\cos[\tfrac{1}{2}t_1\sqrt{(g/a)}] \text{ and } s_2=4a\cos[\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}].$$

But $s_1=s_2$, being the condition for the two particles to meet.

$$\therefore \quad 4a\cos[\tfrac{1}{2}t_1\sqrt{(g/a)}]=4a\cos[\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}]$$

or $\quad \cos[\tfrac{1}{2}t_1\sqrt{(g/a)}]=\cos[\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}]$

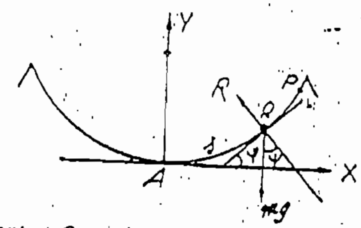
or $\quad \tfrac{1}{2}(t_1-t)\sqrt{(g/a)}=2\pi-\tfrac{1}{2}t_1\sqrt{(g/a)} \qquad [\because \cos(2\pi-\alpha)=\cos\alpha]$

or $\quad t_1\sqrt{(g/a)}=2\pi+\tfrac{1}{2}t\sqrt{(g/a)}$ or $t_1=2\pi\sqrt{(a/g)}+\tfrac{1}{2}t$.

Ex. 38. *A particle starts from rest at any point P in the arc of a smooth cycloid $s=4a\sin\psi$ whose axis is vertical and vertex A downwards; prove that the time of descent to the vertex is $\pi\sqrt{(a/g)}$.*

Show that if the particle is projected from P downwards along the curve with velocity equal to that with which it reaches A when starting from rest at P, it will now reach A in half the time taken in the preceding case. [Meerut 1973, 82, 86, Rohilkhand 85]

Sol. A particle starts from rest at any point P in the arc of a smooth cycloid whose vertex is A. Let arc $AP=b$.



Let Q be the position of the particle at any time t where arc $AQ=s$ and let ψ be the angle which the tangent at Q to the cycloid makes with the tangent at the vertex A. The tangential equation of motion of the particle at Q is

$$m\frac{d^2s}{dt^2}=-mg\sin\psi.$$

But for the cycloid, $s=4a\sin\psi$. $\qquad \ldots(1)$

$\therefore$ the equation (1) becomes $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and integrating w.r.t. 't', we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A. \qquad \ldots(2)$$

But initially at the point P, we have $s=b$ and $v=0$.

$$\therefore \quad 0=-\frac{g}{4a}b^2+A \quad \text{or} \quad A=\frac{g}{4a}b^2.$$

$$\therefore \quad v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{4a}b^2=\frac{g}{4a}(b^2-s^2). \qquad \ldots(3)$$

Taking square root of (3), we get

$$ds/dt=-\tfrac{1}{2}\sqrt{(g/a)}\sqrt{(b^2-s^2)},$$

where the –ive sign has been taken because the particle is moving in the direction of s decreasing.

$$\therefore \quad dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(b^2-s^2)}}. \qquad \ldots(4)$$

Let t_1 be the time taken by the particle to reach the vertex A where $s=0$. Then integrating (4) from P to A, we have

$$\int_0^{t_1}dt=-2\sqrt{(a/g)}\int_b^0\frac{ds}{\sqrt{(b^2-s^2)}},$$

$$\therefore \quad t_1=2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{b}\right]_b^0=2\sqrt{(a/g)}[\cos^{-1}0-\cos^{-1}1]$$

$=2\sqrt{(a/g)}[\tfrac{1}{2}\pi-0]=\pi\sqrt{(a/g)}$, which proves the first result.

If v_1 is the velocity with which the particle reaches the vertex A, then at A, $v=v_1$ and $s=0$. So from (3), we have

$$v_1^2=\frac{g}{4a}(b^2-0^2)=\frac{g}{4a}b^2.$$

Second case. Now suppose the particle starts from P with velocity v_1 where $v_1^2=(g/4a)\,b^2$. Then applying the initial condition $s=b$ and $v=v_1$ in (2), we have

$$v_1^2=-\left(\frac{g}{4a}\right)b^2+A$$

or $$A=v_1^2+\left(\frac{g}{4a}\right)b^2=\left(\frac{g}{4a}\right)b^2+\left(\frac{g}{4a}\right)b^2=\frac{g}{2a}b^2.$$

For this new value of A, (2) becomes

$$\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{2a}b^2=\frac{g}{4a}(2b^2-s^2).$$

$$\therefore \frac{ds}{dt}=-\tfrac{1}{2}\sqrt{(g/a)}\sqrt{(2b^2-s^2)}$$

or $$dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(2b^2-s^2)}}. \quad \ldots(5)$$

Let t_2 be the time taken by the particle to reach the vertex A in this case. Then integrating (5) from P to A, we have

$$\int_0^{t_2} dt=-2\sqrt{(a/g)}\int_b^0 \frac{ds}{\sqrt{(2b^2-s^2)}}$$

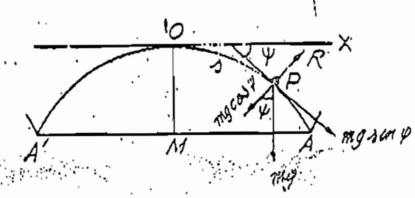
$$\therefore t_2=2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{b\sqrt{2}}\right]_b^0$$

$=2\sqrt{(a/g)}\,[\cos^{-1}0-\cos^{-1}(1/\sqrt{2})]=2\sqrt{(a/g)}\,(\tfrac{1}{2}\pi-\tfrac{1}{4}\pi)$

$=2\sqrt{(a/g)}\cdot\tfrac{1}{4}\pi=\tfrac{1}{2}\pi\sqrt{(a/g)}=\tfrac{1}{2}t_1$, which proves the second result.

§ 8. **Motion on the outside of a smooth cycloid with its axis vertical and vertex upwards.** *A particle is placed very close, to the vertex of a smooth cycloid whose axis is vertical and vertex upwards and is allowed to run down the curve, to discuss the motion.*

[Meerut 1979; Kanpur 77]



Let a particle of mass m, starting from rest at O, slide down the arc of a smooth cycloid whose axis OM is vertical and vertex O is upwards. Let P be the position of the particle at time t such that arc $OP=s$ and the tangent at P to the cycloid makes an angle ψ with the tangent at the vertex O. The forces acting on the particle at P are : (i) weight mg acting vertically downwards and (ii) the reaction R acting along the outwards drawn normal.

$\therefore$ The equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\psi \quad \ldots(1)$$

and $$m\frac{v^2}{\rho}=mg\cos\psi-R. \quad \ldots(2)$$

Also for the cycloid, $s=4a\sin\psi$. $\quad \ldots(3)$

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}s^2+A.$$

Initially at O, $s=0$ and $v=0$. $\therefore A=0$.

$$\therefore v^2=\frac{g}{4a}s^2=\frac{g}{4a}(4a\sin\psi)^2=4ag\sin^2\psi. \quad \ldots(4)$$

From (2) and (4), we have

$$R=mg\cos\psi-\frac{mv^2}{\rho}$$

$$=mg\cos\psi-\frac{m\cdot 4ag\sin^2\psi}{4a\cos\psi} \quad \left[\because \rho=\frac{ds}{d\psi}=4a\cos\psi\right]$$

$$=\frac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi). \quad \ldots(5)$$

The equation (4) gives the velocity of the particle at any position and the equation (5) gives the reaction of the cycloid on the particle at any position. The pressure of the particle on the curve is equal and opposite to the reaction of the curve on the particle.

When the particle leaves the cycloid, we have $R=0$

i.e., $\dfrac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi)=0$

i.e., $\sin^2\psi=\cos^2\psi$, i.e., $\tan^2\psi=1$

i.e., $\tan\psi=1$ i.e., $\psi=45°$.

Hence the particle will leave the curve when it is moving in a direction making an angle 45° downwards with the horizontal.

[Meerut 1983]

Illustrative Examples

Ex. 39. *If a particle starts from the vertex of a cycloid whose axis is vertical and vertex upwards, prove that its velocity at any point varies as the distance of that point from the vertex measured along the arc.* [Kanpur 1975]

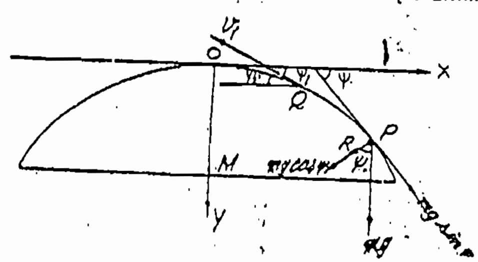
Sol. Proceed as in § 8. From the equation (4), the velocity v at any point P is given by

$$v^2=(g/4a)\, s^2,$$

$\therefore \quad v=\sqrt{(g/4a)}\,.\,s \quad$ or $\quad v\propto s.$

Hence the velocity varies as the distance measured along the arc.

Ex. 40. *A cycloid is placed with its axis vertical and vertex upwards and a heavy particle is projected from the cusp up the concave side of the curve with velocity $\sqrt{(2gh)}$; prove that the latus rectum of the parabola described after leaving the arc is $(h^2/2a)$, where a is the radius of the generating circle.* [Rohilkhand 1987]



Sol. Let a particle of mass m be projected with velocity $\sqrt{(2gh)}$ from the cusp A up the concave side of the cycloid. If P is the position of the particle after any time t such that arc $OP=s$, the equations of motion along the tangent and normal are

$$m\,(d^2s/dt^2)=mg\sin\psi, \quad \ldots(1)$$

and

$$m\,(v^2/\rho)=R+mg\cos\psi. \quad \ldots(2)$$

[Note that here the reaction R of the curve acts along the inwards drawn normal and the tangential component of mg acts in the direction of s increasing.]

For the cycloid, $\quad s=4a\sin\psi.$

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=\dfrac{g}{4a}\,s.$

Multiplying both sides by $2\,(ds/dt)$ and then integrating, we have

$$v^2=(ds/dt)^2=(g/4a)\,s^2+A.$$

Initially at A, $s=4a$ and $v=\sqrt{(2gh)}$.

$\therefore \quad A=2gh-4ag.$

$$\therefore \quad v^2=\frac{g}{4a}\,s^2+2gh-4ag=\frac{g}{4a}\,(4a\sin\psi)^2+2gh-4ag$$
$$=4ag\sin^2\psi+2gh-4ag=2gh-4ag\,(1-\sin^2\psi)$$
$$=2gh-4ag\cos^2\psi. \quad \ldots(4)$$

From (2) and (4), we have

$$R=\frac{m}{4a\cos\psi}\,(2gh-4ag\cos^2\psi)-mg\cos\psi \quad [\because\ \rho=ds/d\psi=4a\cos\psi]$$
$$=\frac{mg}{2a\cos\psi}\,(h-2a\cos^2\psi)-mg\cos\psi$$
$$=\frac{mg}{2a\cos\psi}\,[h-2a\cos^2\psi-2a\cos^2\psi]$$
$$=\frac{mg}{2a\cos\psi}\left[h-4a\cos^2\psi\right] \quad \ldots(5)$$

Suppose the particle leaves the cycloid at the point Q where $\psi=\psi_1$. Then putting $\psi=\psi_1$ and $R=0$ in (5), we have

$$h-4a\cos^2\psi_1=0$$

or

$$\cos^2\psi_1=h/4a. \quad \ldots(6)$$

If v_1 is the velocity at Q, then from (4), we have

$$v_1^2=2gh-4ag\cos^2\psi_1=2gh-4ag\,.\,(h/4a)=gh.$$

$\therefore$ the particle leaves the cycloid at the point Q with velocity $v_1=\sqrt{(gh)}$ inclined at an angle ψ_1 to the horizontal given by (6). Subsequently it describes a parabolic path.

The latus rectum of the parabolic path described after Q

$=(2/g)$ (square of the horizontal velocity at Q)

$=(2/g)\,(v_1^2\cos^2\psi_1)=(2/g)\,(gh)\,(h/4a)=h^2/2a.$

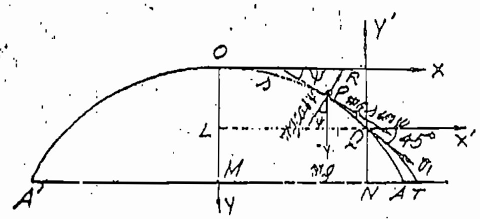
Ex. 41. *A particle is placed very near the vertex of a smooth cycloid whose axis is vertical and vertex upwards, and is allowed to run down the curve. Prove that it will leave the curve when it has fallen through half the vertical height of the cycloid.*

[Lucknow 1981; Gorakhpur 78; Allahabad 79; Meerut 77, 79, 80, 84P; Agra 86]

Also prove that the latus rectum of the parabola subsequently described is equal to the height of the cycloid. [Kanpur 1973]

Also show that it falls upon the base of the cycloid at a distance $(\frac{1}{2}\pi+\sqrt{3})\,a$ from the centre of the base, a being the radius of the generating circle. [Kanpur 1984]

Sol. Let a particle of mass m, starting from rest at O, slide down the arc of a smooth cycloid whose axis OM is vertical and vertex O is upwards. Let P be the position of the particle at any time t such that arc $OP=s$. If the tangent at P makes an angle ψ with the horizontal, then the equations of motion of the particle along the tangent and normal at P are



$$m\frac{d^2s}{dt^2}=mg\sin\psi, \qquad \ldots(1)$$

and
$$m\frac{v^2}{\rho}=mg\cos\psi-R. \qquad \ldots(2)$$

Also for the cycloid, $\quad s=4a\sin\psi. \qquad \ldots(3)$

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=\dfrac{g}{4a}s.$

Multiplying both sides by $2\,(ds/dt)$ and integrating, we have
$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}s^2+A.$$

Initially at O, $s=0$ and $v=0$. $\quad\therefore\quad A=0.$

$$\therefore\quad v^2=\frac{g}{4a}s^2=\frac{g}{4a}(4a\sin\psi)^2=4ag\sin^2\psi. \qquad \ldots(4)$$

From (2) and (4), we have
$$R=mg\cos\psi-\frac{mv^2}{\rho}=mg\cos\psi-m\cdot\frac{4ag\sin^2\psi}{4a\cos\psi}$$
$$[\because\ \rho=ds/d\psi=4a\cos\psi]$$
$$=\frac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi).$$

If the particle leaves the cycloid at the point Q, then at Q, $R=0$. From $R=0$, we have
$$\frac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi)=0$$
or $\quad \sin^2\psi=\cos^2\psi \quad$ or $\quad \tan^2\psi=1$

or $\quad \tan\psi=1 \quad$ or $\quad \psi=45°.$

Thus at Q, we have $\psi=45°$. Putting $\psi=\frac{1}{4}\pi$ in $s=4a\sin\psi$, we have at Q, $s=4a\sin\frac{1}{4}\pi=4a.(1/\sqrt{2})=2\sqrt{2}a$. Again putting $s=2\sqrt{2}a$ in $s^2=8ay$, we have at Q, $y=s^2/8a=8a^2/8a=a$.

Thus $OL=a$. Therefore $LM=OM-OL=2a-a=a$. Hence the particle leaves the cycloid at the point Q, when it has fallen through half the vertical height of the cycloid.

Second part. If v_1 is the velocity of the particle at Q, then from (4), we have $\quad v_1^2=4ag\sin^2 45°=2ag.$

Hence the particle leaves the cycloid at Q with velocity $v_1=\sqrt{(2ag)}$ in a direction making an angle $45°$ downwards with the horizontal. After Q the particle will describe a parabolic path.

Latus rectum of the parabola described after Q
$$=\frac{2v_1^2\cos^2 45°}{g}=\frac{2.2ag.\frac{1}{2}}{g}=2a$$

i.e., the latus rectum of the parabola subsequently described is equal to the height of the cycloid.

Third part. The equation of the parabolic path described by the particle after leaving the cycloid at Q with respect to the horizontal and vertical lines QX' and QY' as the coordinate axes is

$$y=x\tan(-45°)-\frac{gx^2}{2v_1^2\cos^2(-45°)}$$ [Note that here the angle of projection for the motion of the projectile is $-45°$]

or $\quad y=-x-\dfrac{gx^2}{2.\,2ag.\,\frac{1}{2}}$

or $\quad y=-x-\dfrac{x^2}{2a}. \qquad \ldots(5)$

Suppose after leaving the cycloid at Q the particle strikes the base of the cycloid at the point T. Let (x_1, y_1) be the coordinates of T with respect to QX' and QY' as the coordinate axes. Then $x_1=NT$ and $y_1=-QN=-a$.

But the point $T(x_1, -a)$ lies on the curve (5).

$$\therefore\quad -a=-x_1-\frac{x_1^2}{2a}$$

or $\quad x_1^2+2ax_1-2a^2=0.$

$$\therefore\quad x_1=\frac{-2a\pm\sqrt{(4a^2-4.1.(-2a^2))}}{2.1}$$

Neglecting the $-$ive sign because x_1 cannot be negative, we have $\quad x_1=NT=-a+a\sqrt{3}.$

The parametric equations of the cycloid w.r.t. OX and OY as the coordinate axes are

$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta),$$

where θ is the parameter and $\theta=2\psi$.

$\therefore$ At the point Q, where $\psi=\frac{1}{3}\pi$, we have

$$x=LQ=a(2\psi+\sin 2\psi)=a\,[2.\tfrac{1}{3}\pi+\sin(2.\tfrac{1}{3}\pi)]=a\,(\tfrac{2}{3}\pi+1).$$

$\therefore$ the horizontal distance of the point T from the centre M of the base of the cycloid

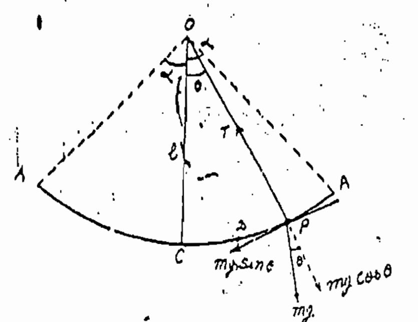
$$=MT=MN+NT=LQ+NT$$

$$=a\,(\tfrac{2}{3}\pi+1)+(-a+a\sqrt{3})=(\tfrac{2}{3}\pi+\sqrt{3})\,a.$$

§ 9. Simple Pendulum.

Definition. *A light inextensible string and a heavy particle of negligible size tied to one end of the string whose other end is attached to a fixed point and oscillating in a vertical plane under gravity through a small angle, are said to form a simple pendulum.*

§ 10. Oscillations of a simple pendulum.



Let m be the mass of the particle and l the length of the string. Let P be the position of the particle and θ be the angle which the string makes with the vertical at any time t. Let OC be the vertical line through the fixed point O and arc $CP=s$.

$$\therefore\quad s=l\theta. \qquad \ldots(1)$$

The forces acting on the particle at time t are :

(i) its weight mg acting vertically downwards,

and (ii) the tension T in the string acting along PO.

$\therefore$ the equation of motion along the tangent at P is

$$m\frac{d^2s}{dt^2}=-mg\sin\theta$$

or $$d^2s/dt^2=-g\sin\theta$$

or $$l\frac{d^2\theta}{dt^2}=-g\sin\theta. \qquad [\because\ s=l\theta,\ \text{from (1)}] \qquad \ldots(2)$$

Case I. When the oscillations are small. [Lucknow 1981]

When the pendulum swings through a small angle on each side of the vertical OC, then θ is very small and hence we can take $\sin\theta=\theta$.

$\therefore$ from (2), we get

$$l\frac{d^2\theta}{dt^2}=-g\theta$$

or $$\frac{d^2\theta}{dt^2}=-\frac{g}{l}\theta,$$

which shows that the motion of the particle is simple harmonic about C. The period T of a small complete oscillation is given by

$$T=2\pi\sqrt{(l/g)}. \qquad \ldots(3)$$

Case II. When the oscillations are not small.

Multiplying both sides of (2) by $2l\,(d\theta/dt)$ and integrating with respect to 't', we get

$$\left(l\frac{d\theta}{dt}\right)^2=2gl\cos\theta+A,$$

where A is a constant of integration.

If the pendulum oscillates through an angle α on each side of the vertical OC, then

$$v=l\,(d\theta/dt)=0,\ \text{when}\ \theta=\alpha.$$

$\therefore$ $0=2gl\cos\alpha+A,$ or $A=-2gl\cos\alpha.$

$$\therefore\quad \left(l\frac{d\theta}{dt}\right)^2=2gl\cos\theta-2gl\cos\alpha$$

or $$\frac{d\theta}{dt}=\sqrt{\left(\frac{2g}{l}\right)}\sqrt{(\cos\theta-\cos\alpha)}$$

or $$dt=\sqrt{\left(\frac{l}{2g}\right)}\frac{d\theta}{\sqrt{(\cos\theta-\cos\alpha)}} \qquad \ldots(4)$$

If t_1 is the time from C (*i.e.*, the lowest point) to the extreme position $\theta=\alpha$, then integrating (4), we get

$$t_1=\sqrt{\left(\frac{l}{2g}\right)}\int_{\theta=0}^{\alpha}\frac{d\theta}{\sqrt{(\cos\theta-\cos\alpha)}}$$

$$=\sqrt{\left(\frac{l}{2g}\right)}\int_0^{\alpha}\frac{d\theta}{\sqrt{((1-2\sin^2\frac{1}{2}\theta)-(1-2\sin^2\frac{1}{2}\alpha))}}$$

$$=\frac{1}{2}\sqrt{\left(\frac{l}{g}\right)}\int_0^{\alpha}\frac{d\theta}{\sqrt{(\sin^2\frac{1}{2}\alpha-\sin^2\frac{1}{2}\theta)}}$$

Substituting $\sin \frac{1}{2}\theta = \sin \frac{1}{2}\alpha \sin \phi$, so that $\frac{1}{2} \cos \frac{1}{2}\theta \, d\theta = \sin \frac{1}{2}\alpha \cos \phi \, d\phi$, we get

$$t_1 = \frac{1}{2} \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \frac{2 \sin \frac{1}{2}\alpha \cos \phi \, d\phi}{\cos \frac{1}{2}\theta \sqrt{(\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\alpha \sin^2 \phi)}}$$

$$= \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \frac{\sin \frac{1}{2}\alpha \cos \phi \, d\phi}{\cos \frac{1}{2}\theta \sin \frac{1}{2}\alpha . \cos \phi} = \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \frac{d\phi}{\cos \frac{1}{2}\theta}$$

$$= \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - \sin^2 \frac{1}{2}\theta)}} = \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \left(1 - \sin^2 \tfrac{1}{2}\alpha \sin^2 \phi\right)^{-1/2} d\phi$$

$$= \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \left[1 + \frac{1}{2} \sin^2 \tfrac{1}{2}\alpha \sin^2 \phi + \frac{1 \cdot 3}{2 \cdot 4} \sin^4 \tfrac{1}{2}\alpha \sin^4 \phi + \dots\right] d\phi$$

$$= \sqrt{\left(\frac{l}{g}\right)} \left[\tfrac{1}{2}\pi + \frac{1}{2} \sin^2 \tfrac{1}{2}\alpha . \frac{1}{2} . \tfrac{1}{2}\pi + \frac{1 \cdot 3}{2 \cdot 4} \sin^4 \tfrac{1}{2}\alpha . \frac{3 \cdot 1}{4 \cdot 2} \tfrac{1}{2}\pi + \dots\right]$$

[By Walli's formula]

$$= \sqrt{\left(\frac{l}{g}\right)} \left[\tfrac{1}{2}\pi + \frac{1}{2^2} . \tfrac{1}{2}\pi \sin^2 \tfrac{1}{2}\alpha + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 . \tfrac{1}{2}\pi \sin^4 \tfrac{1}{2}\alpha + \dots\right]$$

$$= \frac{\pi}{2} \sqrt{\left(\frac{l}{g}\right)} \left[1 + \tfrac{1}{4} \sin^2 \tfrac{1}{2}\alpha + \tfrac{9}{64} \sin^4 \tfrac{1}{2}\alpha + \dots\right] \quad \dots(5)$$

$$= \frac{\pi}{2} \sqrt{\left(\frac{l}{g}\right)} (1 + \tfrac{1}{4} \sin^2 \tfrac{1}{2}\alpha).$$

[Neglecting powers of $\sin \frac{1}{2}\alpha$ higher than 2]

Hence if T_1 is the time of one complete oscillation of a simple pendulum, then a second approximation to the period is given by

$$T_1 = 4t_1 = 2\pi \sqrt{(l/g)} (1 + \tfrac{1}{4} \sin^2 \tfrac{1}{2}\alpha)$$

or $\quad T_1 = T (1 + \frac{1}{4} \sin^2 \frac{1}{2}\alpha)$. $\quad$ [$\because \; T = 2\pi \sqrt{(l/g)}$, from (3)]

Neglecting the powers of α higher than 2, we get

$$T_1 = 2\pi \sqrt{(l/g)} (1 + \tfrac{1}{16}\alpha^2) = T(1 + \tfrac{1}{16}\alpha^2).$$

§ 11. Beat of a pendulum. *A beat of a pendulum means its going from one extreme position of rest to the other position of rest i.e., half of the complete oscillation.*

$\therefore$ the time of a beat $= \frac{1}{2}T = \pi \sqrt{(l/g)}$.

§ 12. The Second's pendulum. *If a simple pendulum oscillates from rest in one second i.e., if the time of one beat of a simple pendulum is one second, then it is called a* second's pendulum *and such a clock is said to be a correct clock.*

Thus for a second's pendulum

$$1 = \pi \sqrt{(l/g)}$$

or $\quad l = \dfrac{g}{\pi^2} = \dfrac{g}{(3 \cdot 1416)^2}$.

In F. P. S. system $g = 32 \cdot 2$, then $l = \dfrac{32 \cdot 2}{(3 \cdot 1416)^2}$

$= 39 \cdot 14$ inches (appr.)

and in C. G. S. system $g = 981$, then $l = \dfrac{981}{(3 \cdot 1416)^2}$

$= 99 \cdot 4$ cm. (appr.).

§ 13. Gain or loss of beats (time) by a clock.

[Lucknow 1975, 77, 79]

The time t_0 of one beat of a clock is given by

$$t_0 = \tfrac{1}{2}T = \pi \sqrt{(l/g)}.$$

Clearly t_0 depends upon the values of l and g. Thus there is a change in the time of a beat of a clock when l and g change, either one or both.

Thus if n is the number of beats in a given time t, then

$$t = n . \pi \sqrt{(l/g)}$$

or $\quad n = \dfrac{t}{\pi} \sqrt{(g/l)}. \quad \dots(1)$

Now we shall determine the loss or gain in the number of beats of a clock when l and g change, either one or both.

Taking log of both sides of (1), we get

$$\log n = \log t - \log \pi + \tfrac{1}{2} (\log g - \log l).$$

Differentiating, $\quad \dfrac{1}{n} \delta n = \dfrac{1}{2g} \delta g - \dfrac{1}{2l} \delta l. \quad \dots(2)$

[$\because$ t and π are constants]

Now the following cases arise :

(a) When g remains constant. [Lucknow 1975, 77, 79]

If g remains constant, then $\delta g = 0$. Therefore from (2), we get

$$\frac{1}{n} \delta n = -\frac{1}{2l} \delta l. \quad \dots(3)$$

$\therefore$ δn is positive or negative according as δl is negative or positive respectively.

Hence there is a gain or loss in the number of beats according as the length of the string is shortened or increased i.e., the clock becomes fast when the length of the pendulum is shortened and the clock becomes slow when the length of the pendulum is increased.

(b) When l remains constant. [Lucknow 1975, 77, 79]

If l remains constant, then $\delta l = 0$. Therefore from (2), we get

$$\frac{1}{n} \delta n = \frac{1}{2g} \delta g. \quad \dots(4)$$

$\therefore$ there is a gain or loss in the number of beats according as g increases or decreases. *Hence the clock becomes fast when g increases and it becomes slow when g decreases.* In other words a

clock becomes fast if it is taken to a place of more gravity and it becomes slow if it is taken to a place of less gravity.

Now we shall discuss the following two situations.

(i) When the pendulum (or clock) is taken to the top of a mountain.

We know that outside the surface of the earth the attraction varies inversely as the square of the distance from the centre of the earth.

Thus at a distance x from the centre of the earth, the attraction is given by μ/x^2.

On the surface of the earth where $x=r$ (*i.e.*, the radius of the earth) the attraction is g.

$\therefore \quad g=\mu/r^2$

or $\quad \log g=\log \mu-2\log r.$

Differentiating, we get

$$\frac{1}{g}\,\delta g=-\frac{2}{r}\,\delta r \qquad [\because \mu \text{ is constant}]$$

or $$\frac{1}{g}\,\delta g=-\frac{2}{r}\,h,$$

where $\delta r=h$ is the height of the mountain.

$\therefore$ from (4), we get $\frac{1}{n}\,\delta n=-\frac{1}{r}\,h$

or $$\delta n=-\frac{n}{r}\,h. \qquad \ldots(5)$$

The negative sign indicates that the number of beats are lost.

Hence the clock becomes slow when it is taken to the top of a mountain.

(ii) When the pendulum (or clock) is taken to the bottom of a mine.

We know that inside the earth the attraction varies as the distance from the centre.

Thus at a distance x from the centre of the earth, the attraction is given by μx.

On the surface of the earth where $x=r$ (*i.e.*, the radius of the earth) the attraction is g.

$\therefore \quad g=\mu r$

or $\quad \log g=\log \mu+\log r.$

Differentiating, we get

$$\frac{1}{g}\,\delta g=\frac{1}{r}\,\delta r. \qquad [\because \mu \text{ is a constant}]$$

If the pendulum is taken to the bottom of a mine of depth d, then $\delta r=-d$.

$$\therefore \quad \frac{1}{g}\,\delta g=-\frac{d}{r}.$$

$\therefore$ from (4), we get $\frac{1}{n}\,\delta n=-\frac{1}{2r}\,d$

or $$\delta n=-\frac{n}{2r}\,d. \qquad \ldots(6)$$

The negative sign indicates that the number of beats are lost.

Hence the clock becomes slow when it is taken to the bottom of a mine.

Illustrative Examples

Ex. 42. *In a simple pendulum, show that the period T is given by*

$$T=2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{1}{4}k^2+\frac{9}{64}k^4+\ldots\right],$$

where $k=\sin\frac{1}{2}\alpha$ and α is the amplitude.

Sol. The time t_1 taken by the pendulum to swing from its lowest position to the position $\theta=\alpha$ (extreme position on one side), is given by

$$t_1=\frac{\pi}{2}\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{1}{4}\sin^2\frac{\alpha}{2}+\frac{9}{64}\sin^4\frac{\alpha}{2}+\ldots\right]$$

[see equation (5), § 10 on page 222]

$\therefore$ period T is given by

$$T=4t_1=2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{1}{4}\sin^2\frac{\alpha}{2}+\frac{9}{64}\sin^4\frac{\alpha}{2}+\ldots\right]$$

$$=2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{1}{4}k^2+\frac{9}{64}k^4+\ldots\right], \text{ where } k=\sin\tfrac{1}{2}\alpha.$$

Ex. 43. *A simple pendulum is started so as to make complete revolution in a vertical plane. Find the least velocity of projection.*

In the subsequent motion, ω_1, ω_2 are the greatest and least angular velocities; and T_1, T_2 are the greatest and least tensions. Prove that when the pendulum makes an angle θ with the vertical, the angular velocity is

$$[\omega_1^2\cos^2\tfrac{1}{2}\theta+\omega_2^2\sin^2\tfrac{1}{2}\theta]^{1/2},$$

and that the tension is

$$T_1\cos^2\tfrac{1}{2}\theta+T_2\sin^2\tfrac{1}{2}\theta. \qquad \text{[Agra 1985]}$$

Sol. [Refer fig. § 2 on page 156].

Let the string be inclined at an angle θ to the vertical at time t. The forces acting on the particle at P are : (i) The tension T in

the string along PO and (ii) the weight mg of the particle acting vertically downwards.

Let l be the length of the string and let arc $AP=s$. The equations of motion of the particle along the tangent and normal at P are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta, \quad \ldots(1)$$

and $$m\frac{v^2}{l}=T-mg\cos\theta. \quad \ldots(2)$$

Also $s=l\theta$, $\ldots(3)$

From (1) and (3), we get $l\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Mulutiplying both sides by $2l\,(d\theta/dt)$ and integrating w.r.t. 't', we get

$$v^2=\left(l\frac{d\theta}{dt}\right)^2=2gl\cos\theta+A,$$

where A is a constant of integration.

If the particle is projected with velocity u from the lowest point A, then $v=u$, when $\theta=0$.

$\therefore\quad u^2=2gl+A \quad$ or $\quad A=u^2-2gl.$

$$\therefore\quad v^2=\left(l\frac{d\theta}{dt}\right)^2=2gl\cos\theta+u^2-2gl. \quad \ldots(4)$$

Substituting in (2), we get

$$T=\frac{m}{l}\left[v^2+lg\cos\theta\right]$$

$$=\frac{m}{l}\left[u^2-2gl+3gl\cos\theta\right]. \quad \ldots(5)$$

The pendulum will make complete revolution if neither the velocity nor the tension vanishes before the particle reaches the highest point.

At the highest point $\theta=\pi$. So in order to make complete revolution we should have at the highest point

$$v^2=u^2-4gl\geqslant 0$$

and $$T=\frac{m}{l}\left(u^2-5gl\right)\geqslant 0.$$

$\therefore\quad u^2\geqslant 5gl \quad$ or $\quad u\geqslant\sqrt{(5gl)}.$

Hence for the particle to make a complete revolution in the vertical plane, the least velocity of projection $u=\sqrt{(5gl)}$.

For complete solution of the second and third parts of this question proceed as in Ex. 2 (b) on page 162.

Ex. 44. *Show that if the tension of the string when the bob is in its lowest position is k times the tension when the bob is in its highest position, the velocities in these positions being u_1 and u_2 respectively, then*

$$\frac{u_1^2}{u_2^2}=\frac{5k+1}{k+5}.$$

Sol. Let the string be inclined at an angle θ to the vertical at time t. Let u be the velocity of projection of the bob from its lowest position. If v is the velocity and T the tension in the string at time t, then proceeding as in the preceding Ex. 43, we get

$$v^2=u^2-2gl+2gl\cos\theta \quad \ldots(1)$$

and $$T=\frac{m}{l}(u^2-2gl+3gl\cos\theta), \quad \ldots(2)$$

where l is the length of the string.

If u_1, u_2 are the velocities and T_1, T_2 the tensions in the string in its lowest and highest positions respectively, we have

$\theta=0,\ v=u_1,\ T=T_1$

and $\theta=\pi,\ v=u_2,\ T=T_2.$

Putting these values in (1) and (2), we have

$$u_1^2=u^2,\quad u_2^2=u^2-4gl,$$

$$T_1=\frac{m}{l}(u^2+gl), \quad \text{and} \quad T_2=\frac{m}{l}(u^2-5gl).$$

$\therefore\quad u_1^2-u_2^2=4gl, \quad$ or $\quad gl=\frac{1}{4}(u_1^2-u_2^2).$

Also given that $T_1=kT_2$.

$$\therefore\quad \frac{m}{l}(u^2+gl)=k\cdot\frac{m}{l}(u^2-5gl)$$

or $\quad u^2+gl=ku^2-5kgl.$

Substituting $u^2=u_1^2$ and $gl=\frac{1}{4}(u_1^2-u_2^2)$, we get

$$u_1^2+\tfrac{1}{4}(u_1^2-u_2^2)=ku_1^2-\tfrac{5}{4}k(u_1^2-u_2^2)$$

or $\quad 4u_1^2+u_1^2-u_2^2=4ku_1^2-5ku_1^2+5ku_2^2$

or $\quad (5+k)\,u_1^2=(5k+1)\,u_2^2$

or $$\frac{u_1^2}{u_2^2}=\frac{5k+1}{k+5}.$$

Ex. 45. *If a pendulum of length l makes n complete oscillations in a given time, show that if g is changed to $(g+g')$, the number of oscillations gained is $ng'/(2g)$* [Meerut 1979]

Sol. For a pendulum of length l, the time of one complete oscillation T is given by

$$T=2\pi\sqrt{(l/g)}.$$

$\therefore$ n = the number of complete oscillations in a given time t

$$=\frac{t}{T}=\frac{t}{2\pi}\sqrt{\left(\frac{g}{l}\right)}.$$

$\therefore$ $\log n=\log\left(\frac{t}{2\pi}\right)+\frac{1}{2}\log g-\frac{1}{2}\log l.$

Differentiating,

$$\frac{1}{n}\,\delta n=\frac{1}{2g}\,\delta g-\frac{1}{2l}\,\delta l. \qquad \ldots(1)$$

[$\because$ $t/2\pi$ is constant]

If l is fixed then $\delta l=0$ and if g is changed to $(g+g')$, then $\delta g=g'$.

$\therefore$ from (1), we get $\frac{1}{n}\,\delta n=\frac{1}{2g}\cdot g'$

or $\delta n=\frac{ng'}{2g}$.

Hence the number of oscillations gained

$=\delta n=ng'/(2g).$

Ex. 46. *If a pendulum beating seconds at the foot of a mountain, loses 9 seconds a day when taken to its summit, find the height of the mountain assuming the radius of the earth to be 4000 miles and neglecting the attraction of the mountain.*

Sol. For a pendulum beating seconds at the foot of a mountain, n = number of beats in a day $=24\times 60\times 60$.

If h is the height of the mountain and r the radius of the earth then the gain in the number of beats in a day at the top of the mountain is given by

$$\delta n=-\frac{n}{r}\,h. \qquad \ldots(1)$$

[Refer equation (5) of § 13 on page 226]

Here $r=4000$ miles $=4000\times 1760\times 3$ ft.,

and $\delta n=-9$.

$\therefore$ from (1), we have

$$-9=-\frac{24\times 60\times 60}{4000\times 1760\times 3}\,h,$$

or $h=2200$ ft.

Ex. 47. *Find approximately the height of a mountain at the top of which a pendulum which beats seconds at sea level, loses 8 seconds a day. The radius of earth may be taken 4000 miles.*

Sol. Proceed as in the preceding Ex. 46, height of mountain $=1955\cdot5$ ft.

Ex. 48. *A pendulum beats seconds accurately at a place where g is 32 ft./sec². Prove that it will gain 270 seconds per day, if it be taken to a place where g is $32\cdot 2$ ft./sec².*

Sol. For a pendulum which beats seconds accurately, let the number of beats in a day be n. Then $n=24\times 60\times 60$.

When the length of the pendulum remains constant, from the equation (4) of § 13, the number of beats gained in a day is given by

$$\delta n=\frac{n}{2g}\,\delta g. \qquad \ldots(1)$$

Here $g=32$ ft./sec². and $\delta g=32\cdot 2-32=0\cdot 2$ ft./sec².

$\therefore$ from (1), we get

$$\delta n=\frac{24\times 60\times 60\times 0\cdot 2}{2\times 32}=270.$$

Hence the pendulum will gain 270 seconds per day.

Ex. 49. *If a pendulum of length l makes n complete oscillations in a given time, show that, if the length be changed to $l+l'$, the number of oscillations lost is $nl'/(2l)$.* [Meerut 1979]

Sol. For a pendulum of length l, the time of one complete oscillation T is given by $T=2\pi\sqrt{(l/g)}$.

$\therefore$ n = the number of complete oscillations in a given time t

$$=\frac{t}{T}=\frac{t}{2\pi}\sqrt{(g/l)}.$$

$\therefore$ $\log n=\log(t/2\pi)+\frac{1}{2}\log g-\frac{1}{2}\log l.$

Differentiating, $\frac{1}{n}\,\delta n=\frac{1}{2g}\,\delta g-\frac{1}{2l}\,\delta l. \qquad \ldots(1)$

[$\because$ $t/2\pi$ is constant]

If g is fixed, then $\delta g=0$ and if l is changed to $l+l'$, then $\delta l=l'$.

$\therefore$ from (1), we get

$$\frac{1}{n}\,\delta n=0-\frac{1}{2l}\,l', \quad \text{or} \quad \delta n=-\frac{n}{2l}\,l'.$$

Hence the number of oscillations lost in time $t=-\delta n=nl'/(2l)$.

Ex. 50. *A pendulum is carried to the top of a mountain 2640 feet high. How many seconds will it lose per day? By how much its present length be shortened so that it may beat seconds at the summit of the mountain?* [Lucknow 1975, 80]

Sol. For a second's pendulum, let the number of beats in a day be n. Then $n=24\times 60\times 60$.

If r is the radius of the earth then the gain in the number of beats in a day at the top of a mountain of height h is given by

$$\delta n = -\frac{n}{r} h. \qquad \ldots(1)$$

Here $h = 2640$ ft. and $r = 4000 \times 1760 \times 3$ ft.

$\therefore$ from (1), we get

$$\delta n = -\frac{24 \times 60 \times 60}{4000 \times 1760 \times 3} \times 2640 = -10 \cdot 8$$

i.e., the pendulum will lose 10·8 seconds per day.

IInd part. For a second's pendulum, if n be the number of beats in a given time t, we have

$$n = \frac{t}{\pi} \sqrt{(g/l)}.$$

$\therefore$ $\log n = \log (t/\pi) + \frac{1}{2} \log g - \frac{1}{2} \log l.$

Differentiating, $\dfrac{\delta n}{n} = \dfrac{\delta g}{2g} - \dfrac{1}{2l} \delta l.$ ...(2)

The pendulum will give correct time at the top of the mountain if there is neither increase nor decrease in the number of beats there *i.e.*, if $\delta n = 0$.

$\therefore$ putting $\delta n = 0$ in (2), we get

$$0 = \frac{1}{2g} \delta g - \frac{1}{2l} \delta l$$

or $$\frac{\delta g}{g} = \frac{\delta l}{l}. \qquad \ldots(3)$$

Now on the surface of the earth, attraction $= \mu/r^2 = g$.

$\therefore$ $\log g = \log \mu - 2 \log r.$

Differentiating, $\dfrac{1}{g} \delta g = -\dfrac{2}{r} \delta r.$

$\therefore$ from (3), we get $-\dfrac{2}{r_1} \delta r = \dfrac{\delta l}{l}.$

But at the top of a mountain of height h, $\delta r = h$.

$\therefore$ $\dfrac{\delta l}{l} = -\dfrac{2}{r} h$

or $\delta l = -\dfrac{2l}{r} h = -\dfrac{2 \times 2640}{4000 \times 1760 \times 3} l = -\dfrac{l}{4000}.$

Hence the pendulum should be shortened by (1/4000) of its present length. Ans.

Motion in a Resisting Medium

(In a straight line only)

§ 1. Introduction.

It is a well known fact that a body moving in a medium (like air) feels a resistance to its motion which increases with the increase in the velocity of the body. Thus the resistance on a body moving in a medium may be assumed to be equal to some function of the velocity of the body. The resistance of the medium always acts opposite to the direction of motion of the body.

Experimentally it has been found out that when a particle is projected in air, the force of resistance varies as the square of the velocity upto a velocity of 800 ft./sec. and as cube of the velocity between 800 ft./sec and 1350 ft./sec. Beyond this velocity the resistance again varies as the square of the velocity.

Therefore in this chapter we shall mostly discuss the motion of a particle (or body) in a resisting medium where the resistance varies as the square of the velocity.

§ 2. Terminal Velocity.

If a particle falls under gravity in a resisting medium the force of resistance acts vertically upwards on the particle while the force of gravity acts vertically downwards. As the velocity of the particle goes on increasing the force of resistance also goes on increasing. Suppose the force of resistance becomes equal to the force of gravity when the particle has attained the velocity V. Then the resultant downward acceleration of the particle becomes zero and so during its subsequent motion the particle falls with constant velocity V, called the *terminal velocity* or the *limiting velocity*. The terminal velocity is maximum for the downward motion.

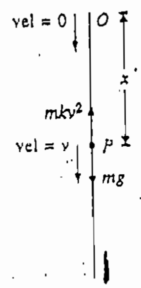
Definition. *If a particle is falling under gravity in a resisting medium, then the velocity V when the downward acceleration is zero is called the* terminal (or limiting) velocity. [Meerut 95]

§ 3. Motion of a Particle Falling Under Gravity.

A particle is falling from rest under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity; to discuss the motion. [Lucknow 80; Meerut 78, 81S, 83P, 84S, 86, 88S, 90S, 97]

Let a particle of mass m fall from rest under gravity from the fixed point O.

Let P be the position of the particle after time t, where $OP = x$. If v is the velocity of the particle at P, then mkv^2 is the resistance of the medium on the particle acting in the upwards direction *i.e.*, in the direction of x decreasing. Here kv^2 is the resistance per unit mass so that the resistance on the particle of mass m is mkv^2.



The weight mg of the particle acts vertically downwards *i.e.*, in the direction of x increasing.

$\therefore$ the equation of motion of the particle at time t is

$$m \frac{d^2x}{dt^2} = mg - mkv^2$$

or
$$\frac{d^2x}{dt^2} = g\left(1 - \frac{k}{g} v^2\right) \qquad \text{...(1)}$$

If V is the terminal velocity, then when $v = V$, $d^2x/dt^2 = 0$.

$\therefore$ from (1), we have, $0 = g\left(1 - \frac{k}{g} V^2\right)$ or $\frac{k}{g} = \frac{1}{V^2}$.

$$\therefore \frac{d^2x}{dt^2} = g\left(1 - \frac{v^2}{V^2}\right) \quad \text{or} \quad \frac{d^2x}{dt^2} = \frac{g}{V^2}(V^2 - v^2) \qquad \text{...(2)}$$

To find the relation between v and x.

The equation (2) can be written as

$$v \frac{dv}{dx} = \frac{g}{V^2}(V^2 - v^2) \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

or
$$\frac{-2g}{V^2} dx = \frac{-2v\,dv}{V^2 - v^2}.$$

Integrating, $\frac{-2g}{V^2} x = \log (V^2 - v^2) + A$, where A is a constant.

But initially at O, when $x = 0$, $v = 0$.

$\therefore$ $0 = \log V^2 + A$ or $A = -\log V^2$.

$$\therefore \frac{-2gx}{V^2} = \log (V^2 - v^2) - \log V^2 = \log \left(\frac{V^2 - v^2}{V^2}\right)$$

or
$$\frac{V^2 - v^2}{V^2} = e^{-2gx/V^2}$$

or
$$V^2 - v^2 = V^2 e^{-2gx/V^2} \text{ or } v^2 = V^2(1 - e^{-2gx/V^2}), \qquad \text{...(3)}$$

which gives the velocity of the particle at any position.

Relation between v and t. Again the equation (2) can be written as

$$\frac{dv}{dt} = \frac{g}{V^2}(V^2 - v^2) \qquad \left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$$

or
$$\frac{g}{V^2} dt = \frac{dv}{V^2 - v^2}.$$

Integrating, $\frac{g}{V^2} t = \frac{1}{2V} \log \frac{V + v}{V - v} + B$, where B is a constant.

Initially at O, when $t = 0$, $v = 0$.

$$\therefore 0 = \frac{1}{2V} \log 1 + B, \text{ or } B = 0.$$

$$\therefore \frac{gt}{V^2} = \frac{1}{2V} \log \frac{V + v}{V - v}$$

or
$$t = \frac{V}{g} \cdot \frac{1}{2} \log \frac{1 + (v/V)}{1 - (v/V)} = \frac{V}{g} \tanh^{-1} \frac{v}{V}$$
$$\left[\because \tanh^{-1} z = \frac{1}{2} \log \frac{1 + z}{1 - z}\right]$$

or
$$\frac{gt}{V} = \tanh^{-1} \frac{v}{V} \text{ or } v = V \tanh (gt/V), \qquad \text{...(4)}$$

which gives the velocity of the particle at any time.

Relation between x and t. Eliminating v between (3) and (4), we have

$$V^2 \tanh^2 (gt/V) = V^2 (1 - e^{-2gx/V^2})$$

or
$$e^{-2gx/V^2} = 1 - \tanh^2 (gt/V) = \text{sech}^2 (gt/V)$$

or
$$e^{2gx/V^2} = \cosh^2 (gt/V)$$

or
$$\frac{2gx}{V^2} = 2 \log \cosh (gt/V) \text{ or } x = \frac{V^2}{g} \log \cosh (gt/V), \qquad \text{...(5)}$$

which gives the position of the particle at any time.

Remark. To evaluate $\int \frac{dv}{V^2 - v^2}$, we can directly apply the formula $\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}$. Remember this formula.

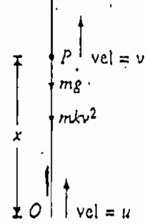
§ 4. Motion of a particle Projected Vertically Upwards.

A particle is projected vertically upwards under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity; to discuss the motion. [Lucknow 79; Meerut 78, 90S, 91, 92]

Let a particle of mass m be projected, vertically upwards from the point O, with velocity u. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are

(i) The force mkv^2 due to resistance acting against the direction of motion *i.e.*, acting vertically downwards.

(ii) The weight mg of the particle also acting vertically downwards.



Both these forces act in the direction of x decreasing. Therefore the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

or $$\frac{d^2x}{dt^2} = -g\left(1 + \frac{k}{g}v^2\right).$$

Let V be the terminal velocity of the particle during its downwards motion *i.e.*, the velocity when the resultant acceleration of the particle during its downwards motion is zero. Then

$$0 = mg - mkV^2, \text{ or } k = g/V^2.$$

Putting this value of k in the above equation of motion of the particle, we get

$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right) \text{ or } \frac{d^2x}{dt^2} = \frac{-g}{V^2}(V^2 + v^2). \qquad \ldots(1)$$

Relation between v and x.

Equation (1) can be written as

$$v\frac{dv}{dx} = \frac{-g}{V^2}(V^2 + v^2) \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

or $$\frac{-2g}{V^2}dx = \frac{2v\,dv}{V^2 + v^2}, \text{ separating the variables.}$$

Integrating, $\dfrac{-2gx}{V^2} = \log(V^2 + v^2) + A$, where A is a constant.

Initially at O, $x = 0$ and $v = u$.

$$\therefore \; 0 = \log(V^2 + u^2) + A$$

or $$A = -\log(V^2 + u^2).$$

$$\therefore \; \frac{-2gx}{V^2} = \log(V^2 + v^2) - \log(V^2 + u^2)$$

or $$x = \frac{V^2}{2g}\log\frac{V^2 + u^2}{V^2 + v^2} \qquad \ldots(2)$$

which gives the velocity of the particle in any position.

Relation between v and t.

Equation (1) can be written as

$$\frac{dv}{dt} = -\frac{g}{V^2}(V^2 + v^2) \qquad \left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$$

or $$dt = \frac{-V^2}{g}\cdot\frac{dv}{V^2 + v^2}, \text{ separating the variables.}$$

Integrating, $t = \dfrac{-V^2}{g}\cdot\dfrac{1}{V}\tan^{-1}\dfrac{v}{V} + B$, where B is a constant

or $$t = \frac{-V}{g}\tan^{-1}\frac{v}{V} + B.$$

Initially at O, when $t = 0$, $v = u$.

$$\therefore \; 0 = -\frac{V}{g}\tan^{-1}\frac{u}{V} + B \text{ or } B = \frac{V}{g}\tan^{-1}\frac{u}{V}.$$

$$\therefore \; t = \frac{V}{g}\left(\tan^{-1}\frac{u}{V} - \tan^{-1}\frac{v}{V}\right), \qquad \ldots(3)$$

which gives the velocity of the particle at any time t.

Relation between x and t.

A relation between x and t can be obtained by eliminating v between (2) and (3).

Illustrative Examples

Ex. 1 (a). *A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance it has described in time t is $(1/\mu V)[\sqrt{(1 + 2\mu V^2 t)} - 1]$ and that its velocity then is $V/\sqrt{(1 + 2\mu V^2 t)}$.* [Meerut 73; 76]

Solution. Take the point of projection O as origin. Let v be the velocity of the particle at time t at a point distant x from the fixed point O. Then the resistance at this point will be $m\mu v^3$, acting in the direction of x decreasing. Here the resistance is the only force acting on the particle during its motion.

$\therefore$ the equation of motion of the particle is

$$m\frac{dv}{dt} = -m\mu v^3$$

or $$\frac{dv}{v^3} = -\mu\,dt.$$

Integrating, $-\dfrac{1}{2v^2} = -\mu t + A$, where A is a constant.

But initially, when $t = 0$, $v = V$; $\therefore A = -\dfrac{1}{2V^2}$.

$$\therefore -\frac{1}{2v^2} = -\mu t - \frac{1}{2V^2}$$

or $\quad 1/v^2 = (2\mu V^2 t + 1)/V^2$ or $v = V/\sqrt{(1 + 2\mu V^2 t)}$, ...(1)

which gives the velocity of the particle at time t.

Since the particle is moving in the direction of x increasing, therefore from the equation (1), we have

$$\frac{dx}{dt} = v = V/\sqrt{(1 + 2\mu V^2 t)}$$

or $\quad dx = V(1 + 2\mu V^2 t)^{-1/2}\, dt.$

Integrating, $x = \dfrac{1}{\mu V}(1 + 2\mu V^2 t)^{1/2} + B$, where B is a constant.

But initially when $t = 0, x = 0$; $\therefore B = -\dfrac{1}{\mu V}$;

$$\therefore x = \frac{1}{\mu V}(1 + 2\mu V^2 t)^{1/2} - \frac{1}{\mu V}$$

or $$x = \frac{1}{\mu V}[\sqrt{(1 + 2\mu V^2 t)} - 1],$$

which gives the distance described in time t.

(b) *A particle is projected with velocity u along a smooth horizontal plane in a medium whose resistance per unit mass is k (velocity), show that the velocity after a time t and the distance s in that time are given by*

$$v = u e^{-kt} \text{ and } s = u(1 - e^{-kt})/k. \quad \text{(Meerut 1989, 91)}$$

Sol. Proceed as in Ex. 1. (a).

Here the equation of motion of the particle is

$$m\frac{dv}{dt} = -mkv \quad \text{or} \quad \frac{dv}{dt} = -kv.$$

$$\therefore \frac{dv}{v} = -k\,dt.$$

Integrating, $\log v = -kt + C_1$.

But initially, when $t = 0, v = u$; $\therefore C_1 = \log u$.

$\therefore \log v = -kt + \log u$, or $\log(v/u) = -kt$

or $\quad v/u = e^{-kt}$ or $v = u e^{-kt}$.

But $v = ds/dt$.

$\therefore ds/dt = u e^{-kt}$ or $ds = u e^{-kt}\,dt$.

Integrating, $s = \dfrac{ue^{-kt}}{-k} + C_2$.

But initially, when $t = 0, s = 0$; $\therefore C_2 = u/k$.

$$\therefore s = \frac{u}{k} - \frac{u e^{-kt}}{k} = \frac{u(1 - e^{-kt})}{k}.$$

Ex. 2. *A particle falls from rest under gravity through a distance x in a medium whose resistance varies as the square of the velocity. If v be the velocity actually acquired by it, v_0 the velocity it would have acquired, had there been no resisting medium and V the terminal velocity, show that*

$$\frac{v^2}{v_0^2} = 1 - \frac{1}{2}\frac{v_0^2}{V^2} + \frac{1}{2.3}\frac{v_0^4}{V^4} - \frac{1}{2.3.4}\frac{v_0^6}{V^6} + \ldots\ldots$$

(Meerut 1985 P)

Sol. If v is the velocity of the particle acquired in falling through a distance x in the given resisting medium, then proceeding as in § 3, it is given by

$$v^2 = V^2\left(1 - e^{-2gx/V^2}\right). \quad \ldots(1)$$

If v_0 is the velocity of the particle acquired in falling freely through a distance x, if there is no resisting medium, then

$$v_0^2 = 0 + 2gx = 2gx. \quad \ldots(2)$$

Substituting $2gx = v_0^2$ in (1), we have

$$v^2 = V^2\left(1 - e^{-v_0^2/V^2}\right)$$

$$= V^2\left[1 - \left\{1 - \frac{v_0^2/V^2}{1!} + \frac{v_0^4/V^4}{2!} - \frac{v_0^6/V^6}{3!} + \ldots\ldots\right\}\right]$$

$$= V^2\left[\frac{v_0^2/V^2}{1!} - \frac{v_0^4/V^4}{2!} + \frac{v_0^6/V^6}{3!} - \ldots\ldots\right]$$

$$= v_0^2\left[1 - \frac{1}{2!}\cdot\frac{v_0^2}{V^2} + \frac{1}{3!}\cdot\frac{v_0^4}{V^4} - \frac{1}{4!}\cdot\frac{v_0^6}{V^6} + \ldots\right]$$

or $$\frac{v^2}{v_0^2} = 1 - \frac{1}{2}\frac{v_0^2}{V^2} + \frac{1}{2.3}\frac{v_0^4}{V^4} - \frac{1}{2.3.4}\frac{v_0^6}{V^6} + \ldots\ldots$$

Ex. 3. *A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\dfrac{V^2}{g}[\lambda - \log(1 + \lambda)]$, where V is the terminal velocity of the particle and λV is the initial velocity.*

(Lucknow 1981; Meerut 71, 76, 84P, 86, 87P, 88, 92S, 93)

Sol. Suppose a particle of mass m is projected vertically upwards from O with velocity λV in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are,

(i) The force mkv due to the resistance acting vertically downwards *i.e.*, against the direction of motion of the particle, and

(ii) the weight mg of the particle acting vertically downwards.

Since both these forces act in the direction of x decreasing, therefore the equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2}=-mg-mkv$$

or $$\frac{d^2x}{dt^2}=-g\left(1+\frac{k}{g}v\right). \quad ...(1)$$

Now V is given to be the terminal velocity of the particle during its downward motion. Then V is the velocity of the particle when during the downward motion its acceleration is zero. If the particle falls vertically downwards, the resistance acts vertically upwards. Therefore the equation of motion of the particle in downward motion is

$$m\frac{d^2x}{dt^2}=mg-mkv. \quad ...(2)$$

Putting $v=V$ and $d^2x/dt^2=0$ in (2), we get

$$0=mg-mkV \quad \text{or} \quad k=g/V.$$

Substituting this value of k in (1), the equation of motion of the particle in the upward motion is

$$\frac{d^2x}{dt^2}=-g\left(1+\frac{v}{V}\right)$$

or $$v\frac{dv}{dx}=-\frac{g}{V}(V+v), \quad \left[\because \frac{d^2x}{dt^2}=v\frac{dv}{dx}\right]$$

or $$dx=-\frac{V}{g}\frac{v\,dv}{V+v}, \text{ separating the variables}$$

or $$dx=-\frac{V}{g}\left\{\frac{(v+V)-V}{v+V}\right\}dv=-\frac{V}{g}\left(1-\frac{V}{V+v}\right)dv.$$

Integrating, we have

$$x=-\frac{V}{g}\{v-V\log(V+v)\}+A, \text{ where } A \text{ is a constant.}$$

But initially when $x=0$, $v=\lambda V$ (given).

$$\therefore \quad 0=-\frac{V}{g}\{\lambda V-V\log(V+\lambda V)\}+A$$

or $$A=\frac{V}{g}[\lambda V-V\log\{V(1+\lambda)\}].$$

$$\therefore \quad x=\frac{V}{g}\left[\lambda V-v-V\log\frac{V(1+\lambda)}{(V+v)}\right], \text{ giving the velocity of}$$

the particle at any position.

If h is the greatest height attained by the particle, we have $v=0$ when $x=h$.

$$\therefore \quad h=\frac{V}{g}\left[\lambda V-V\log\frac{V(1+\lambda)}{V}\right]=\frac{V^2}{g}[\lambda-\log(1+\lambda)].$$

Ex. 4. *A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Find the greatest height attained by the particle.* (Meerut 96, 97)

Sol. Suppose a particle of mass m is projected vertically upwards from a point O with velocity u in a medium whose resitance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP=x$ and let v be the velocity of the particle at P. Then proceeding as in Ex. 3, the equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2}=-mg-mkv$$

or $$\frac{d^2x}{dt^2}=-(g+kv) \quad \text{or} \quad v\frac{dv}{dx}=-(g+kv).$$

$$\therefore \quad dx=-\frac{v}{g+kv}dv=-\frac{1}{k}\frac{kv}{g+kv}dv$$

$$=-\frac{1}{k}\frac{(g+kv)-g}{g+kv}dv=-\frac{1}{k}\left[1-\frac{g}{g+kv}\right]dv.$$

Integrating, we get

$$x=-\frac{1}{k}\left[v-\frac{g}{k}\log(g+kv)\right]+A, \text{ where } A \text{ is a constant.}$$

But initially when $x=0$, we have $v=u$.

$$\therefore \quad 0=-\frac{1}{k}\left[u-\frac{g}{k}\log(g+ku)\right]+A$$

or $$A=\frac{1}{k}\left[u-\frac{g}{k}\log(g+ku)\right].$$

$$\therefore \quad x=-\frac{1}{k}\left[v-\frac{g}{k}\log(g+kv)\right]+\frac{1}{k}\left[u-\frac{g}{k}\log(g+ku)\right]$$

$$=\frac{1}{k}(u-v)-\frac{g}{k^2}\log\frac{g+ku}{g+kv}, \text{ giving the velocity of the particle at}$$

any position.

If h is the greatest height attained by the particle, we have $v=0$ when $x=h$.

$$\therefore \quad h=\frac{u}{k}-\frac{g}{k^2}\log\left(\frac{g+ku}{g}\right)=\frac{u}{k}-\frac{g}{k^2}\log\left(1+\frac{ku}{g}\right).$$

Ex. 5. *A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is* $\dfrac{gm^2}{\mu^2}\left[e^{-(\mu/m)t}-1+\dfrac{\mu t}{m}\right]$ (Meerut 1975, 79, 83, 87S 88S, 90S)

Sol. Let a particle of mass m falling under gravity be at a distance x from the starting point, after time t. If v is its velocity at this point, then the resistance on the particle is μv acting vertically upwards *i.e.*, in the direction of x decreasing. The weight mg of the particle acts vertically downwards *i.e.*, in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = mg - \mu v$$

or $$\frac{dv}{dt} = g - \frac{\mu}{m}v, \qquad \left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$$

or $$dt = \frac{dv}{g - (\mu/m)v}.$$

Integrating, we have

$$t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + A, \text{ where } A \text{ is a constant.}$$

But initially when $t = 0, v = 0$; $\therefore$ $A = (m/\mu)\log g$.

$$\therefore \quad t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + \frac{m}{\mu}\log g$$

or $$t = -\frac{m}{\mu}\log\left\{\frac{g - (\mu/m)v}{g}\right\}$$

or $$-\frac{\mu t}{m} = \log\left(1 - \frac{\mu}{gm}v\right) \quad \text{or} \quad 1 - \frac{\mu}{gm}v = e^{-\mu t/m}$$

or $$v = \frac{dx}{dt} = \frac{gm}{\mu}(1 - e^{-\mu t/m}) \quad \text{or} \quad dx = \frac{gm}{\mu}(1 - e^{-\mu t/m})\,dt.$$

Integrating, we have

$$x = \frac{gm}{\mu}\left[t + \frac{m}{\mu}e^{-\mu t/m}\right] + B, \qquad \ldots(1)$$

where B is a constant.

But initially when $t = 0, x = 0$.

$$\therefore \quad 0 = \frac{gm}{\mu}\left[\frac{m}{\mu}\right] + B. \qquad \ldots(2)$$

Subtracting (2) from (1), we have

$$x = \frac{gm}{\mu}\left[\frac{m}{\mu}e^{-\mu t/m} - \frac{m}{\mu} + t\right] = \frac{gm^2}{\mu^2}\left\{e^{-(\mu t/m)} - 1 + \frac{\mu t}{m}\right\}.$$

Ex. 6. *Discuss the motion of a particle projected upwards with a velocity u in a medium whose resistance varies as the velocity.* (Meerut 89)

Sol. Suppose a particle of mass m is projected vertically upwards from a point O with velocity u in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are

(i) The force mkv due to the resistance, acting vertically downwards *i.e.*, against the direction of motion of the particle, and

(ii) the weight mg of the particle acting vertically downwards.

Since both these forces act in the direction of x decreasing, therefore the equation of motion of the particle in upwards motion at time t is

$$m\frac{d^2x}{dt^2} = -mg - mkv$$

or $$\frac{d^2x}{dt^2} = -(g + kv). \qquad \ldots(1)$$

If the particle moves downwards in the same resisting medium and its velocity is v at time t at distance x from the starting point, then its equation of motion in downwards motion will be

$$m\frac{d^2x}{dt^2} = mg - mkv \quad \text{or} \quad \frac{d^2x}{dt^2} = g - kv.$$

If V is the terminal velocity of the particle during its downward motion, then

$$0 = g - kV \quad \text{or} \quad k = g/V.$$

$\therefore$ the equation of motion (1) in upwards motion becomes

$$\frac{d^2x}{dt^2} = -\left(g + \frac{g}{V}v\right) = \frac{-g(V + v)}{V}. \qquad \ldots(2)$$

Relation between v and x.

The equation (2) can be written as

$$v\frac{dv}{dx} = -\frac{g}{V}(V + v)$$

or $$dx = -\frac{V}{g}\frac{v}{v + V}dv = -\frac{V}{g}\frac{(v + V) - V}{v + V}dv$$

$$= -\frac{V}{g}\left[1 - \frac{V}{v + V}\right]dv.$$

Integrating, $x = -\dfrac{V}{g}[v - V\log(v + V)] + A$, where A is a constant.

But initially at O, $x = 0$ and $v = u$.

$$\therefore \quad A = \frac{V}{g}[u - V\log(u + V)].$$

$$\therefore \quad x = -\frac{V}{g}[v - V\log(v + V)] + \frac{V}{g}[u - V\log(u + V)]$$

or $$x = \frac{V}{g}\left[(u - v) + V\log\left(\frac{v + V}{u + V}\right)\right], \qquad \ldots(3)$$

which gives the velocity of the particle at any position.

Relation between v and t.

The equation (2) can also be written as

$$\frac{dv}{dt} = -\frac{g}{V}(v+V).$$

$$\therefore \quad dt = -\frac{V}{g}\frac{dv}{v+V}.$$

Integrating, $t = -\frac{V}{g}\log(v+V) + B$, where B is a constant.

But initially at O, $t = 0$ and $v = u$.

$$\therefore \quad B = \frac{V}{g}\log(u+V).$$

$$\therefore \quad t = -\frac{V}{g}\log(v+V) + \frac{V}{g}\log(u+V)$$

or $$t = \frac{V}{g}\log\frac{u+V}{v+V}, \qquad \text{...(4)}$$

which gives the velocity of the particle at any time t.

Relation between x and t.

From (4), we have

$$\log\frac{u+V}{v+V} = \frac{gt}{V} \quad \text{or} \quad \frac{u+V}{v+V} = e^{gt/V}$$

or $$v + V = (u+V)e^{-gt/V}$$

or $$v = \frac{dx}{dt} = -V + (u+V)e^{-gt/V}$$

or $$dx = [-V + (u+V)e^{-gt/V}]\,dt.$$

Integrating, we get

$$x = -Vt - \frac{V}{g}(u+V)e^{-gt/V} + C, \text{ where } C \text{ is a constant.}$$

Initially at O, $x = 0$ and $t = 0$.

$$\therefore \quad C = \frac{V}{g}(u+V).$$

$$\therefore \quad x = -Vt - \frac{V}{g}(u+V)e^{-gt/V} + \frac{V}{g}(u+V)$$

or $$x = -Vt + \frac{V}{g}(u+V)[1 - e^{-gt/V}], \qquad \text{...(5)}$$

which gives the distance covered by the particle at any time t.

Ex. 7. *Discuss the motion of a particle falling under gravity in a medium whose resistance varies as the velocity.* (Meerut 1992S, 93S)

Sol. Suppose a particle of mass m starts at rest from a point O and falls vertically downwards in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P.

The forces acting on the particle at P are

(i) The force mkv due to the resistance acting vertically upwards *i.e.*, against the direction of motion of the particle, and

(ii) the weight mg of the particle acting vertically downwards.

By Newton's second law of motion the equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2} = mg - mkv$$

or $$\frac{d^2x}{dt^2} = g - kv. \qquad \text{...(1)}$$

If V is the terminal velocity of the particle during its downward motion, then from (1)

$$0 = g - kV \quad \text{or} \quad k = g/V.$$

Putting $k = g/V$ in (1), we get

$$\frac{d^2x}{dt^2} = g - \frac{g}{V}v = \frac{g}{V}(V - v). \qquad \text{...(2)}$$

Relation between v and x.

The equation (2) can be written as

$$v\frac{dv}{dx} = \frac{g}{V}(V - v)$$

or $$dx = \frac{V}{g}\frac{v}{V-v}dv = -\frac{V}{g}\frac{-v}{V-v}dv$$

$$= -\frac{V}{g}\frac{(V-v) - V}{V-v}dv = -\frac{V}{g}\left[1 - \frac{V}{V-v}\right]dv.$$

Integrating, $x = -\frac{V}{g}[v + V\log(V-v)] + A$, where A is a constant.

But initially at O, $x = 0$ and $v = 0$.

$$\therefore \quad A = \frac{V^2}{g}\log V.$$

$$\therefore \quad x = -\frac{V}{g}v - \frac{V^2}{g}\log(V-v) + \frac{V^2}{g}\log V$$

or $$x = -\frac{V}{g}v + \frac{V^2}{g}\log\frac{V}{V-v}, \qquad \text{...(3)}$$

which gives the velocity of the particle at any position.

Relation between v and t.

The equation (2) can also be written as

$$\frac{dv}{dt} = \frac{g}{V}(V - v).$$

$$\therefore \quad dt = \frac{V}{g}\frac{dv}{V-v}.$$

Integrating, we have

$$t = -\frac{V}{g}\log(V - v) + B, \text{ where } B \text{ is a constant.}$$

Initially at O, $t = 0$ and $v = 0$.

$$\therefore \quad B = \frac{V}{g}\log V.$$

$$\therefore \quad t = -\frac{V}{g}\log(V - v) + \frac{V}{g}\log V$$

or $$t = \frac{V}{g}\log\frac{V}{V - v}, \qquad \ldots(4)$$

which gives the velocity of the particle at any time t.

Relation between x and t.

From (4), we have

$$\log\frac{V}{V - v} = \frac{gt}{V} \quad \text{or} \quad \frac{V}{V - v} = e^{gt/V}$$

or $$V - v = Ve^{-gt/V}$$

or $$v = V[1 - e^{-gt/V}]$$

or $$\frac{dx}{dt} = V[1 - e^{-gt/V}]$$

or $$dx = V[1 - e^{-gt/V}]\,dt.$$

Integrating, we get

$$x = Vt + \frac{V^2}{g}e^{-gt/V} + C, \text{ where } C \text{ is a constant.}$$

Initially at O, $x = 0$ and $t = 0$.

$$\therefore \quad C = -\frac{V^2}{g}.$$

$$\therefore \quad x = Vt + \frac{V^2}{g}e^{-gt/V} - \frac{V^2}{g}$$

or $$x = Vt + \frac{V^2}{g}(e^{-gt/V} - 1), \qquad \ldots(5)$$

which gives the distance fallen through in time t.

Ex. 8. *A particle is projected vertically upwards with velocity u, in a medium where resistance is kv^2 per unit mass for velocity v of the particle. Show that the greatest height attained by the particle is* $\frac{1}{2k}\log\frac{g + ku^2}{g}$. (Meerut 1979)

Sol. Let a particle of mass m be projected vertically upwards from a point O with velocity u. If v is the velocity of the particle at time t at a distance x from the starting point O, then the resistance on the particle is mkv^2 in the downward direction *i.e.*, in the direction of x decreasing. The weight mg of the particle also acts vertically downwards. So the equation of motion of the particle during its upward motion is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

or $$v\frac{dv}{dx} = -(g + kv^2), \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

or $$\frac{2kv\,dv}{g + kv^2} = -2k\,dx, \text{ separating the variables.}$$

Integrating, $\log(g + kv^2) = -2kx + A$, where A is a constant.

But initially $x = 0$, $v = u$; $\therefore \quad A = \log(g + ku^2)$.

$$\therefore \quad \log(g + kv^2) = -2kx + \log(g + ku^2)$$

or $$2kx = \log(g + ku^2) - \log(g + kv^2)$$

or $$x = \frac{1}{2k}\log\frac{g + ku^2}{g + kv^2} \qquad \ldots(1)$$

which gives the velocity of the particle at a distance x.

If h is the greatest height attained by the particle then at $x = h$, $v = 0$. Therefore from (1), we have

$$h = \frac{1}{2k}\log\frac{g + ku^2}{g}.$$

Ex. 9. *A particle is projected vertically upwards with a velocity V and the resistance of the air produces a retardation kv^2, where v is the velocity. Show that the velocity V' with which the particle will return to the point of projection is given by*

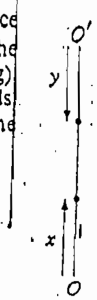
$$\frac{1}{V'^2} = \frac{1}{V^2} + \frac{k}{g}.$$

(Allahabad 1979; Meerut 73, 80, 82S, 84, 86P, 88, 93)

Sol. Let a particle of mass m be projected vertically upwards with a velocity V.

If v is the velocity of the particle at time t, at a distance x from the starting point, the resistance there is mkv^2 in the downward direction (*i.e.*, in the direction of x decreasing). The weight mg of the particle also acts vertically downwards.

$\therefore$ the equation of motion of the particle in the upward motion is



$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

or $$v\frac{dv}{dx} = -(g + kv^2)$$

or $$\frac{2kv\,dv}{g + kv^2} = -2k\,dx.$$

Integrating, $\log(g+kv^2) = -2kx + A$, where A is a constant.
Initially when $x=0, v=V$; $\therefore A = \log(g+kV^2)$.

$\therefore \quad \log(g+kv^2) = -2kx + \log(g+kV^2)$

or $\quad x = \dfrac{1}{2k}\log\dfrac{g+kV^2}{g+kv^2}$.

If h is the maximum height attained by the particle, then $v=0$, when $x=h$.

$$\therefore \quad h = \frac{1}{2k}\log\frac{g+kV^2}{g}. \quad \ldots(1)$$

Now from the highest point O' the particle falls vertically downwards.

Let y be the depth of the particle below the highest point O' after time t and v be the velocity there. Then the resistance at this point is mkv^2 acting in the vertically upwards direction.

$\therefore$ the equation of motion of the particle during its downward motion is

$$m\frac{d^2y}{dt^2} = mg - mkv^2$$

or $\quad v\dfrac{dv}{dy} = g - kv^2 \quad$ or $\quad \dfrac{-2kv\,dv}{g-kv^2} = -2k\,dy$.

Integrating, $\log(g-kv^2) = -2ky + B$, where B is a constant.
At the highest point O', $y=0, v=0$; $\therefore B = \log g$.

$\therefore \quad \log(g-kv^2) = -2ky + \log g$

or $\quad y = \dfrac{1}{2k}\log\dfrac{g}{g-kv^2}$.

If the particle returns to the point of projection O with velocity V', then $v = V'$ when $y = h$.

$$\therefore \quad h = \frac{1}{2k}\log\frac{g}{g-kV'^2}. \quad \ldots(2)$$

From (1) and (2), equating the values of h, we have

$$\frac{1}{2k}\log\frac{g+kV^2}{g} = \frac{1}{2k}\log\frac{g}{g-kV'^2}$$

or $\quad \dfrac{g+kV^2}{g} = \dfrac{g}{g-kV'^2}$

or $\quad (g+kV^2)(g-kV'^2) = g^2$

or $\quad -gkV'^2 + gkV^2 - k^2V^2V'^2 = 0$.

Dividing by $kgV^2V'^2$, we have

$$-\frac{1}{V^2} + \frac{1}{V'^2} - \frac{k}{g} = 0 \quad \text{or} \quad \frac{1}{V'^2} = \frac{1}{V^2} + \frac{k}{g}.$$

Ex. 10. *A particle falls from rest in a medium in which the resistance is kv^2 per unit mass. Prove that the distance fallen in time t is $(1/k)\log\cosh\{t\sqrt{(gk)}\}$.*

If the particle were ascending, show that at any instant its distance from the highest point of its path is $(1/k)\log\sec\{t\sqrt{(gk)}\}$, where t now denotes the time it will take to reach its highest point.

Sol. When the particle is falling vertically downwards, let x be its distance from the starting point after time t. If v is its velocity at this point, then the resistance on the particle is mkv^2 in the vertically upwards direction. The weight mg of the particle acts vertically downwards.

$\therefore$ the equation of motion of the particle during the downward motion is

$$m\frac{d^2x}{dt^2} = mg - mkv^2 \quad \text{or} \quad \frac{d^2x}{dt^2} = g - kv^2$$

or $\quad \dfrac{dv}{dt} = g - kv^2 \quad \left[\because \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt}\right]$

or $\quad \dfrac{dv}{g-kv^2} = dt \quad$ or $\quad \dfrac{dv}{k[(g/k)-v^2]} = dt$.

Integrating, we get

$$\frac{1}{k}\cdot\frac{1}{\sqrt{(g/k)}}\tanh^{-1}\frac{v}{\sqrt{(g/k)}} = t + C_1.$$

But initially when $t=0, v=0$; $\quad \therefore C_1 = 0$.

$$\therefore \quad \frac{1}{\sqrt{(gk)}}\tanh^{-1}\frac{v}{\sqrt{(g/k)}} = t$$

or $\quad \tanh^{-1}\dfrac{v}{\sqrt{(g/k)}} = t\sqrt{(gk)} \quad$ or $\quad \dfrac{v}{\sqrt{(g/k)}} = \tanh\{t\sqrt{(gk)}\}$

or $\quad v = \sqrt{\left(\dfrac{g}{k}\right)}\cdot\dfrac{\sinh\{t\sqrt{(gk)}\}}{\cosh\{t\sqrt{(gk)}\}}$

or $\quad \dfrac{dx}{dt} = \sqrt{\left(\dfrac{g}{k}\right)}\cdot\dfrac{1}{\sqrt{(gk)}}\cdot\dfrac{\sqrt{(gk)}\cdot\sinh\{t\sqrt{(gk)}\}}{\cosh\{t\sqrt{(gk)}\}}$

or $\quad dx = (1/k)\cdot\dfrac{\sqrt{(gk)}\cdot\sinh\{t\sqrt{(gk)}\}}{\cosh\{t\sqrt{(gk)}\}}\,dt$.

Integrating, we get

$$x = (1/k)\cdot\log\cosh\{t\sqrt{(gk)}\} + C_2.$$

But initially when $t=0, x=0$.

$\therefore \quad 0 = (1/k)\cdot\log\cosh 0 + C_2 = (1/k)\cdot\log 1 + C_2 = 0 + C_2$.

$\therefore \quad C_2 = 0$.

$\therefore \quad x = (1/k)\log\cosh\{t\sqrt{(gk)}\}$, which proves the first part of the question.

Vertically Upwards Motion. When the particle is ascending vertically upwards, let y be its distance from the starting point after time T. If v is its velocity at this point, then the resitance is mkv^2 in the downward direction. The weight mg of the particle also acts vertically downwards.

$\therefore$ the equation of motion of the particle during the upward motion is

$$m\frac{d^2y}{dT^2}=-mg-mkv^2 \quad \text{or} \quad \frac{d^2y}{dT^2}=-(g+kv^2)$$

or $$\frac{dv}{dT}=-(g+kv^2) \qquad \left[\because \frac{d^2y}{dT^2}=\frac{dv}{dT}\right]$$

or $$\frac{dv}{g+kv^2}=-dT \quad \text{or} \quad \frac{dv}{k[(g/k)+v^2]}=-dT.$$

Integrating, we get

$$\frac{1}{k}\cdot\frac{1}{\sqrt{(g/k)}}\tan^{-1}\frac{v}{\sqrt{(g/k)}}=-T+C_1.$$

Let t_1 be the time from the point of projection to reach the highest point. Then $T=t_1, v=0$.

$\therefore$ $$0=-t_1+C_1 \quad \text{or} \quad C_1=t_1.$$

$\therefore$ $$\frac{1}{\sqrt{(gk)}}\tan^{-1}\frac{v}{\sqrt{(g/k)}}=t_1-T$$

or $$\tan^{-1}\frac{v}{\sqrt{(g/k)}}=(t_1-T)\sqrt{(gk)}$$

or $$\frac{v}{\sqrt{(g/k)}}=\tan\{(t_1-T)\sqrt{(gk)}\}$$

or $$v=\frac{dy}{dT}=\sqrt{\left(\frac{g}{k}\right)}\cdot\tan\{(t_1-T)\sqrt{(gk)}\}. \quad ...(1)$$

If h is the greatest height attained by the particle and x be the depth below the highest point of the point distant y from the point of projection, then

$$x=h-y. \quad ...(2)$$

Also if t denotes the time from the distance y from the point of projection to reach the highest point, then

$$t=t_1-T. \quad ...(3)$$

From (2), we have $dx=-dy$
and from (3), we have $dt=-dT$.

$\therefore$ $$\frac{dx}{dt}=\frac{dy}{dT}.$$

$\therefore$ from (1), we have

$$\frac{dx}{dt}=\sqrt{\left(\frac{g}{k}\right)}\cdot\tan\{t\sqrt{(gk)}\}.$$

Integrating, we get

$$x=\sqrt{\left(\frac{g}{k}\right)}\cdot\frac{\log\sec\{t\sqrt{(gk)}\}}{\sqrt{(gk)}}+C_2$$

or $$x=(1/k)\log\sec\{t\sqrt{(gk)}\}+C_2. \qquad [\because \textstyle\int\tan x\,dx=\log\sec x]$$

But from (2) and (3), it is obvious that $x=0$, when $t=0$.

$\therefore$ $0=(1/k)\log\sec 0+C_2$ or $C_2=0$.

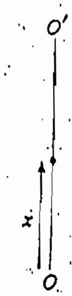
$\therefore$ $x=(1/k)\log\sec\{t\sqrt{(gk)}\}$, which gives the required distance x of the particle from the highest point.

Ex. 11. *A particle of unit mass is projected vertically upwards with velocity V in a medium for which the resistance is kv when the speed of the particle is v. Prove that the particle returns to the point of projection with speed V_1 such that*

$$V+V_1=\frac{g}{k}\log\left(\frac{g+kV}{g-kV_1}\right).$$

(Meerut 1974)

Sol. Let x be the distance of the particle of unit mass from the starting point O at time t in its upward motion. If v is its velocity at this point, then the resistance is kv. The weight $1.g$ of the particle acts vertically downwards.



$\therefore$ the equation of motion of the particle during its upward motion is

$$1.\frac{d^2x}{dt^2}=-g-kv$$

or $$v\frac{dv}{dx}=-(g+kv)$$

or $$dx=-\frac{v\,dv}{kv+g}$$

or $$k\,dx=-\frac{kv\,dv}{kv+g}$$

or $$k\,dx=-\frac{(kv+g)-g}{kv+g}dv=-\left(1-\frac{g}{kv+g}\right)dv.$$

Integrating, $kx=-v+\frac{g}{k}\log(kv+g)+A$, where A is a constant.

But initially when $x=0, v=V$; $\therefore$ $A=V-\frac{g}{k}\log(kV+g)$.

$\therefore$ $$kx=-v+\frac{g}{k}\log(kv+g)+V-\frac{g}{k}\log(kV+g)$$

or $$x = \frac{V - v}{k} + \frac{g}{k^2} \log \left(\frac{kv + g}{kV + g} \right).$$

Let h be the maximum height attained by the particle. Then at the highest point O', $x = h$ and $v = 0$.

$\therefore$ $$h = \frac{V}{k} + \frac{g}{k^2} \log \left(\frac{g}{kV + g} \right). \quad ...(1)$$

Now after coming to instantaneous rest at O', the particle begins to fall vertically downwards. If v be its velocity at the point distant y from O' after time t (measured from the instant it started from O'), then the resistance there is kv in the upward direction and the weight $1.g$ acts vertically downwards.

$\therefore$ the equation of motion of the particle during its downward motion is

$$1 \cdot \frac{d^2y}{dt^2} = g - kv \quad \text{or} \quad v \frac{dv}{dy} = g - kv \quad \text{or} \quad dy = \frac{v\,dv}{g - kv}$$

or $$k\,dy = \frac{kv\,dv}{g - kv} = \frac{g - (g - kv)}{g - kv} dv = \left(\frac{g}{g - kv} - 1 \right) dv.$$

Integrating, $ky = -(g/k) \log (g - kv) - v + B$, where B is a constant.

But at O', $y = 0$ and $v = 0$; $\therefore$ $B = (g/k) \log g$.

$\therefore$ $$ky = \frac{g}{k} \log g - \frac{g}{k} \log (g - kv) - v$$

or $$y = -\frac{v}{k} + \frac{g}{k^2} \log \left(\frac{g}{g - kv} \right).$$

If the particle returns to the point O with velocity V_1, then at O, $v = V_1$ and $y = h$.

$\therefore$ $$h = -\frac{V_1}{k} + \frac{g}{k^2} \log \left(\frac{g}{g - kV_1} \right). \quad ...(2)$$

From (1) and (2), we have

$$\frac{V}{k} + \frac{g}{k^2} \log \left(\frac{g}{g + kV} \right) = -\frac{V_1}{k} + \frac{g}{k^2} \log \left(\frac{g}{g - kV_1} \right)$$

or $$V + V_1 = \frac{g}{k} \left[\log \left(\frac{g}{g - kV_1} \right) - \log \left(\frac{g}{g + kV} \right) \right]$$

or $$V + V_1 = \frac{g}{k} \log \left(\frac{g + kV}{g - kV_1} \right).$$

Ex. 12. *A particle of unit mass is projected vertically upwards with velocity v_0 in a medium for which the resitance is kv when the speed of the particle is v, show that the distance coverd when the velocity is v is given by*

$$x = \frac{v_0 - v}{k} + \frac{g}{k^2} \log \left(\frac{kv + g}{kv_0 + g} \right).$$

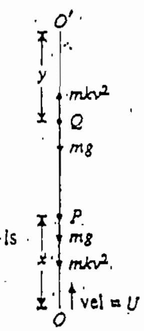
(Meerut 94)

Sol. For complete solution of this problem proceed as in the first part of Ex. 11. Simply replace V by v_0.

Ex. 13. *A particle projected upwards with a velocity U, in a medium whose resistance varies as the square of the velocity, will return to the point of projection with velocity $v_1 = \frac{UV}{\sqrt{(U^2 + V^2)}}$ after a time $\frac{V}{g} \left(\tan^{-1} \frac{U}{V} + \tanh^{-1} \frac{v_1}{V} \right)$, where V is the terminal velocity.*

[Meerut 86S, 96; Kanpur 88]

Solution. Upward motion. Let a particle of mass m be projected vertically upwards from the point O with velocity U. If v is the velocity of the particle at time t at the point P such that $OP = x$, then the resistance at P is mkv^2 acting vertically downwards. Since the weight mg of the particle also acts vertically downwards, therefore the equation of motion of the particle is



$$m \frac{d^2x}{dt^2} = -mg - mkv^2$$

or $$\frac{d^2x}{dt^2} = -(g + kv^2) \quad ...(1)$$

or $$v \frac{dv}{dx} = -(g + kv^2)$$

or $$\frac{2kv\,dv}{g + kv^2} = -2k\,dx.$$

Integrating, $\log (g + kv^2) = -2kx + A$, where A is a constant.

But initially at O, $x = 0$, $v = U$;

$\therefore$ $A = \log (g + kU^2)$.

$\therefore$ $\log (g + kv^2) = -2kx + \log (g + kU^2)$,

or $$x = \frac{1}{2k} \log \left(\frac{g + kU^2}{g + kv^2} \right).$$

If $OO' = h$ is the maximum height attained by the particle, then at O', $x = h$ and $v = 0$.

$\therefore$ $$h = \frac{1}{2k} \log \left(\frac{g + kU^2}{g} \right). \quad ...(2)$$

Now to find the time from O to O', we write the equation (1) as

$$\frac{dv}{dt} = -(g + kv^2) = -k \left(\frac{g}{k} + v^2 \right)$$

or $\quad u^2 \cot^2 \alpha - v_1^2 = u^2 \cot^2 \alpha \cdot \cos^2 \alpha$

or $\quad v_1^2 = u^2 \cot^2 \alpha \cdot (1 - \cos^2 \alpha) = u^2 \cot^2 \alpha \sin^2 \alpha = u^2 \cos^2 \alpha$

or $\quad v_1 = u \cos \alpha$,

i.e., the particle returns to the point of projection with velocity $v_1 = u \cos \alpha$. This proves the first part of the question. Again to find the time from O' to O, the equation (4) can be written as

$$\frac{dv}{dt} = gu^{-2} \tan^2 \alpha \, (u^2 \cot^2 \alpha - v^2)$$

or
$$dt = \frac{u^2}{g} \cot^2 \alpha \cdot \frac{dv}{u^2 \cot^2 \alpha - v^2}.$$

Let t_2 be time from O' to O. Then from O' to O, t varies from 0 to t_2 and v varies from 0 to $u \cos \alpha$. Therefore integrating from O' to O, we have

$$\int_0^{t_2} dt = \frac{u^2}{g} \cot^2 \alpha \int_{v=0}^{v=u \cos \alpha} \frac{dv}{u^2 \cot^2 \alpha - v^2}.$$

$$\therefore \quad t_2 = \frac{u^2 \cot^2 \alpha}{2gu \cot \alpha} \cdot \left[\log \frac{u \cot \alpha + v}{u \cot \alpha - v}\right]_0^{u \cos \alpha}$$

$$= \frac{u}{2g} \cot \alpha \cdot \left[\log \frac{u \cot \alpha + u \cos \alpha}{u \cot \alpha - u \cos \alpha} - \log 1\right]$$

$$= \frac{u}{2g} \cot \alpha \cdot \log \frac{1 + \sin \alpha}{1 - \sin \alpha} = \frac{u}{2g} \cot \alpha \cdot \log \frac{(1 + \sin \alpha) \cdot (1 - \sin \alpha)}{(1 - \sin \alpha) \cdot (1 - \sin \alpha)}$$

$$= \frac{u}{2g} \cot \alpha \cdot \log \frac{(1 - \sin^2 \alpha)}{(1 - \sin \alpha)^2} = \frac{u}{2g} \cot \alpha \cdot \log \left(\frac{\cos \alpha}{1 - \sin \alpha}\right)^2$$

$$= \frac{u}{g} \cot \alpha \cdot \log \frac{\cos \alpha}{1 - \sin \alpha}.$$

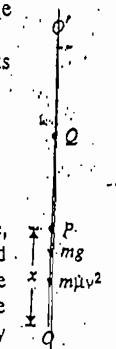
$\therefore$ the required time $= t_1 + t_2 = \dfrac{u}{g} \cot \alpha \left[\alpha + \log \dfrac{\cos \alpha}{1 - \sin \alpha}\right]$.

Ex. 15. *A heavy particle is projected vertically upwards in a medium the resistance of which varies as the square of velocity. If it has a kinetic energy K in its upwards path at a given point, when it passes the same point on the way down, show that its loss of energy is $\dfrac{K^2}{K + K'}$, where K' is the limit to which the energy approaches in its downwards course.*

[Meerut 85S; Lucknow 79]

Sol. Let a particle of mass m be projected vertically upwards with a velocity u from the point O. If v is the velocity of the particle at time t at the point P such that $OP = x$, then the resistance at P is $m\mu v^2$

acting vertically downwards. The weight mg of the particle also acts vertically downwards.



$\therefore$ The equation of motion of the particle during its upwards motion is

$$m \frac{d^2x}{dt^2} = -mg - m\mu v^2$$

or
$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{\mu}{g} v^2\right) \qquad \ldots(1)$$

If H is the maximum height attained by the particle, then at the highest point O' the particle comes to rest and starts falling vertically downwards. If y is the distance fallen in time t from O' and v is the velocity of the particle at this point, then the resistance is $m\mu v^2$ acting vertically upwards.

$\therefore$ the equation of motion of the particle during its downward motion is

$$m \frac{d^2y}{dt^2} = mg - m\mu v^2 \quad \text{or} \quad \frac{d^2y}{dt^2} = g - \mu v^2. \qquad \ldots(2)$$

If V is the terminal velocity of the particle during its downward motion, then $d^2y/dt^2 = 0$ when $v = V$. Therefore $0 = g - \mu V^2$

or
$$\frac{\mu}{g} = \frac{1}{V^2}. \qquad \ldots(3)$$

$\therefore$ From (2), the equation of motion of the particle in downward motion is

$$\frac{d^2y}{dt^2} = g\left(1 - \frac{1}{V^2} v^2\right) \quad \text{or} \quad v \frac{dv}{dy} = \frac{g}{V^2}(V^2 - v^2)$$

or
$$\frac{-2v\,dv}{V^2 - v^2} = -\frac{2g}{V^2} dy.$$

Integrating, $\log (V^2 - v^2) = -\dfrac{2g}{V^2} y + A$, where A is a constant.

But at O', $y = 0$ and $v = 0$; $\therefore$ $A = \log V^2$.

$$\therefore \quad \log (V^2 - v^2) = -\frac{2g}{V^2} y + \log V^2$$

or
$$\frac{2gy}{V^2} = \log V^2 - \log (V^2 - v^2)$$

or
$$y = \frac{V^2}{2g} \log \left(\frac{V^2}{V^2 - v^2}\right). \qquad \ldots(4)$$

If v_1 is the velocity of the particle at the point Q at distance h from O', when falling downwards, then from (4),

$$h = \frac{V^2}{2g} \log\left(\frac{V^2}{V^2 - v_1^2}\right). \quad \ldots(5)$$

Upward Motion. When the particle is moving upwards from O, then from (1) with the help of (3), the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right) \text{ or } v\frac{dv}{dx} = -\frac{g}{V^2}(V^2 + v^2)$$

or $$\frac{2v\,dv}{V^2 + v^2} = -\frac{2g}{V^2}dx.$$

Integrating, $\log(V^2 + v^2) = -\frac{2g}{V^2}x + B$, where B is a constant.

But at O, $x = 0$, $v = u$; $\therefore B = \log(V^2 + u^2)$.

$\therefore \log(V^2 + v^2) = -\frac{2g}{V^2}x + \log(V^2 + u^2)$

or $$x = \frac{V^2}{2g}\log\left(\frac{V^2 + u^2}{V^2 + v^2}\right). \quad \ldots(6)$$

$\therefore$ If v_2 is the velocity of the particle at the point Q in its upward motion then at Q, $x = OQ = H - h$, $v = v_2$.

$$\therefore H - h = \left(\frac{V^2}{2g}\right)\log\left(\frac{V^2 + u^2}{V^2 + v_2^2}\right). \quad \ldots(7)$$

Since H is the maximum height attained by the particle therefore putting $x = H$ and $v = 0$ in (6), we get

$$H = \frac{V^2}{2g}\log\left(\frac{V^2 + u^2}{V^2}\right). \quad \ldots(8)$$

Substituting the values of h and H from (5) and (8) in (7), we get

$$\frac{V^2}{2g}\log\frac{V^2 + u^2}{V^2} - \frac{V^2}{2g}\log\frac{V^2}{V^2 - v_1^2} = \frac{V^2}{2g}\log\frac{V^2 + u^2}{V^2 + v_2^2}$$

or $$\log\frac{(V^2 + u^2)}{V^2} - \log\left(\frac{V^2 + u^2}{V^2 + v_2^2}\right) = \log\left(\frac{V^2}{V^2 - v_1^2}\right),$$

or $$\log\left\{\left(\frac{V^2 + u^2}{V^2}\right) \cdot \left(\frac{V^2 + v_2^2}{V^2 + u^2}\right)\right\} = \log\frac{V^2}{V^2 - v_1^2}$$

or $$\frac{V^2 + v_2^2}{V^2} = \frac{V^2}{V^2 - v_1^2}$$

or $$(V^2 + v_2^2)(V^2 - v_1^2) = V^4$$

or $$(V^2 + v_2^2)V^2 - (V^2 + v_2^2)v_1^2 = V^4$$

or $$v_1^2 = \frac{v_2^2 V^2}{V^2 + v_2^2}. \quad \ldots(9)$$

Now the kinetic energy K of the particle at the point Q at depth h below O' during its upward motion $= \frac{1}{2}mv_2^2$ and the K.E. at Q during downward motion $= \frac{1}{2}mv_1^2$.

Also the terminal K.E. $= \frac{1}{2}mV^2$.

The requried loss of K.E. $= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

$$= \frac{1}{2}m\left[v_2^2 - \frac{v_2^2 V^2}{V^2 + v_2^2}\right], \text{ substituting for } v_1^2 \text{ from (9)}$$

$$= \frac{m}{2} \cdot \frac{v_2^4}{V^2 + v_2^2} = \frac{(\frac{1}{2}mv_2^2)^2}{\frac{1}{2}mV^2 + \frac{1}{2}mv_2^2} = \frac{K^2}{K' + K},$$

where $K' = \frac{1}{2}mV^2$ = limiting K.E. in the medium.

Ex. 16. *If the resistance vary as the 4th power of the velocity, the energy of m lbs. at a depth x below the highest point when moving in a vertical line under gravity will be E tan (mgx/E) when rising, and E tanh (mgx/E) when falling, where E is the terminal energy in the medium.*

(Allahabad 1977; Meerut 77, 94)

Sol. Let a particle of mass m be projected vertically upwards with a velocity u in the given resisting medium. If v is the velocity of the particle at time t at the point whose distance is y from the starting point O, then the resistance on the particle is $m\mu v^4$ acting vertically downwards.

$\therefore$ the equation of motion of the particle during its upward motion is

$$m\frac{d^2y}{dt^2} = -mg - m\mu v^4$$

or $$\frac{d^2y}{dt^2} = -(g + \mu v^4). \quad \ldots(1)$$

If h is the maximum height attained by the particle, then at the highest point, say O', the particle will come to rest and will start falling

downwards. If x is the distance fallen in time t from O' and v is the velocity of the particle at this point, then resistance is $m\mu v^4$ acting vertically upwards.

$\therefore$ the equation of motion of the particle during its downward motion is

$$m\frac{d^2x}{dt^2}=mg-m\mu v^4$$

or $$\frac{d^2x}{dt^2}=g-\mu v^4. \qquad ...(2)$$

If V is the terminal velocity, then $0=g-\mu V^4$

or $$\frac{\mu}{g}=\frac{1}{V^4}. \qquad ...(3)$$

$\therefore$ from (2), the equation of motion of the particle when moving vertically downwards is

$$\frac{d^2x}{dt^2}=g\left(1-\frac{1}{V^4}v^4\right)=\frac{g}{V^4}(V^4-v^4) \qquad ...(4)$$

or $$v\frac{dv}{dx}=\frac{g}{V^4}(V^4-v^4)$$

or $$\frac{2v\,dv}{V^4-v^4}=\frac{2g}{V^4}dx$$

or $$\frac{dz}{V^4-z^2}=\frac{2g}{V^4}dx, \text{ putting } v^2=z \text{ so that } 2v\,dv=dz.$$

Integrating, $\frac{1}{V^2}\tanh^{-1}\frac{z}{V^2}=\frac{2g}{V^4}x+A$, where A is a constant

or $$\frac{1}{V^2}\tanh^{-1}\frac{v^2}{V^2}=\frac{2gx}{V^4}+A.$$

But at O', $x=0$ and $v=0$; $\therefore$ $A=0$.

$\therefore$ $$\frac{1}{V^2}\tanh^{-1}\frac{v^2}{V^2}=\frac{2gx}{V^4} \quad\text{or}\quad \tanh^{-1}\frac{v^2}{V^2}=\frac{2gx}{V^2}$$

or $$v^2=V^2\tanh(2gx/V^2).$$

$\therefore$ the K.E. at a depth x below the highest point when moving downwards is $=\frac{1}{2}mv^2=\frac{1}{2}mV^2\tanh(2gx/V^2)$

$$=\tfrac{1}{2}mV^2\tanh(mgx/\tfrac{1}{2}mV^2)=E\tanh(mgx/E),$$

where $E=\frac{1}{2}mV^2=$ the terminal energy in the medium.

Upward motion. When the particle is moving upwards from O then from (1) with the help of (3), the equation of motion of the particle is

$$\frac{d^2y}{dt^2}=-g\left(1+\frac{v^4}{V^4}\right)$$

or $$v\frac{dv}{dy}=-\frac{g}{V^4}(V^4+v^4)$$

or $$\frac{2v\,dv}{V^4+v^4}=-\frac{2g}{V^4}dy$$

or $$\frac{dz}{V^4+z^2}=-\frac{2g}{V^4}dy, \text{ putting } v^2=z \text{ so that } 2v\,dv=dz.$$

Integrating, $\frac{1}{V^2}\tan^{-1}\frac{z}{V^2}=-\frac{2gy}{V^4}+B$, where B is a constant

or $$\frac{1}{V^2}\tan^{-1}\frac{v^2}{V^2}=-\frac{2gy}{V^4}+B.$$

But at O, $y=0$, $v=u$; $\therefore$ $B=\frac{1}{V^2}\tan^{-1}\frac{u^2}{V^2}$.

$\therefore$ $$\frac{1}{V^2}\tan^{-1}\frac{v^2}{V^2}=-\frac{2gy}{V^4}+\frac{1}{V^2}\tan^{-1}\frac{u^2}{V^2}$$

or $$\frac{2gy}{V^2}=\tan^{-1}\frac{u^2}{V^2}-\tan^{-1}\frac{v^2}{V^2}. \qquad ...(5)$$

At the highest point O', $y=h$ and $v=0$.

$\therefore$ $$\frac{2gh}{V^2}=\tan^{-1}\frac{u^2}{V^2}. \qquad ...(6)$$

If v_1 is the velocity during the upward motion at a depth x below the highest point O', i.e., at a height $y=h-x$ from the starting point O, then from (5), we have

$$\frac{2g}{V^2}(h-x)=\tan^{-1}\frac{u^2}{V^2}-\tan^{-1}\frac{v_1^2}{V^2}. \qquad ...(7)$$

Subtracting (7) from (6), we have

$$\frac{2gx}{V^2}=\tan^{-1}\left(\frac{v_1^2}{V^2}\right) \quad\text{or}\quad v_1^2=V^2\tan\left(\frac{2gx}{V^2}\right).$$

$\therefore$ the K.E. at a depth x below the highest point when rising is

$$=\tfrac{1}{2}mv_1^2=\tfrac{1}{2}mV^2\tan\left(\frac{2gx}{V^2}\right)$$

$$=\tfrac{1}{2}mV^2\tan(mgx/(\tfrac{1}{2}mV^2))=E\tan(mgx/E).$$

Ex. 17. *A particle moving in a straight line is subjected to a resistance kv^3, where v is the velocity. Show that if v is the velocity at time t when the distance is s, $v=u/(1+kus)$ and $t=(s/u)+\frac{1}{2}ks^2$, where u is the initial velocity.* (Meerut 1983, 93S, 95)

Sol. Suppose a particle of mass m starts with velocity u from a point O. Let v be the velocity of the particle at a distance s from the point O at time t. Then the resitance on the particle is mkv^3 acting against the direction of motion of the particle.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2s}{dt^2} = -mkv^3$$

or $$v\frac{dv}{ds} = -kv^3, \quad \left[\because \frac{d^2s}{dt^2} = v\frac{dv}{ds}\right]$$

or $$\frac{dv}{v^2} = -k\,ds.$$

Integrating, $-\frac{1}{v} = -ks + A$, where A is a constant.

But initially when $s = 0, v = u$; $\therefore$ $A = -1/u$.

$\therefore$ $-\frac{1}{v} = -ks - \frac{1}{u} = -\frac{kus+1}{u}$

or $v = u/(1 + kus)$,

which proves the first part of the question.

Now $$v = \frac{ds}{dt} = \frac{u}{1+kus}.$$

$\therefore$ $$dt = \left(\frac{1+kus}{u}\right)ds = \left(\frac{1}{u} + ks\right)ds.$$

Integrating, $t = \left(\frac{s}{u} + \frac{1}{2}ks^2\right) + B$, where B is a constant.

But $t = 0, s = 0$; $\therefore$ $B = 0$.

$\therefore$ $$t = \frac{s}{u} + \frac{1}{2}ks^2.$$

Ex. 18. *A heavy particle is projected vertically upwards with velocity U in a medium, the resistance of which varies as the cube of the particle's velocity. Determine the height to which the particle will ascend.* (Meerut 1980)

Sol. Let a particle of mass m be projected vertically upwards with velocity U. If v be the velocity of the particle at time t at the point whose distance is x from the starting point O, then the resistance on the particle is $m\mu v^3$ acting vertically downwards. Also the weight mg of the particle acts vertically downwards.

$\therefore$ The equation of motion of the particle during its upwards motion is

$$m\frac{d^2x}{dt^2} = -mg - m\mu v^3$$

or $$v\frac{dv}{dx} = -g - \mu v^3. \quad \text{...(1)}$$

If the particle is moving downwards in the given resisting medium then the resistance will act vertically upwards and the equation of motion will be

$$m\,(d^2x/dt^2) = mg - m\mu v^3.$$

If V is the terminal velocity, then $d^2x/dt^2 = 0$, when $v = V$.

$\therefore$ $0 = g - \mu V^3$ or $\mu/g = 1/V^3$.

$\therefore$ from (1), we have

$$v\frac{dv}{dx} = -g\left(1 + \frac{v^3}{V^3}\right) = -\frac{g}{V^3}(V^3 + v^3)$$

or $$dx = -\left(\frac{V^3}{g}\right)\frac{v\,dv}{v^3 + V^3}$$

or $$dx = -\frac{V^3}{g}\frac{v\,dv}{(v+V)(v^2 - vV + V^2)}. \quad \text{...(2)}$$

Now let $$\frac{v}{(v+V)(v^2 - vV + V^2)} = \frac{A}{v+V} + \frac{Bv + C}{v^2 - Vv + V^2} \quad \text{...(3)}$$

or $v = A(v^2 - vV + V^2) + (Bv + C)(v + V)$.

Equating the coefficients of like powers of v from the two sides, we get

$$0 = A + B;\ 1 = -AV + BV + C \text{ and } 0 = AV^2 + CV.$$

Solving, we have, $A = -1/3V$, $B = 1/3V$ and $C = 1/3$.

Substituting in (3), we have

$$\frac{v}{(v+V)(v^2 - vV + V^2)} = -\frac{1}{3V(v+V)} + \frac{v+V}{3V(v^2 - Vv + V^2)}.$$

$\therefore$ from (2), we have

$$dx = -\frac{V^3}{g}\left[-\frac{1}{3V(v+V)} + \frac{v+V}{3V(v^2 - Vv + V^2)}\right]dv$$

$$= -\frac{V^2}{3g}\left[-\frac{1}{v+V} + \frac{\frac{1}{2}(2v - V) + \frac{3}{2}V}{(v^2 - Vv + V^2)}\right]dv$$

$$= -\frac{V^2}{3g}\left[-\frac{1}{v+V} + \frac{(2v - V)}{2(v^2 - Vv + V^2)} + \frac{3V}{2(v^2 - Vv + V^2)}\right]dv$$

$$= -\frac{V^2}{3g}\left[-\frac{1}{v+V} + \frac{(2v - V)}{2(v^2 - Vv + V^2)} + \frac{3V}{2}\cdot\frac{1}{(v - \frac{1}{2}V)^2 + (\sqrt{3}V/2)^2}\right]dv.$$

Integrating, we have

$$x = -\frac{V^2}{3g}\left[-\log(v+V) + \tfrac{1}{2}\log(v^2 - Vv + V^2) + \frac{3V}{2}\cdot\frac{2}{\sqrt{3}V}\tan^{-1}\frac{v - \frac{1}{2}V}{(\sqrt{3}V/2)}\right] + D,$$

where D is a constant

or $$x = -\frac{V^2}{3g}\left[-\log(v+V) + \tfrac{1}{2}\log(v^2 - Vv + V^2) + \sqrt{3}\tan^{-1}\frac{2v - V}{\sqrt{3}V}\right] + D. \quad \text{...(4)}$$

But initially when $x = 0, v = U$.

$$\therefore \quad 0 = -\frac{V^2}{3g}\left[-\log(U+V) + \tfrac{1}{2}\log(U^2 - UV + V^2) + \sqrt{3}\tan^{-1}\frac{2U-V}{\sqrt{3}V}\right] + D. \quad \ldots(5)$$

Subtracting (5) from (4), we have

$$x = \frac{V^2}{3g}\left[-\log(U+V) + \log(v+V) + \tfrac{1}{2}\log(U^2 - UV + V^2) - \tfrac{1}{2}\log(v^2 - Vv + V^2) + \sqrt{3}\left(\tan^{-1}\frac{2U-V}{\sqrt{3}V} - \tan^{-1}\frac{2v-V}{\sqrt{3}V}\right)\right]$$

or $$x = \frac{V^2}{3g}\left[\log\frac{v+V}{U+V} + \tfrac{1}{2}\log\frac{U^2 - UV + V^2}{v^2 - Vv + V^2} + \sqrt{3}\left(\tan^{-1}\frac{2U-V}{\sqrt{3}V} - \tan^{-1}\frac{2v-V}{\sqrt{3}V}\right)\right]$$

If h is the height to which the particle will ascend, then $v = 0$, when $x = h$.

$$\therefore \quad h = \frac{V^2}{3g}\left[\log\left(\frac{V}{U+V}\right) + \tfrac{1}{2}\log\frac{U^2 - UV + V^2}{V^2} + \sqrt{3}\left\{\tan^{-1}\frac{2U-V}{\sqrt{3}V} - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right\}\right]$$

or $$h = \frac{V^2}{3g}\left[\log\frac{(U^2 - UV + V^2)^{1/2}}{(U+V)} + \sqrt{3}\left(\tan^{-1}\frac{2U-V}{\sqrt{3}V} + \frac{\pi}{6}\right)\right]$$

Ex. 19. *A particle of mass m falls from rest at a distance a from the centre of the earth, the motion meeting with a small resistance proportional to the square of the velocity v and the retardation being μ for unit velocity; show that the kinetic energy at a distance x from the centre is*

$$mgr^2\left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu\log\left(\frac{a}{x}\right)\right],$$

the square of μ being neglected and r is the radius of the earth.

[Meerut 90, 95BP]

Sol. Let a particle of mass m fall from rest at a distance a from the centre O of the earth. If v is the velocity of the particle at time t at the point P whose distance from the centre of the earth is x *i.e.*, $OP = x$, then the two accelerations *i.e.*, the forces acting on the unit mass of the particle at P are:

(i) The attraction of the earth towards its centre $= \lambda/x^2$. But on the surface of the earth, the attraction (acceleration) is g and $x = r =$ the radius of the earth.

$$\therefore \lambda/r^2 = g \text{ or } \lambda = r^2 g.$$

$\therefore$ the attraction of the earth towards the centre (*i.e.*, in the direction of x decreasing) is r^2g/x^2.

(ii) The resistance of the medium on the particle $= kv^2$, acting against the direction of motion. But for $v = 1$, the retardation due to the resistance is μ.

$$\therefore \mu = k \cdot 1^2 \text{ or } k = \mu.$$

$\therefore$ the retardation on the particle due to the resistance of the medium is μv^2 acting in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\frac{r^2g}{x^2} + \mu v^2$$

or $$v\frac{dv}{dx} = -\frac{r^2g}{x^2} + \mu v^2 \text{ or } \frac{1}{2}\frac{d(v^2)}{dx} = -\frac{r^2g}{x^2} + \mu v^2$$

or $$\frac{d(v^2)}{dx} - (2\mu)v^2 = -\frac{2r^2g}{x^2}, \quad \ldots(1)$$

which is a linear differential equation in v^2.

$$\text{I.F.} = e^{\int -2\mu\,dx} = e^{-2\mu x}.$$

$\therefore$ the solution of (1) is

$$v^2e^{-2\mu x} = C - \int \frac{2r^2g}{x^2}e^{-2\mu x}\,dx, \text{ where } C \text{ is a constant}$$

or $$v^2(1 - 2\mu x) = C - 2r^2g \cdot \int \frac{1}{x^2}(1 - 2\mu x)\,dx,$$

[expanding $e^{-2\mu x}$ and neglecting the squares and higher powers of μ]

or $$v^2(1 - 2\mu x) = C - 2r^2g \cdot \int\left(\frac{1}{x^2} - \frac{2\mu}{x}\right)dx$$

or $$v^2(1 - 2\mu x) = C + 2r^2g\left(\frac{1}{x} + 2\mu\log x\right). \quad \ldots(2)$$

But initially at $x = a, v = 0$.

$$\therefore 0 = C + 2r^2g\left(\frac{1}{a} + 2\mu\log a\right). \quad \ldots(3)$$

Subtracting (3) from (2), we have

$$v^2(1 - 2\mu x) = 2r^2g\left(\frac{1}{x} - \frac{1}{a} + 2\mu\log x - 2\mu\log a\right)$$

or $$v^2 = 2r^2g\left[\frac{1}{x} - \frac{1}{a} - 2\mu\log\left(\frac{a}{x}\right)\right] \cdot (1 - 2\mu x)^{-1}$$

$$= 2r^2 g\left[\frac{1}{x} - \frac{1}{a} - 2\mu \log\left(\frac{a}{x}\right)\right] \cdot (1 + 2\mu x)$$

[Expanding by binomial theorem and neglecting the squares and higher powers of μ]

$$= 2r^2 g\left[\frac{1}{x} - \frac{1}{a} + 2\mu x\left(\frac{1}{x} - \frac{1}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right]$$

[Neglecting μ^2]

$$= 2r^2 g\left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right].$$

$\therefore$ the kinetic energy of the particle at a distance x from the centre

$$= \frac{1}{2} mv^2$$

$$= mgr^2\left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right].$$

Ex. 20. *A particle moves from rest at a distance a from a fixed point O under the action of a force to O equal to μ times the distance per unit of mass. If the resistance of the medium in which it moves be k times the square of the velocity per unit mass, then show that the square of its velocity when it is at a distance x from O, is* $\frac{\mu x}{k} - \frac{\mu a}{k} e^{2k(x-a)} + \frac{\mu}{2k^2}[1 - e^{2k(x-a)}]$. *Show also that when it first comes to rest it will be at a distance b given by*

$$(1 - 2bk) e^{2bk} = (1 + 2ak) e^{-2ak}.$$

Sol. Let a particle of mass m start from rest at a distance a from the fixed point O. If v is the velocity of the particle at time t at the point P such that $OP = x$, then the two forces acting on the particle are :

(i) the force $m\mu x$ towards O (*i.e.*, in the direction of x decreasing) ;

and (ii) the resistance of the medium $= mkv^2$ acting opposite to the direction of motion *i.e.*, in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -m\mu x + mkv^2$$

or $\quad v\frac{dv}{dx} = -\mu x + kv^2 \quad$ or $\quad \frac{1}{2}\frac{d}{dx}(v^2) = -\mu x + kv^2$

or $\quad \frac{d}{dx}(v^2) - 2kv^2 = -2\mu x, \qquad \ldots(1)$

which is a linear differential equation in v^2.

$\therefore$ I.F. $= e^{\int -2k\,dx} = e^{-2kx}$.

$\therefore$ the solution of the equation (1) is

$v^2 \cdot e^{-2kx} = C - \int 2\mu x\, e^{-2kx}\, dx$, where C is a constant

$$= C - 2\mu\left[x\frac{e^{-2kx}}{-2k} - \int 1 \cdot \frac{e^{-2kx}}{-2k}\, dx\right] \quad \text{[integrating by parts]}$$

$$= C - 2\mu\left[-\frac{x}{2k}e^{-2kx} - \frac{e^{-2kx}}{(-2k)^2}\right]$$

or $\quad v^2 e^{-2kx} = C + \frac{\mu}{k}\left[xe^{-2kx} + \frac{e^{-2kx}}{2k}\right]. \qquad \ldots(2)$

But initially when $x = a$, $v = 0$.

$\therefore \quad 0 = C + \frac{\mu}{k}\left[ae^{-2ka} + \frac{e^{-2ka}}{2k}\right]. \qquad \ldots(3)$

Subtracting (3) from (2), we have

$$v^2 e^{-2kx} = \frac{\mu}{k}\left[xe^{-2kx} + \frac{e^{-2kx}}{2k} - ae^{-2ka} - \frac{e^{-2ka}}{2k}\right]$$

or $\quad v^2 = \frac{\mu x}{k} + \frac{\mu}{2k^2} - \frac{\mu a}{k}e^{2k(x-a)} - \frac{\mu}{2k^2}e^{2k(x-a)}$

or $\quad v^2 = \frac{\mu x}{k} - \frac{\mu a}{k}e^{2k(x-a)} + \frac{\mu}{2k^2}\left[1 - e^{2k(x-a)}\right]. \qquad \ldots(4)$

which proves the first part of the question.

Let V be the velocity of the particle at the centre of force O so that $v = V$ when $x = 0$. Then from (4), we have

$$V^2 = -\frac{\mu a}{k}e^{-2ka} + \frac{\mu}{2k^2}[1 - e^{-2ka}] \neq 0$$

i.e., the particle does not come to rest at the centre of force O. Therefore the particle moves to the left of O with velocity V. As the particle moves to the left of O, the force of attraction and the resistance of the medium will act towards O, and therefore the velocity of the particle will go on decreasing. If the particle comes to instantaneous rest at a distance b from O on its left, then $v = 0$, when $x = -b$.

$\therefore$ from (4), we have

$$0 = -(\mu b/k) - (\mu a/k)e^{2k(-b-a)} + (\mu/2k^2)[1 - e^{2k(-b-a)}]$$

or $\quad \frac{\mu}{2k^2} - \frac{\mu b}{k} = \left(\frac{\mu a}{k} + \frac{\mu}{2k^2}\right)e^{-2kb-2ka}$

or $\quad (1 - 2bk) = (2ak + 1)e^{-2ak} \cdot e^{-2bk}$

or $\quad (1 - 2bk)e^{2bk} = (1 + 2ak)e^{-2ak}$.

Ex. 21. *What do you understand by 'terminal velocity' ? Give reasons that the terminal velocity obtained from vertically downward motion is also used for the motion vertically upwards. Why is it so ?*

(Meerut 95)

Sol. Suppose a particle falls under gravity in a resisting medium. The force of resistance acts vertically upwards on the particle while the force of gravity acts vertically downwards. As the velocity of the particle goes on increasing the force of resistance also goes on increasing. Suppose the force of resistance becomes equal to the weight of the particle when it has attained the velocity V. Then the resultant downward acceleration of the particle becomes zero and so during the subsequent motion the particle falls with constant velocity V, called the terminal velocity.

Thus if a particle is falling under gravity in a resisting medium, then the velocity V when the force of resistance on the particle becomes equal to the weight of the particle so that the downward acceleration of the particle is zero is called the **terminal velocity.**

If a particle is projected vertically upwards in a resisting medium, then both the force of resistance and the weight of the particle act vertically downwards *i.e.,* act against the direction of motion. Consequently the velocity of the particle goes on decreasing and becomes zero when the particle reaches the point of maximum height. Thus in the upwards motion the question of terminal velocity does not arise. The terminal velocity in a resisting medium arises only in the downward motion.

If a particle is projected vertically upwards in a resisting medium and we have to change the constant of proportionality of the force of resistance in terms of the terminal velocity in that resisting medium, then first we write the equation of motion of the particle during its downward motion and using this equation the value of constant of proportionality of the force of resistance is expressed in terms of terminal velocity and is then used in the equation of upward motion.

Ex. 22. *A heavy particle is projected vertically downwards with an initial velocity U, in a resisting medium. Discuss the behaviour of its velocity when U is less than, equal to or greater than the terminal velocity V in the medium.* (Allahabad 1987)

Sol. If the initial velocity U is less than the terminal velocity V, then the velocity of the particle goes on increasing till it becomes equal to the terminal velocity V. After attaining this terminal velocity V, the particle will move with constant velocity V.

If the initial velocity U is equal to the terminal velocity V, then the particle will continue moving with this constant velocity U.

If the initial velocity of projection U is greater than the terminal velocity V, then at first the velocity of the particle goes on decreasing till it becomes equal to the terminal velocity V. After attaining this terminal velocity V, the particle will move with constant velocity V.

Ex. 23. *A particle is projected vertically upwards. Prove that if the resistance of air were constant and equal to $(1/n)$th of its weight, the time of ascent and descent would be as*

$$(n-1)^{1/2} : (n+1)^{1/2}.$$

(Allahabad 1987)

Sol. Suppose a particle of mass m is projected vertically upwards from O with velocity u. Let P be its position at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are,

(i) The weight mg of the particle acting vertically downwards, and

(ii) the force of resistance $(1/n)\,mg$ acting vertically upwards.

The equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2} = -mg - \frac{1}{n}mg$$

or $$\frac{d^2x}{dt^2} = -\left(\frac{n+1}{n}\right)g. \qquad \ldots(1)$$

The equation (1) can be written as

$$\frac{dv}{dt} = -\left(\frac{n+1}{n}\right)g.$$

$\therefore$ $$dt = -\left(\frac{n}{n+1}\right)\frac{1}{g}\,dv. \qquad \ldots(2)$$

Let O' be the point of maximum height *i.e.,* the velocity of the particle becomes zero at O'. Let $OO' = h$ and let t_1 be the time of ascent from O to O'. Then from (2), we have

$$\int_0^{t_1} dt = -\left(\frac{n}{n+1}\right)\frac{1}{g}\int_u^0 dv = \left(\frac{n}{n+1}\right)\frac{1}{g}\int_0^u dv$$

or $$t_1 = \left(\frac{n}{n+1}\right)\frac{u}{g}. \qquad \ldots(3)$$

Again the equation (1) can also be written as

$$v\frac{dv}{dx} = -\left(\frac{n+1}{n}\right)g.$$

$\therefore$ $$dx = -\left(\frac{n}{n+1}\right)\frac{1}{g}\,v\,dv.$$

Integrating from O to O', we get

$$\int_0^h dx = -\left(\frac{n}{n+1}\right)\frac{1}{g}\int_u^0 v\,dv$$

or $$h = \left(\frac{n}{n+1}\right)\frac{1}{g}\cdot\frac{u^2}{2}.$$

During the downwards motion from O' to O, the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = g - \frac{g}{n} = \left(\frac{n-1}{n}\right)g. \qquad \ldots(4)$$

The equation (4) can be written as

$$v\frac{dv}{dx}=\left(\frac{n-1}{n}\right)g.$$

$$\therefore \qquad dx=\left(\frac{n}{n-1}\right)\frac{1}{g}v\,dv. \qquad \ldots(5)$$

Suppose the particle reaches back O with velocity u_1. Then integrating (5) from O' to O, we get

$$\int_0^h dx=\left(\frac{n}{n-1}\right)\frac{1}{g}\int_0^{u_1} v\,dv$$

or $$h=\left(\frac{n}{n-1}\right)\frac{1}{g}\cdot\frac{u_1^2}{2}$$

or $$\left(\frac{n}{n+1}\right)\frac{1}{g}\cdot\frac{u^2}{2}=\left(\frac{n}{n-1}\right)\frac{1}{g}\cdot\frac{u_1^2}{2}, \qquad \text{substituting for } h$$

or $$u_1^2=\left(\frac{n-1}{n+1}\right)u^2 \qquad \text{or} \qquad u_1=\sqrt{\left(\frac{n-1}{n+1}\right)}\,u.$$

Now the equation (4) can also be written as

$$\frac{dv}{dt}=\left(\frac{n-1}{n}\right)g.$$

$$\therefore \qquad dt=\left(\frac{n}{n-1}\right)\frac{1}{g}dv. \qquad \ldots(6)$$

Let t_2 be the time of descent from O' to O. Then integrating (6) from O' to O, we get

$$\int_0^{t_2} dt=\left(\frac{n}{n-1}\right)\frac{1}{g}\int_0^{u_1} dv$$

or $$t_2=\left(\frac{n}{n-1}\right)\frac{1}{g}u_1=\left(\frac{n}{n-1}\right)\frac{1}{g}\cdot\sqrt{\left(\frac{n-1}{n+1}\right)}\,u$$

$$=\frac{u}{g}\cdot\frac{n}{\sqrt{(n-1)}\sqrt{(n+1)}}.$$

$$\therefore \qquad \frac{t_1}{t_2}=\left(\frac{n}{n+1}\right)\frac{u}{g}\cdot\frac{g\sqrt{(n-1)}\sqrt{(n+1)}}{un}=\frac{\sqrt{(n-1)}}{\sqrt{(n+1)}}.$$

Hence $t_1 : t_2=(n-1)^{1/2} : (n+1)^{1/2}$.

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